The stability improvement of spinning top in collision motion

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Abstract. This paper worked on spinning tops' stability improvement with enlightenning and heuristic result. The behavior of the conventional spinning top has been investigated from a player's view and from designer's perspective. A simplified theoretical model is established focusing on precession of the spinning top and simulation on Ansys platform has been proformed as well. It confirmed and verified our stability assessing formula. Worth to mention, a new indicator which characterizes the stability performance of the spinning top have been proposed. This indicator takes the form of ratio of moment of inertia to mass which helps improve spinning top stability along with fast angular velocity and a slight friction.

Keywords: spinning top, torque impulse, stability, moment to mass ratio

1. Introduction

Spinning top, has been a gadget or little toy in entertaining and recreation for numerous children in their childhood. As people whip the top, the top can be balanced upon its axis and spin freely. It also catches the attention of physicists. Dating back to 1895, Felix Klein firstly described a top as a fixed body its mass is distributed symmetrically about its axis and it has one point along the axis of the body that is fixed in space [1]. To making a spinning top, people usually carves the wood in an upside-down circular cone shape. It's polished at the bottom-most part into a sharp pin that is in contact with the ground and that provide the upward normal force.

Up to now, some works have been made on the motion analysis of spinning tops. Leonard Euler had experimented with the spinning top with a top rotating freely without any external torque. Based on this simulation, he then concluded the famous theory of Eulerian free procession in 1751. In 1811, Joseph-Louis Lagrange, a famous mathematician who created the Lagrange Error Bound theory, derived the conclusion that the spinning top with a symmetric shape will move about a fixed point which lies on the symmetric axis and the center of gravity also is on the symmetric axis. He also pointed out that tops that have special limitations such as frictional force or velocity constraints are classified as non-holonomic tops [2]. Jellett deduced an equation about the rise of the top. It turns out that the slipping of the top caused the increase in angular velocity [3]. Hugenholtz uses the calculation of the spinning part and the rolling part of the tops to conclude that the friction in the rolling part of the tops is bigger than the spinning part. By considering the cases of no friction and graphing the determinants of friction, he explains the rise of the top [4]. As a result, both Jellett's equation and Hugenholtz's explanation indicated that for the spinning of a heavy symmetric top, friction surprisingly raises the top instead of ceasing the top.

However, previous works on this topic not bring with a concise formula and brief characterizing of spinning top stability. This paper attempts to provide a parameter as indicator to improve and characterize its stability.

2. Mechanics

2.1. Classical Rotational Motion

Translational motion of centroid is governed by Newton's Second Law,

$$F_{\text{net}} = ma_{cm} \tag{1}$$

Rotational Motion In the inertial frame with the time derivative of angular momentum is given by:

$$\frac{dL}{dt} = N \tag{2}$$

where N is the net external torque.

$$L = I\omega \tag{3}$$

In the body-fixed frame, the time derivative of angular momentum is governed by [7]:

$$\frac{d}{dt}(I\omega) + \omega \times (I\omega) = N \tag{4}$$

$$I\frac{d\omega}{dt} + \omega \times (I\omega) = r \times mg \tag{5}$$

If the top is symmetric about the z-axis (rotational symmetry), the moment of inertia tensor can be simplified as the form:

$$\mathbf{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \tag{6}$$

where $I_{xx} = I_{yy} \neq I_{zz}$.

2.2. Euler angles equations of motion

The motion of a spinning top can be described using Euler angles (θ, ϕ, ψ), where:

θ is the inclination angle of the top's axis relative to the vertical axis, ϕ is the precession angle, ψ is the spin angle of the top itself [8].

The equations of motion for the Euler angles are [9]:

$$\dot{\phi} = \Omega_P = \frac{Mgh}{I\omega} \tag{7}$$

$$\dot{\psi} = \omega - \dot{\phi}\cos(\theta) \tag{8}$$

$$\ddot{\theta} = -\frac{Mgh}{I}\sin(\theta) \tag{9}$$

2.3. Newton-euler equation

For any rigid body, the motion can be written in a form of ordinary differential equation of Newton-Euler Equations:

$$\Sigma F = \frac{d}{dt}(P) \tag{10}$$

$$\Sigma\tau = \frac{d}{dt}(L) \tag{11}$$

Where P is the linear momentum and L is the angular momentum.

