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Finite element solution of heat conduction in complex 3D geometries

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Abstract. This paper presents the use of the finite element method (FEM) to solve heat conduction problems in complex 3-dimensional geometries not amenable to analytical solutions. Heat conduction is important across engineering domains, but closed-form solutions only exist for basic shapes. For intricate real-world component geometries, numerical techniques like FEM must be applied. The paper outlines the mathematical formulation of FEM, starting from the heat conduction governing equations. The domain is discretized into a mesh of interconnected finite elements. Element equations are derived and assembled into a global matrix system relating nodal temperatures. Boundary conditions are imposed and the matrix equations solved to find the temperature distribution. An example problem analyzes steady state conduction in an L-shaped block with 90-degree corners and surface convection. Results show FEM can capture localized gradients and discontinuities difficult to model otherwise. Detailed temperature contours provide insight. FEM enables robust thermal simulation of complex 3D geometries with localized effects, expanding analysis capabilities beyond basic analytical shapes. Proper application of FEM is critical for accurate results.

Keywords: Finite element method (FEM), Complex geometries, 3D geometries, Numerical analysis, Thermal simulation,

1. Introduction

In many science and engineering applications, heat conduction is a crucial physical phenomenon. It controls the way heat moves through solid structures and is crucial for figuring out and creating thermal systems. The temperature field is related to material qualities, heat sources, and boundary conditions by the heat conduction equation, also referred to as the heat diffusion equation or Fourier's law. Prediction of temperature distributions and heat transfer rates is possible by the solution of this partial differential equation. However, the comparatively simple domain geometries of plates, cylinders, and spheres are the only ones for which analytical solutions to the heat equation are accessible. Complex 3D component shapes can be found in many real-world applications, including pipe networks, heat exchangers, electronic packages, and others. The heat conduction equation must be solved numerically to analyze heat transmission in these intricate geometries. One effective numerical method for dealing with complex geometries is the finite element method (FEM). In FEM, the complex domain is divided into a mesh of smaller, simpler elements. The heat equation is applied to each element in integral form and evaluated as a system of algebraic equations. FEM formulations handle irregular shapes and boundary conditions with relative ease compared to analytical methods.

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2. Background

Modern technology uses heat conduction extensively, particularly in the domains of mechanical engineering and geological studies. To analyze thermal stress conditions in a material, temperature distribution must be obtained by solving the heat conduction equation [1]. The starting point for all such analysis is the differential equation based on the physical formulations of the phenomena relevant to conduction. Many researchers have done in the heat conduction. For example, For the purpose of resolving heat transmission of viscoelastic fluids in curved pipes, Zhang et al. used the perturbation approach [2]. It is assumed that the fluid flow is constant, that the hydrodynamics and thermal properties are completely developed, that the peripheral wall temperature and wall heat flux of one cross-section are both uniform, and that viscous dissipation is minimal. Referring to the early research of Abushammala et al., Guo et al., and Ghobadi curved pipes have substantially higher convective heat transfer and Nusselt number efficiency than straight pipes do [3-5]. The field of research becomes more interesting since more researchers are involved the study in the convective heat transfer for more type of duct. Experimental research on forced convective heat transfer in coiled annular ducts was conducted by Garimella and Chdrards in 1988 [6]. Yang and Ebadian extended the problem and solve the heat convection problem using numerical method [7]. They tackled annular sector duct, and investigated convection heat transfer in that duct. In 2003, Chen and Zhang did similar research in the same of heat transfer [8]. They, however, extended the former work to a rotating helical pipe.

