Proceedings of CONF-CDS 2025 Symposium: Application of Machine Learning in Engineering
DOI: 10.54254/2755-2721/2025.AST27534

Tietze Extension Does Not Always Work in Constructive Mathematics If Closed Sets Are Defined as Sequentially Closed Sets

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Abstract. We prove that Tietze Extension does not always exist in constructive mathematics if closed sets on which the function we are extending are defined as sequentially closed sets. Firstly, we take a discrete metric space as our topological space. Now all sets open and sequentially closed. Then, we form an unextendible algorithmic function transforming positive integers to 0 and 1, looking at the preimages of these values as our sequentially closed sets. Then we show that if the Tietze theorem conclusion holds for these closed sets then the unextendible function is extendible thus giving us a contradiction. Hence, topology in constructive mathematics have great differences compared to standard topology on Euclidean space. In addition, different definition of special topological space may have converse result on the same theory. Hence, topology in constructive mathematics have great differences compared to standard topology on Euclidean space. In addition, different definition of special topological space may have converse result on the same theory.

Keywords: Constructive Mathematics, Tietze Extension theorem, unextendible function

1. Introduction

Constructive mathematics is different from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase “there exists” as “we can construct”. In order to work constructively, we need to re-interpret not only the existential quantifier but all the logical connectives and quantifiers as instructions on how to construct a proof of the statement involving these logical expressions. There are mathematical schools that study constructive mathematics. The difference between the two schools is that Russian school allows for the principle of constructive choice also known as Markov Principle. It says that if one can refute that the set is empty then one can find an element of the set. The Markov Principle is not allowed by the American school [1-3].

Constructive real numbers (CRN) were defined by Alan Turing (1936), who defined a real is constructive if there exists a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that [4,5]:

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$$\forall k \in \mathbb{N} \ |x - f(x)| < 2^{-k}$$

This operational definition provided by him rejects the classical continuum and establishes computability as the bedrock of existence [1,4,5]. Constructive functions mean that a function on constructive numbers $f : \mathbb{R}_c \to \mathbb{R}_C$ can maps CRNs to CRNs and it is algorithmic. Of course, equivalent CNRs should be mapped to equivalent CRNs.

Markov Tseitin theorem says that all constructive functions are continuous, and Zaslavskii says that the closed bounded interval is not compact in the sense of open covers definition but is compact in terms of existence of finite ϵ -net for each ϵ , see Kushner [2]

In classical mathematics, the Tietze Extension Theorem is a fundamental result in topology, asserting that any real-valued continuous function defined on a closed subset of a normal topological space can be extended to a continuous function on the whole space [6,7]. Its proof relies non-constructively on the Axiom of Choice and Law of Excluded Middle [6].

However, constructive mathematics adopts a stricter view on existence proofs: mathematical objects must be explicitly constructed by an algorithm, and proofs must avoid non-constructive principles. This paradigm shift, led by Brouwer's intuitionism [8], Bishop's constructive analysis [1], and also developed by Markov [3] and Shanin [6], reveals that many classical theorems fail under constructive scrutiny. In particular, the validity of extension theorems like Tietze's becomes highly sensitive to the precise definitions of topological concepts. A critical point of divergence arises in the definition of closed sets. While classical topology typically defines closed sets as complements of open sets, constructive approaches often employ alternative characterizations, such as sequentially closed sets, to better align with computability and explicit definability. These two definitions of closed sets are equivalent in point set topology but not in constructive topology [1,2].

This paper investigates the status of the Tietze Extension Theorem in a constructive setting where closed sets are defined as sequentially closed. We demonstrate that, contrary to the classical case, the Tietze theorem does not always hold constructively under this definition. Our approach centers on a specific counterexample: we construct a metric space X and a sequentially closed subset $A \subseteq X$ with a continuous function $f : A \to \mathbb{R}$ that admits no continuous extension to X . The counter example uses natural features of constructive logic, including the undecidability of disjunctions and the inability to uniformly decide convergence properties, which obstruct the extension process.

This result emphasizes a deeper tension between classical and constructive topology: definitions equivalent in classical topology may bifurcate in constructive settings, leading to divergent theorem validity. It also highlights the necessity of carefully reevaluating foundational tools when transitioning to constructive frameworks. Our work contributes to the broader program of constructive analysis by clarifying the limitations of extension theorems and emphasizing the role of definability in continuity principles.

