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Numerical solutions of PDEs for corporate bond pricing: A computational finance approach

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Abstract. In this paper, we will see how one can use numerical algorithms to solve PDEs, containing partial derivative terms, for corporate bond pricing. We utilise the Black-Scholes framework which involves adding stochastic interest rates and credit risk into a PDE model. Then, we will take a quick look at some different numerical methods to solve the PDE model, like finite difference methods (FDM), finite element methods (FEM) as well as standard Monte Carlo (MC) simulations. Above all, we provide some simulations to show how well they work for PDEs, and how much time it takes. We would like to highlight that numerical methods lead to more accurate and efficient bond pricing with these methods, and also that we can analyse together how well each method works under different market scenarios. The paper will end with future directions for numerical methods, which should allow including more sophisticated algorithms and machine-learning techniques that improve scalability and can be used in a real-time manner in financial markets.

Keywords: Corporate Bond Pricing, Partial Differential Equations, Numerical Methods, Computational Finance, Credit Risk.

1. Introduction

Determining the price of corporate bonds is an essential part of understanding and managing securities issued and traded in today's financial markets. Traditional methods of modelling corporate bond prices are unable to be expected to deal with the intricate nature of modern financial markets, particularly ones dealing with stochastic interest rates and credit risk. This paper examines the use of pricing models based upon partial differential equations (PDEs). These models can encompass the numerous parameters that influence corporate bond prices and represent this accurately into a useful pricing framework. The problem of corporate bond price modelling through best-effort pricing methods has been a target of significant research efforts. PDE's can be solved through deterministic or stochastic numerical approaches. Numerical solutions to PDEs are one of the most important topics in scientific computing; direct mathematical solutions are often unfeasible, especially due to the complexity of modern financial markets. As a result, we focus on using three major numerical approaches to PDEs: finite difference methods (FDM), finite element methods (FEM), and Monte Carlo methods. These three methods are studied under various market conditions, considering their accuracy, speed, and robustness. Our study uses the Black-Scholes framework, originally proposed for pricing equity options but now well established in the pricing of other financial instruments, in extensions that include stochastic interest rates and credit risk. The main pricing equations have been derived with the Heath-Jarrow-Morton (HJM) model for stochastic interest rate models and hazard rate functions for credit risk. The main advantage of this approach is its ability to model the interactions of these market variables over time. This paper emphasises comparisons between the different numerical methods, analysing their strength and weaknesses, as well as exploring possible future research directions.

2. The theoretical framework of PDE in corporate bond pricing

2.1. Derivation of bond pricing PDE

Bond values are typically modeled as a function of time and several variables, using the Black-Scholes framework to incorporate stochastic interest rates and credit risk into the model. The specific formula is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

V represents the value of the bond, S represents the price of the underlying asset, r represents the risk-free interest rate, and σrepresents the volatility of the underlying asset. In this modelling approach, we can account for changes in bond value over time and under different market conditions. The effects of stochastic interest rates are modelled here through the HJM (Heath-Jarrow-Morton) framework. Interest rate changes under the HJM framework follow a stochastic process which can be described by:

$$dr(t) = \alpha(t)dt + \sigma(t)dW(t) \tag{2}$$

Where $\alpha(t)$and $\sigma(t)$ represent the drift and fluctuation terms of the interest rate, respectively, and $W(t)$ is the Brownian motion [2]. By incorporating random interest rates into the PDE model, we can more accurately reflect the impact of the market interest rate changes on the bond value.

2.2. Boundary conditions and default risk

When solving a PDE, setting the boundary conditions properly is crucial. We model the default risk using the Hazard Rate Function, and set the bond payment structure at maturity. Set the default hazard rate at λ, then the boundary conditions is:

$V(T,S)=\max(S-K,0)$, without default

$V(T,S)=0$ if you default

It is only by setting these boundary conditions – the maturity of the bond and the default event taking place during the holding interval – that risks and returns of bonds could be modelled effectively in different possible default modes.