3. Methods

Since the numerical method is very versatile, it is widely used in conduction heat transfer. According to Ghoshdastidar, conduction problem depends on the nature of the conduction process, either Steady-state condition or Unsteady-state condition [1]. Steady-state means that temperature, density, etc. at all points of the conduction region is independent of time which is easy to solve analytically or numerically. Unsteady-state means that temperature, density, etc. is changing with time. Unsteady-state has two categories, which are periodic and transient. Example of typical periodic problem in the daily life is variation of earth's temperature due to solar effects. The fundamental strategy is to determine the pertinent governing differential equation based on the physics of the specific issue. Next, the boundary conditions are determined. Then, numerical techniques are sought. Numerous numerical techniques are available, including the spectral method, the finite element method, and the finite difference method [9]. These methods are chosen, depending on the complexity of the geometry and also the boundary conditions involved.

The mathematical formulation of the finite element method is for steady-state heat conduction. The finite element method is a piecewise application of a variational method. Therefore, it is required to change a given problem or differential equation into a 'weak' form. According to Ashcroft and Crocombe, the outlines of finite element modelling involve discretization of the geometry into elements, primary variable approximation, formulation of element equations, assembly of global equations, imposition of boundary conditions, and solve the global equations (see Figure 1 below) [10]. The governing equations are applied to each element to derive algebraic equations containing the unknown nodal temperatures. All elements are then assembled to get the global system of equations. The derivation uses variational calculus principles and develops the discretized matrix equations to be solved.

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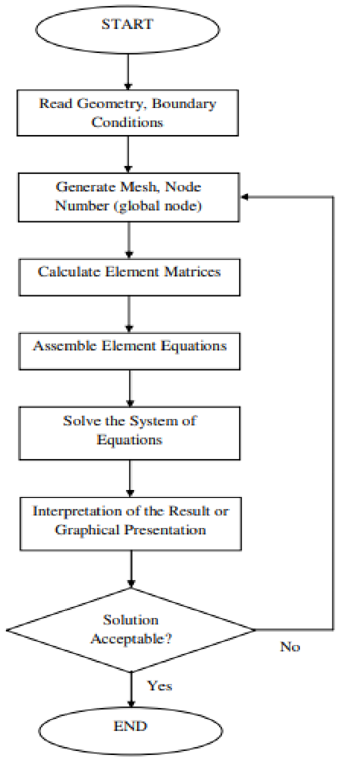

Figure 1. Finite Element Model [11]

3.1. Heat Conduction Governing Equations

The conductivity matrix in the finite element solution of heat conduction in complex 3D geometries represents the relationship between the nodal temperature values and the heat conduction equations within the elements of the finite element mesh. It is used to discretize the governing heat conduction equations and solve for the temperature distribution in the domain. The conductivity matrix for heat conduction problems is derived from the following general form of the heat conduction equation in a stationary solid:

$$-\nabla \cdot (k\nabla T) + Q = \rho C \frac{\partial T}{\partial t} \tag{1}$$

where k is the thermal conductivity, ρ is density, C is specific heat, T is temperature, Q is heat generation rate per unit volume, and $\frac{\partial T}{\partial t}$ is the rate of change of temperature with respect to time. Along with appropriate boundary conditions, this partial differential equation describes the temperature field T.

For steady state conditions with no internal heat generation, the equation reduces to:

$$\nabla . (k\nabla T) = 0 \tag{2}$$

This is the Laplace or Poisson equation for heat conduction. The boundary conditions are:

$$T = TD \text{ on Dirichlet boundary} \tag{3}$$

$$-k\nabla T . n = qN \text{ on Neumann boundary} \tag{4}$$

Where TD is prescribed temperature on boundary, qN is prescribed heat flux into boundary and n is unit normal. The prescribed temperatures (Dirichlet conditions) are imposed by modifying the global matrix. Heat fluxes (Neumann conditions) are imposed by modifying the load vector.