2. Background

A normal topological space is a space in which any two disjoint closed sets have disjoint open neighborhoods and more over every singular point set is closed.

First consider the particular case of the Tietze extension theorem (two closed sets version), with values equal to zero and one on the two sets [6].

Let X be a normal topological space, and let $A, B \subseteq X$ be disjoint closed subsets. Then:

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Any continuous function $f: A \cup B \rightarrow \mathbb{R}$ such that f is 0 on A and 1 on B, can be extended to a continuous function $F: X \rightarrow \mathbb{R}$ such that $F|_A = f|_A$ and $F|_B = f|_B$.

In particular, if f maps A to a constant $a \in \mathbb{R}$ and B to a constant $b \in \mathbb{R}$, then there exists a continuous function $F: X \rightarrow \mathbb{R}$ such that:

$$\mathrm{F}(\mathrm{x}) = \mathrm{a} \text{ for all } \mathrm{x} \in \mathrm{A}$$

$$\mathrm{F}(\mathrm{x}) = \mathrm{b} \text{ for all } \mathrm{x} \in \mathrm{B}$$

This last statement is a well-known Urysohn Lemma needed to prove general Tietze extension theorem in point set topology.

We define a topological space $X = (\Omega, \tau)$, where $\Omega = \mathbb{N}$ with a discrete metric as our topological space, and use the distance function:

$$\mathrm{d}\left(\mathrm{x}, \mathrm{y}\right) = \begin{cases} 1 \; when \; x \neq y \\ 0 \; when \; x = y \end{cases}$$

In the space X, every subset is closed and also open.

3. Main results

In this part, we will state and prove Lemma 1: Every subset of X is sequentially closed, and Theorem 1: There exists a constructive function that cannot be extended to the entire space, in order to solve this problem.

3.1. Lemma 1

Every subset of X is sequentially closed.

Proof: Take any set $\mathrm{E} \subseteq \mathrm{X}$. If $\mathrm{E} = \emptyset$, then there is no sequence in the set, thus it is sequentially closed. If $\mathrm{E} \neq \emptyset$, then let $\{\mathrm{x_n}\}$ be a convergent sequence in E that converges to x. Since the metric is discrete, the sequence has to stabilize after a certain moment with all the points in it being the same. Thus E contains the limit of the sequence. $\mathrm{Q.E.D.}$

We then state a theorem 1:

3.2. Theorem 1

There exists a constructive function that cannot be extended to the whole space.

Proof: There exists a computable function f on positive integers that takes values only 0 and 1 that does not have an everywhere defined computable extension, this result first proved by Turing can be found in Vershagin Shen [4]. Let $\mathrm{f(x)}$ be the unextendible function. Let A be the set of all points where f is 0 and B be the set of all points where f is 1. Both sets are sequentially closed and the function f is continuous when viewed as a function on $\mathrm{A} \cup \mathrm{B}$. This function does not admit a continuous computable extension to the whole space X.

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Remark: Our topological space is normal indeed. Take any closed $C, D \subset X$, which $C \cap D = \emptyset$. Since any subset of X is open, $C' = C$ and $D' = D$ are open, and $C \subseteq C'$, $D \subseteq D'$, $C' \cap D' = \emptyset$. Therefore, for any C and D, we can find two disjoint neiborhoods C' and D', so the space is normal. It is of course all clear that every singular point set is closed.

If the conclusion of the Tietze theorem holds then we can extend this $f(x)$ to continuous constructive function with values in constructive real numbers. We argue by contradiction and let $F : X \to CRN$ be that program.

Let g : All CRN $\to [0,1]$ be a program that does one step of the program computing the rational approximation of the CRNs and get the output as a.

$$g\left(x\right) = \begin{cases} 0 \; when \; a < 0.5 \\ 1 \; when \; a \geq 0.5 \end{cases}$$

Define $h(x) = g(F(x))$, so it is a composition of two programs applying g to the result of F. Note that the program h always termiantes on all inputs. $h(x)$ is an extension of $f(x)$ and has values of 0 and 1 only.

Hence, we get an extension of an unextendible function which is a contradiction. $Q.E.D.$

Remark: the program g is not well defined as a constructive function on the space of all CRNs because different programs giving equivalent CRNs can be mapped to different 0,1 values. However, the composition h is still well defined so there is no gap in the proof. Since for different programs of the CRN $F(x)$, the corresponding x must be different, so for every x, there is only one possible $F(x)$, and one only possible program that computes CRNs (no matter if it has any other equivalent CRNs or not), hence giving one only possible result in the first step of the program, and only one possible $h(x) = g(F(x))$ as the result.