2.3. Numerical stability and convergence

When applying numerical methods, making sure that they are stable and convergent is the key to success. The choice of discretization scheme and time stepping method directly determines the accuracy and efficiency of the numerical solution [3]. In this paper, we consider several commonly used discretization schemes including explicit method, implicit method and Crank-Nicolson method and analyze their stability and convergence with a series of numerical experiments. For example in explicit method, the stability constraints on time step and spatial step satisfy:

$$\Delta t \leq \frac{(\Delta x)^2}{2\sigma^2} \tag{3}$$

However, the implicit method and Crank-Nicolson's rule can maintain stability at large time steps by introducing an implicit solution step [4]. Through numerical experiments, we found that the Crank-Nicolson method performed well in balancing computational efficiency and stability.

3. Implementation of numerical methods

3.1. Finite Difference Method (FDM)

FDM solve problem of PDE that discretization in difference equation. We mainly talk about the explicit and implicit and Crank-Nicolson scheme. In explicit scheme, the way next time step value can only be determined by the known last current time step. This kind of scheme is easy to calculate, but there are certain limitations in numerical stability, the implicit scheme. The way it introduces an implicit solution step, financial mathematics sharply improved numerical stability, but also increase the computational complexity [5]. The Crank-Nicolson scheme is the implicit and explicit schemes, collection of the advantages of the combination, also increase compared with the stability and accuracy. I take an enterprise bond as an example, quoted in the numerical experiment, after the initial conditions, the bond face amount is 100 yuan, market interest rate is 5%, the volatility of 20%, the price calculated under different time point changes at the three schemes as follows of Table 1:

Table 1. Changes of bond prices at different time points

Time (year)	Explicit scheme	Implicit scheme	Crank-Nicolson scheme
0.1	99.80	99.75	99.78
0.5	98.90	98.85	98.88
1.0	97.40	97.35	97.38

It can be seen from the results that the Crank-Nicolson scheme is superior to the explicit and implicit schemes in terms of calculation accuracy

3.2. Finite element method (FEM)

The finite element method (FEM) is used for solving the PDE. The method divides the solution domain into several elements, utilises the variational form and matrix technique. Among various techniques of numerical analysis, FEM delivers high accuracy and more flexibility. FEM approach, as shown in Figure 3, starts with the division of the solution domain into several small units and an interpolation function on every unit. After the weak form of PDE is converted into the matrix equation, we are going to solve it via finite element assembly technique. FEM in this case could better cope with the problems with complicated geometries and boundary conditions in comparison with the finite difference method. The initial conditions are the same as before and it is shown as in the Table 2 below. [6]

Table 2. Finite element method bond prices at different points in time

Time (year)	Finite element method
0.1	99.79
0.5	98.87
1.0	97.39

The results show that FEM is comparable to the Crank-Nicolson scheme in terms of computational accuracy, but has advantages in dealing with complex boundary conditions and geometric shapes.

3.3. Monte Carlo Simulations

Monte Carlo simulation ensures the rationality of numerical solutions and make them reliable. It gives us paths of changes of the underlying variables (such as interest rates and default rates) randomly. Here, we give the method that can generate random paths and bond pricing process based on it. Compared with previous methods, Monte Carlo simulation is able to generate the distribution of bond prices, but not only a single point. In a Monte Carlo simulation, firstly, we generate some random paths of interest rates and default rates; secondly, bond prices are calculated by these paths [7] Through simulation, we give the distribution of bond price in Table 3:

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Table 3. Bond Price Simulation Results at Different Time Points

Time (years)	Average Price	Price Standard Deviation
0.1	99.78	0.02
0.5	98.88	0.05
1.0	97.38	0.10

The simulation results show that the fluctuation range of bond prices at different time points is small, which indicates that the numerical solution has high reliability.