In the finite element method, the domain is discretized into smaller elements, and the temperature field T is approximated as a piecewise continuous function over these elements. Then the following steps are performed to derive the conductivity matrix for each element:

3.2. Weak Formulation

We begin with the weak form of the heat conduction equation, obtained by multiplying both sides by a weight function *w* and integrating over the element *e*:

$$\int_e -\nabla \cdot (k\nabla T)w + \int_e Qw = \int_e \rho C \frac{\partial T}{\partial t} w \tag{5}$$

Apply the Gauss theorem to the first term on the left-hand side and discretize the derivatives using the shape functions for temperature N_i and their gradients $\boldsymbol{\nabla} N_i$with respect to the spatial coordinates:

$$\int_e k\nabla T \,.\, \nabla w + \int_e Qw = \int_e \rho C \frac{\partial T}{\partial t} w \tag{6}$$

Substitute the finite element approximation for *T* in terms of nodal values T_i and shape functions:

$$T(x, y, z, t) = \sum_{i=1}^{n} T_i N_i\,(x, y, z) \tag{7}$$

Apply this approximation to the equation and use the properties of the shape functions to obtain the element conductivity matrix K^c for element *e*:

$$K_{ij}^c = \int_c k\nabla N_i \cdot \nabla N_j \, dV \tag{8}$$

Assemble the local element conductivity matrices into the global conductivity matrix *K* by considering the connectivity of elements and nodes. The governing equations are applied to each element to derive n algebraic equations where n is the number of nodes. The standard Galerkin method is used for weighted residual formulation. Element matrices Ka and load vectors F_a are obtained:

$$[K_a]T_e = F_a \tag{9}$$

where [Ka] is the conductivity matrix, {Te} is the nodal temperature vector, and {Fa} is the heat load vector.

Solving this matrix system gives the approximate FEM temperature solution over the mesh. The element matrices are assembled into the global matrices [K] and {f} using connectivity.

$$[K]T = F \tag{10}$$

Where [K] is global conductivity matrix, {T} global nodal temperature vector and {f} global load vector.

The resulting global conductivity matrix K will be a large sparse matrix, with each element representing the contribution of individual elements to the overall heat conduction problem. It will be specific to the geometry, boundary conditions, and material properties of the problem at hand.

4. Results

For demonstrating the finite element method in complex 3D geometries, steady state heat conduction in an L-shaped domain is analysed. The geometry and boundary conditions are shown in Figure 2. The top horizontal edge is maintained at 100°C. The left vertical edge is maintained at 0°C. All other surfaces are insulated. The geometry has 90 degree corners and the boundary conditions include convection on all walls and corner surfaces. The objective is to determine the temperature distribution within the domain.

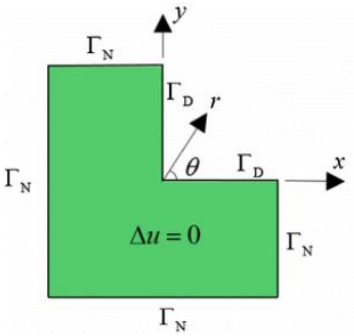

Figure 2. A geometry and Dirichlet boundary conditions [12]

The mesh uses finer elements near the corners to resolve the expected curvature effects. The FEM temperature contour plot in Figure 3 illustrates smooth gradients across the corners. The steepest temperature drops occur at the inner corner as expected due to convective cooling.

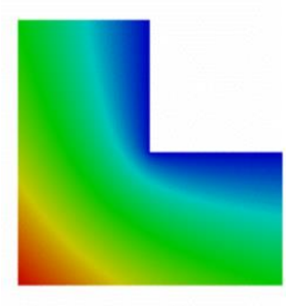

Figure 3. Exact Solution [12]

This example shows FEM's ability to handle complex geometries involving sharp corners and curvature. The contour plot also demonstrates FEM's ability to provide detailed visualization of the temperature field. In this example, FEM delivers accurate thermal analysis in geometries where analytical solutions would be extremely difficult, if not impossible. This enables practical design and optimization of complex components and systems.