2.4. Collision

Some assumptions are set to simplify the analysis and calculation.

1. Collision duration is very short in time. So, we treat such it as a torque impulse [5].
2. It’s assumed that the object is spinning before collision and spinning + precession after collision. Nutation is neglected.

During time interval of collision is from t_1 to t_2, the impulse and torque impulse can be expressed respectively as

$$\int_{t_1}^{t_2} \Sigma F dt = \int_{t_1}^{t_2} \frac{d}{dt}(P) dt = P\Big|_{t_1}^{t_2} = P(t_2) - P(t_1) \tag{12}$$

$$\int_{t_1}^{t_2} \Sigma\tau dt = \int_{t_1}^{t_2} \frac{d}{dt}(L) dt = L\Big|_{t_1}^{t_2} = L(t_2) - L(t_1) \tag{13}$$

The angular impulse-momentum principle states:

$$J = \Delta L \tag{14}$$

Where J is the angular momentum. For a rigid body,

$$L = I\omega \tag{15}$$

where I is the inertia tensor and ω is the angular velocity vector.

The change in angular momentum due to the impulse torque is:

$$J = I(\omega_f - \omega_i) \tag{16}$$

where, impulse Torque (J) is the integral of torque over the very short duration (Δt) of the impulse.

ω_{i} is the angular velocity at the moment before the impulse.

ω_f is the angular velocity at the moment after the impulse.

The impulse can be approximated by using Dirac delta function, which

$$\int_0^\infty \delta(t)dt = 1 \tag{17}$$

Therefore, the very small duration in the collision can be recognized as an impulse [6].

Accordingly the resultant impulsive force can be expressed as

$$\Sigma F\Delta t = P(t^+) - P(t^-) \tag{18}$$

and the resultant impulsive moment can be recognized as

$$\Sigma\tau\Delta t = L(t^+) - L(t^-) \tag{19}$$

The change in angular momentum will eventually change the nutation angle.

2.5. Precession

Precession is the slow rotation of the spinning top's axis around a vertical axis which is the major motion in spinning top. The external torque is from collision and from gravity. The processional angular velocity is:

$$\Omega_P = \frac{Mgh}{I\omega} \tag{20}$$

where: Ω_P is the precessional angular velocity, M is the mass of the top, g is the gravitational acceleration, h is the distance from the center of mass to the pivot point, I is the moment of inertia of the top about its spin axis, ω is the spin angular velocity of the top

2.6. Nutation

Nutation refers to the small periodic oscillations superimposed on the precession motion of the spinning top's axis. Nutation typically arises from perturbations in the initial conditions. The nutation angular frequency (Ω_N) can be approximately given by:

$$\Omega_N = \sqrt{\frac{Mgh}{I}} \tag{21}$$

where: Ω_N is the nutation angular frequency

2.7. Inclination after collision

The impulse in collision makes the variation of translational momentum as

$$F \cdot \Delta t = 2M \cdot \Delta v_n \tag{22}$$

The torque impulse in collision makes variation of angular momentum as

$$J = F \cdot \Delta t \cdot h = 2M \cdot \Delta v_n \cdot h \tag{23}$$

Where h is the vertical distance from the impact point to the spinning tip on the floor. Accordingly, the angular velocity change is

$$\Delta L = L \cdot \tan\,\theta \tag{24}$$

$$2M \cdot \Delta v_n \cdot h = I \cdot \omega \cdot \tan\,\theta \tag{25}$$

Hence, we derive the angular inclination as

$$\theta = \tan^{-1}\frac{2M\cdot\Delta v_n\cdot h}{I\cdot\omega} = \tan^{-1}\frac{2\Delta v_n\cdot h}{\gamma\cdot\omega} \tag{26}$$

$$\gamma = I/M \tag{27}$$

Where we introduce a new variable $\gamma = I/M$, which is the moment-to-mass ration. It's noteable that the deviation angle after collision is inversely proportional to γ in small angle approximation. We take γ as a measure to characterize the stability feature for spinning top.