Remark: In Munkres there are two versions of the Titze extension theorem where the values of the function are in bounded interval and in the whole real line respectively. Above we prove that the version with values in the whole real line does not hold in constructive mathematics when we interpret closed sets as being sequentially closed. The other version of the Tietze theorem also does not hold in the constructive mathematics world for similar reasons.

Urysohn lemma states that if a topological space is normal then any two disjoint closed subsets can be separated by a continuous function. In fact, Urysohn lemma is a very important step in proving Tietze Extension theorem.

Specifically, let X be a normal topological space and let $A, B \subseteq X$ be disjoint closed subsets of X. Then there exists a continuous function:

$$f : X \longrightarrow [0,1]$$

So that:

$f(a) = 0$ for all $a \in A$

$f(b) = 1$ for all $b \in B$

Our solution also prove that Urysohn lemma conclusion does not always exist in constructive mathematics if closed sets are defined as sequentially closed sets.

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4. Conclusion

Through the prove let all sets open and sequentially closed make unextendible function is extendible thus giving us a contradiction. So, that Tietze Extension does not always exist in constructive mathematics when closed sets are defined as sequentially closed sets.

However, since closed sets have other definition in constructive mathematics, we pose out an open question rely on another definition of closed sets in topology. Does the Tietze theorem hold if we define closed sets as the complements of constructive open sets? To our point of view, for this interpretation of closed sets, our answer may be different compared to closed sets defined as consequentially closed sets.

Remark: Constructive open sets means that for every point there is a program giving you an open ball in the set containing the point. In addition, Lacombe open sets means that it is the union of a computable sequence of rational open balls where the enumeration is effectively given and where membership of a point in the set is semi-decidable [2]. Lacombe open sets are particular cases of constructive open sets. for example, in discrete metric space since each single points are open sets, we can choose an unenumerable union of points, which is constructive open sets but not Lacombe open sets. (an instance of an unenumerable set is a complement of enumerable undecidable set, see Post theorem of Vinogradov Schen [9])

Acknowledgement

We are grateful to the Neoscholar company that organized the CIS program during which this research result was obtained. We are thankful to Viktor Chernov and Vladimir Chernov for posing the question and to our TA Nan for the help during the program. Shun Ding, Yang Wan, Luofei Wang and Siqi Xiao contributed equally to this work and should be considered co-first authors.

References

[1] E. Bishop, D. Bridges: Constructive analysis Grundlehren der Mathematis chen Wissenschaften [Fundamental Principles of Mathematical Sciences], 279 Springer-Verlag, Berlin (1985)

[2] B. A. Kushner: Lectures on constructive mathematical analysis (in Russian) Monographs in Mathematical Logic and Foundations of Mathematics. Iz- dat. "Nauka", Moscow, 1973. 447 pp., English translation in Translations of Math ematical Monographs, 60 American Mathematical Society, Providence, R.I. (1984). v+346 pp. ISBN: 0-8218-4513-6

[3] A. A. Markov: On constructive mathematics (in Russian) Trudy Mat. Inst. Steklov 67 (1962), 8–14. English translation in Amer. Math. Soc. Transl. 2, 98 (1971)

[4] A. M. Turing: On computable numbers, with an application to the Entschei dungsproblem, Proc. Lond. Math. Soc., ser. 2, 42 (1936), 230–265 [17] A.M. Turing: Corrections, Proc. Lond. Math. Soc., ser. 2, 43 (1937), 544-546 cite boith papers of turing

[5] A. M. Turing: Corrections, Proc. Lond. Math. Soc., ser. 2, 43 (1937), 544–546

[6] Munkres, J. (2017). Topology (Classic Version, 2nd ed.) Pearson College Div

[7] Tietze, H. (1915). Über Funktionen, die auf einer abgeschlossenen Menge stetig sind

[8] Brouwer, L. E. J. (1907). Over de grondslagen der wiskunde. Maas & van Suchtelen.

[9] Vereshchagin, N. K., & Shen, A. (2002). Computable Functions. Translated from Russian by V. N. Dubrovskiy. Student Mathematical Library, vol. 19, American Mathematical Society