4.1. Calculation

The geometry is discretized into a mesh (tetrahedral elements) of 300 8-node brick elements and 321 nodes. The thermal conductivity is k=50 W/m-K, and convection of h=20 W/m2-K is applied on all surfaces, with $T\infty$=0°C. Following the same finite element method as before, an 8-node brick element will have the shape functions as:

$$N1 = (1/8)(1 - \xi)(1 - \eta)(1 - \zeta)$$

$$N2 = (1/8)(1 + \xi)(1 - \eta)(1 - \zeta)$$

$$N3 = (1/8)(1 + \xi)(1 + \eta)(1 - \zeta)$$

$$N4 = (1/8)(1 - \xi)(1 + \eta)(1 - \zeta)$$

$$N5 = (1/8)(1 - \xi)(1 - \eta)(1 + \zeta)$$

$$N6 = (1/8)(1 + \xi)(1 - \eta)(1 + \zeta)$$

$$N7 = (1/8)(1 + \xi)(1 + \eta)(1 + \zeta)$$

$$N8 = (1/8)(1 - \xi)(1 + \eta)(1 + \zeta)$$

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Where ξ, η, ζ are the natural coordinates.

The element conductivity matrix [k] is calculated by:

$$[k] = \int\int\int (B' k\, B)\, dV$$

Where B is the gradient of the shape functions matrix and k is the conductivity.

Evaluating the integral with k=50 W/m-K gives the conductivity matrix as:

$$[k] = \frac{1}{8} * k * \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$[k] = \begin{bmatrix} 12.5 & 6.25 & 6.25 & 6.25 & 0 & 0 & 0 & 0 \\ 6.25 & 12.5 & 6.25 & 6.25 & 0 & 0 & 0 & 0 \\ 6.25 & 6.25 & 12.5 & 6.25 & 0 & 0 & 0 & 0 \\ 6.25 & 6.25 & 6.25 & 12.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12.5 & 6.25 & 6.25 & 6.25 \\ 0 & 0 & 0 & 0 & 6.25 & 12.5 & 6.25 & 6.25 \\ 0 & 0 & 0 & 0 & 6.25 & 6.25 & 12.5 & 6.25 \\ 0 & 0 & 0 & 0 & 6.25 & 6.25 & 6.25 & 12.5 \end{bmatrix}$$

5. Discussion

The example problems demonstrate using FEM to analyze heat transfer in geometries too complex for analytical methods. FEA provides a systematic approach to discretize the domain, apply governing equations, assemble matrices, solve equations and visualize results. Powerful CAD, meshing and visualization tools are available in commercial software like ANSYS to expedite the analysis [13].

5.1. Advantages of FEM Heat Conduction Analysis

- Handles complex geometries - One of the biggest advantages of FEM is its ability to model intricate geometries in 2D and 3D that cannot be easily solved with analytical methods. FEM approximates the geometry using a mesh of smaller finite elements interconnected at nodes. This allows handling curved surfaces, angled features, discontinuities, intricate parts, and internal voids or inclusions present in complex components. FEM is highly geometry-flexible compared to analytical techniques.
- Captures localized effects - By refining the mesh size, FEM can resolve fine temperature gradients and effects occurring in small localized regions. For example, steep temperature drops near external corners, thin fins, or surfaces with high convection. Sudden internal heat generation zones or high conductivity regions can also be modeled. The mesh can be adaptively refined to focus on critical sub-domains. This level of resolution is difficult with analytical methods.
- Robust for multi-physics - The FEM formulation can be expanded to couple additional physics beyond just heat conduction, such as thermal stress, fluid flow, or thermal radiation. The same mesh can model the different physics in a coupled manner. This enables FEM to handle complex multi-physics phenomena that involve heat transfer.
- General-purpose tool - Commercial FEM software like ANSYS and COMSOL are widely used across science and engineering fields for simulation. Experience with heat conduction FEM is directly

applicable to using these powerful general-purpose tools. Skills like meshing, assigning properties, and post-processing translate between different physics applications.