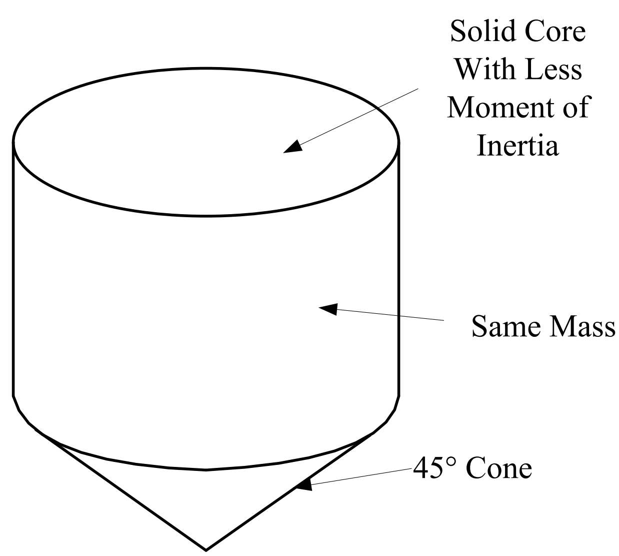
3. Simulation and verification

3.1. Simulation setup

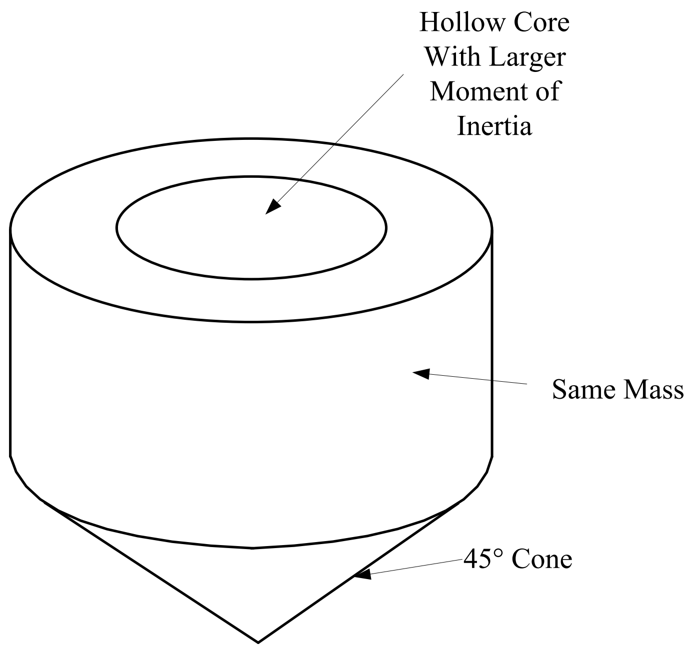
We use two samples of spinning top for comparison in collision stability performance on the floor, which geometry and parameters are shown in Figure 1 and Table 1. Initially, each sample has a certain translational velocity in horizontal direction toward a rigid wall. The rigid wall is set solely for collision purpose with the moving spinning top and bounding it back after collision. Such top-wall collision is better than top-top collision since the wall is not moving and provide a stable reference, whereas top-top collision will experience their random relative collision point and angle. The first sample has solid core with the same mass but a less moment of inertia. The second sample has a harrow core with the same mass but a larger moment of inertia. We model the motion dynamics of these two samples on Ansys simulation software. Before collision they are all heading to the wall. After the short-period collision, they are all rebounded back and move in opposite direction. And at this time each of them will have an inclined angle in vertical direction, which is the result of collision impulse induced angular momentum change. Suppose the they suffer the same torque impulse, why do they have different inclined angles? The reason is from the precession mechanics. If the sample has a large angular momentum at the beginning, such impulse will incur a small inclination. But if the same has a less angular momentum at the beginning, the same impulse will incur a larger amount of inclination. That's the secret of the stability behavior in collision process.