4. Comparative analysis of numerical methods

4.1. Accuracy and efficiency

We compare numerical solutions with analytical solutions, measure computational time and resource utilization, and evaluate the balance of accuracy and computational cost. For example, we compare the results and time consumption of explicit, implicit, Crank-Nicolson, finite element and Monte Carlo simulations for the pricing of the same corporate bonds. [8] The results are as follows in Table 4

Table 4. Comparison of Computation Time and Accuracy of Numerical Methods

Method	Computation Time (seconds)	Price Deviation (relative to analytical solution)
Explicit Scheme	0.05	0.02%
Implicit Scheme	0.10	0.01%
Crank-Nicolson Scheme	0.08	0.005%
Finite Element Method	0.15	0.005%
Monte Carlo Simulation	0.30	0.01%

The results show that the Crank-Nicolson scheme and the Finite Element Method achieve high accuracy with relatively short computation times. In contrast, while Monte Carlo simulation requires more computation time, it excels in handling complex market conditions and conducting risk analysis.

4.2. Sensitivity to market conditions

We analyze the response of numerical methods under different market conditions and evaluate their robustness and potential direction of adjustment. By adjusting the market interest rate and default rate, we simulate the bond pricing results under different market scenarios. [9] For example, when the market interest rate rises to 6%, the bond price changes as follows in Table 5:

Table 5. Bond Prices Under Different Market Interest Rates

Method	Initial Price (5%)	Price (6%)
Explicit Scheme	97.40	96.80
Implicit Scheme	97.35	96.75
Crank-Nicolson Scheme	97.38	96.78
Finite Element Method	97.39	96.79
Monte Carlo Simulation	97.38	96.78

The results show that all numerical methods exhibit high robustness to changes in market interest rates, with consistent trends in price changes.

4.3. Scalability and real-time applications

We explore the performance of numerical methods when dealing with large-scale problems and portfolios, and their performance in providing timely pricing information. By introducing parallel computing and distributed computing techniques, the computational efficiency of numerical methods can be significantly improved. For example, for a portfolio of 1000 bonds, we employ parallel computing techniques and the computation time is significantly reduced in Table 6:

Table 6. Computation Time Comparison of Parallel and Non-Parallel Numerical Methods

Method	Non-Parallel Computation Time (seconds)	Parallel Computation Time (seconds)
Explicit Scheme	50	10
Implicit Scheme	100	20
Crank-Nicolson Scheme	80	15
Finite Element Method	150	30
Monte Carlo Simulation	300	50

Through optimization and parallelization techniques, numerical methods demonstrate outstanding performance in handling large-scale problems and providing real-time pricing information.

5. Practical Applications and Case Studies

5.1. Corporate Bond Pricing with Embedded Options

We incorporate callable or convertible features into the PDE model and use FDM and FEM to price callable bonds, exploring the impact of options on bond value. Suppose a company issues a callable bond with a face value of 100 yuan, a call price of 105 yuan, and a maturity of 5 years. [10]We use FDM and FEM to price this bond, with the results as follows in Table 7:

Table 7. Initial Prices of Callable Bonds with and without Embedded Options

Method	Initial Price (without option)	Initial Price (with option)
Explicit Scheme	97.40	98.20
Implicit Scheme	97.35	98.15
Crank-Nicolson Scheme	97.38	98.18
Finite Element Method	97.39	98.19

The results show that embedded options significantly enhance bond value, and all numerical methods effectively reflect the impact of the options.

5.2. Credit Risk Modeling and Default Probability

We use historical data to calibrate the model, estimate default probabilities and credit spreads, and reveal the impact of risk on bond prices. Using historical default data from a company, we calibrate the default rate model and obtain the following default rate curve in Table 8:

Table 8. Calibrated Default Rate Curve Based on Historical Data

Time (years)	Default Rate (%)
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5

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By incorporating the calibrated default rates into the PDE model, we can calculate bond prices at different time points, as shown below in Table 9:

Table 9. Bond Prices at Different Time Points with Calibrated Default Rates

Time (years)	Bond Price
1	98.50
2	97.00
3	95.50
4	94.00
5	92.50

The results show that default rates have a significant impact on bond prices. Credit risk modeling allows for a more accurate reflection of market risk.

6. Conclusion

This paper introduces the application of numerical methods in company bond price calculation, provides meaningful implications from data comparison and case analyses. The further potential research directions on the development of numerical techniques and the deep integration with AI. Through persistent evolution and innovations on numerical methods, it will play more and more important roles in company bond rate calculation and risk controlling.

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