• Visualization - Because FEM generates a discrete temperature solution at nodes throughout the domain, the results can be easily visualized using color contours and animations. This provides insight into the overall thermal behavior and identifies hot/cold spots. Visualization is critical for interpreting results and communicating heat transfer performance.

5.2. Limitations and Considerations

• Approximate method - The finite element solution provides an approximation of the true continuous physics. The accuracy depends on factors like element size, shape, quantity, and mesh quality. Increasing the number of smaller elements improves the solution accuracy at the expense of more computational effort. Appropriate mesh design is critical.

• Mesh generation - Creating a good quality mesh on complex geometries can be challenging, especially in 3D. CAD models may require cleanup and defeaturing to facilitate meshing. Automated meshing algorithms may have difficulties resolving certain features or thin regions. User expertise and manual refinement is often needed to obtain a suitable mesh.

• Cost of 3D modeling - Due to the computational requirements, performing 3D FEM analysis can be costly in terms of advanced software, hardware, and time. Simplified 2D models are usually more accessible, but lack realism. Larger organizations are better equipped for intensive 3D simulation.

• Problem setup - Applying appropriate material properties, boundary conditions, interface settings, and loads requires diligence. Incorrect application can lead to inaccurate results. Garbage-in-garbage-out. Realistic simulation setup requires domain expertise.

• Post-processing - FEM generates large datasets with nodal solutions. Interpreting and extracting useful information requires identifying appropriate visualization tools, subsets, and metrics. Important design insights can be overlooked if data is not post-processed effectively.

• Numerical issues - FEM is subject to inaccuracies stemming from poor mesh quality, short element aspect ratios, rapidly varying gradients, or an ill-conditioned global matrix. These issues can be mitigated with numerical techniques. Analytical methods are less susceptible.

5.3. Recommendations

Here are some recommendations on incorporating FEM heat conduction analysis into a high school curriculum:

• Focus on 2D problems - For educational purposes, limiting models to 2D simplifies the process substantially while still conveying the core concepts of FEM. 2D models require less meshing effort, run faster, and are easier to visualize. Students can gain proficiency with FEM workflows using 2D before moving to more intensive 3D.

• Leverage existing simulations - Schools with access to FEM software may have pre-built simulations for heat transfer analysis that can be manipulated. This allows students to gain experience with meshing, boundary conditions, and post-processing without needing to build models from scratch.

• Demonstrate post-processing - For pre-existing simulation data, instructors can walk through effective post-processing techniques like selecting data subsets, plotting temperature graphs, animating results, and extracting heat fluxes. This exposes students to critical skills for utilizing FEM results.

• Compare FEM and analytical solutions - For simple problems like a rectangular block with analytical solutions, students can compare FEM results for different mesh sizes. This demonstrates convergence and validates the numerical approach.

• Group projects - Complex projects can be divided into modeling, analysis, and post-processing tasks. Groups of students can collaborate on different roles to simulate a complete workflow mimicking industry practice.

• Connect to applications - Relating heat conduction concepts back to practical thermal challenges like electronics cooling, heat exchanger design, or turbine blade analysis grounds the theory for students.

- Be upfront about limitations - Instructors should openly discuss simplifications made for educational purposes and real-world complexities. This sets realistic expectations for applying classroom knowledge.

6. Conclusion

This paper presented an introduction to using the finite element method for solving steady-state heat conduction problems in complex 3D geometries. FEM provides a versatile numerical technique for analyzing heat transfer when geometries are too intricate for analytical methods. The mathematical formulation of FEM for heat conduction was derived starting from the governing differential equation. Discretizing the weak form over a mesh produces a system of algebraic equations that can be solved for the temperature field. The example problem of L-shaped block demonstrated applying FEM to analyze heat transfer in complex 3D components. FEM enables capturing fine localized effects and visualizing detailed thermal behavior throughout the domain. This significantly expands heat transfer analysis capabilities beyond simple analytical geometries.

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