3.2. Simulation verification

In the ANASYS model, two samples are set on a flat surface with the initial horizontal velocity towards the rigid wall to simulate the dynamics in collision. In collision moment, the spinning top is in vertical direction, the wall has a bulge at the same height to ensure the collision point occur at the same height. It's an elastic collision and the horizontal motion is the same for these two tops. Thus, the torque impulse will be the same for both tops. So the inclination angle after collision will be the criteria for us the distinguish their different stability. So larger of inclination, the less stable it is. And vice versa.



(a) Spinning sample A



(b) Spinning sample B

Figure 1. Two samples of spinning top

Table 1. Main parameters of the model

	Spinning Top A	Spinning B
Mass	5.91 kg	5.92 kg
Inertia	8.61e-3 kg m^2	1.51e-2 kg m^2
Outer diameter		
Bulk Modulus	1.79e11 Pa	1.79e11 Pa
Shear Modulus	8.27e10 Pa	8.27e10 Pa
Poisson's Ratio	0.3	0.3
Friction Coefficient	0.1	0.1
Initial spinning angular velocity	30 rad/s	30 rad/s
Initial translational velocity	200mm/s	200mm/s

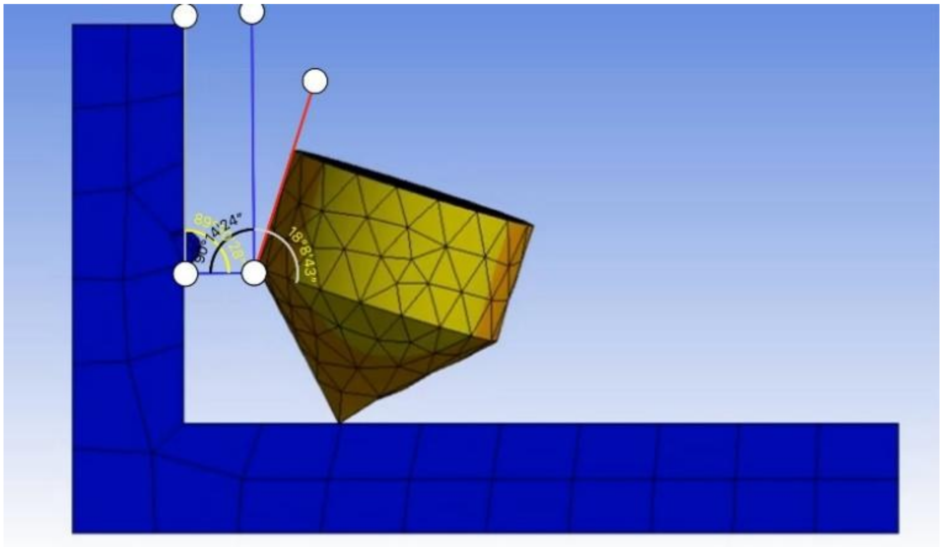
3.3. Simulation result

The simulation is run on ANSYS software, the inclination has been measured respectively for two tops. The result is shown in Figure 2, from which we notice the sample A has a larger inclination, where the sample B has a less inclination. Qualitatively we may say the sample B is more stable than A in collision. Further we make some calculation to check if equation (1) and parameter (2) really help identify the stability. More specifically we evaluate the parameter γ for two tops respectively and compare them with their inclination. Surprisingly we find that their inclination ration is inverse proportional to their ratio. They are in a good agreement with equation (1). So it implies that such parameter can really be an indicator to reflect the stability features.
The ratio of top B's rotational inertia to top A's rotational inertia is calculated, as:

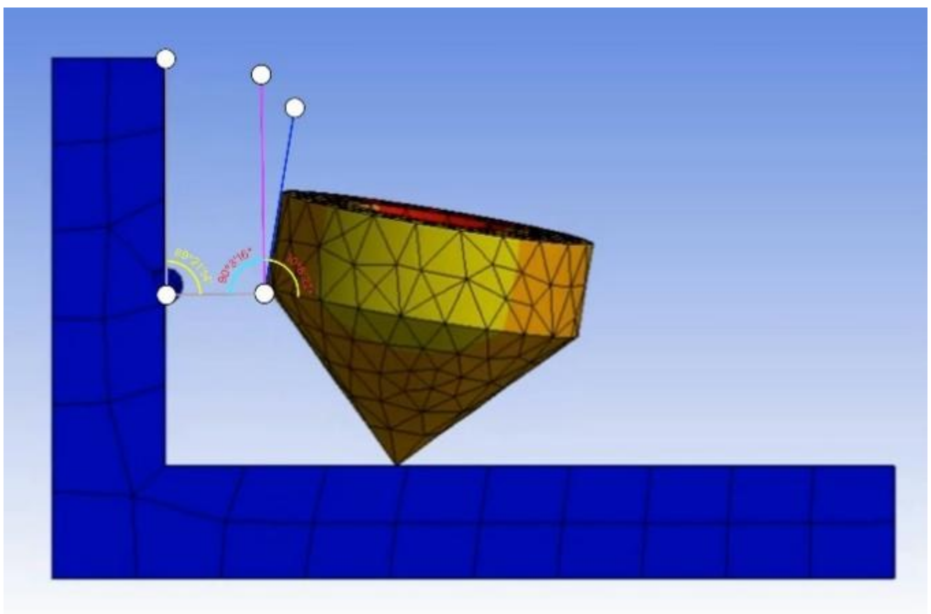
$$\frac{\gamma_2}{\gamma_1} = \frac{I_2}{I_1} = \frac{1.51\times10^{-2}\ \text{kg}\cdot \text{m}^2}{8.61\times10^{-3}\ \text{kg}\cdot \text{m}^2} \approx 1.754 \tag{28}$$

The inclination ratio of sine between top A and top B is

$$\frac{\sin\,\theta_1}{\sin\,\theta_2} = \frac{\sin\,18°}{\sin\,10°} \approx 1.780 \tag{29}$$



(a) Inclination angle of top A (about 18°)



(b) Inclination angle of top B (about 10°)
Figure 2. Inclination of two top samples after collision

By comparing the two ratios, it is concluded that γ is inversely proportional to sin θ.

$$\sin\ \theta \propto \frac{1}{I} \tag{30}$$

Therefore, we take parameter $\gamma = I/M$ as a stability evaluation parameter in top collision dynamics.

4. Conclusion and discussion

Based on theoretical modelling and simulation, we successfully find a route to improve the stability of spinning top and propose a new parameter, i.e., moment-mass-ratio to characterize the stability feature. The simulation result is in a good agreement with the theoretical expectation and support our proposed parameter. Though just a simple toy, it is still enlightening and inspiring for educational and recreational activities.

References

[1] Klein, F., & Sommerfeld, A. (2012). *The Theory of the Top Volume III: Perturbations. Astronomical and Geophysical Applications.* Birkhäuser Basel.

[2] Brecher, K. (2014). Top-ology: a torque about tops. *Bridges Seoul Conference Proceedings*, 51-58.

[3] Jellett, J. H. (1872). *A Treatise on the Theory of Friction*. Hodges, Foster a. Company.

[4] Hugenholtz, N. M. (1952). On tops rising by friction. *Physica*, *18*(8-9), 515-527.

[5] Huang, W., & Li, S. (2023). Hamilton's principle for rigid body collision problems and its applications. *Scientia Sinica Physica, Mechanica & Astronomica*, *53*(5), 254511.

[6] HUANG, W., SU, Q., & LI, S. (2023). Method of analytical mechanics for rigid body collision. *Vibration and Shock*, *42*(15), 42-47. doi:10.13465/j.cnki.jvs.2023.15.006

[7] Chatterjee, A. (1997). *Rigid body collisions: some general considerations, new collision laws, and some experimental data.* Cornell University.

[8] Garstenauer, H., & Kurka, D. I. D. G. (2006). A unified framework for rigid body dynamics. https://github.com/Helikin/Physics-Engine-Thesis

[9] Wan, C. J., Coppola, V. T., & Bernstein, D. S. (1995). Global asymptotic stabilization of the spinning top. *Optimal Control Applications and Methods*, *16*(3), 189-215.