Three-dimensional tilings of blocks and bracelets and related $\{2, 1, 2\}$ sequences

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Abstract. This paper discusses the combinatorial interpretation of the H_n numbers, where each Hn denotes the number of ways to tile a $2 \times 2 \times n$ block with $2 \times 2 \times 1$ "plates" and 6-block "L" shapes. It then investigates a closely related tiling sequence, which is tiling a $2 \times 2 \times n$ bracelet with the same two building blocks, and discusses its relation with Hn. The recursive equation for both integer sequences are found using one to one correspondence, induction and Newton's Sum. Additionally, in the case of H_n numbers, its related "Lucas Style" sequence P_n is found. The relationship between the P_n numbers, the Bn numbers and the Hn numbers is similar to that of the Lucas numbers and the Fibonacci numbers. Finally, identities concerning H_n's generating function and its relations with other existing $\{2, 1, 2\}$ sequences are discussed, and a theorem that generalizes the generating function of all tiling sequences is proposed and proved.

Keywords: three-dimensional tilings, building blocks, integer sequences.

1. Introduction

The Fibonacci numbers are a well-known sequence recursively defined by $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$. The Lucas numbers are also recursively defined by $L_n = L_{n-1} + L_{N-1}$, where $L_0 = 0$ and $L_1 = 2$. However, in the book Proofs that Really Count [1] written by Arthur T. Benjamin and Jennifer J. Quinn, visual interpretations of these numbers were given: f_n counts the number of ways to tile a $1 \times (n - 1)$ board with 1×1 squares and 1×2 dominoes, and L_n counts the number of ways to tile a $1 \times (n - 1)$ bracelet, also with squares and dominoes (an example shown below).

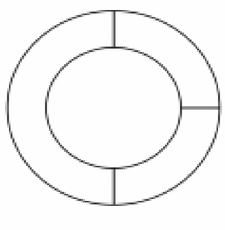

Figure 1. tiling a 1 × 4 bracelet with two squares and a domino.

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These tilings can all be considered as two-dimensional tilings, yet a paper written by Qianyu Guo in 2020 [2] discussed a three-dimensional tiling — tiling a $2 \times 2 \times n$ block with $1 \times 1 \times 1$ cubes and $2 \times 2 \times 1$ plates. This paper is thus inspired and will discuss tiling both a $2 \times 2 \times n$ block and a $2 \times 2 \times n$ bracelet with $2 \times 2 \times 1$ plates and 6-block "L" shapes (also known as "hinges" on the OEIS [3]). The two shapes used for tilings are shown below. We will call the two generated sequences the Hn numbers and the Bn numbers, respectively. To be more percise, let us define H_n[1][2] as the number of ways to tile a $2 \times 2 \times n$ block with $2 \times 2 \times 1$ plates and "L"s, an B_n as the number of ways to tile a $2 \times 2 \times n$ bracelet with $2 \times 2 \times 1$ "plates" and "L"s.

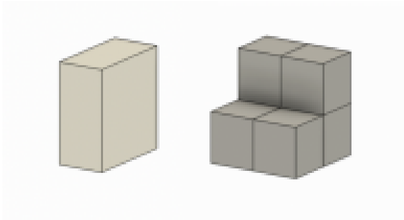

Figure 2. 2 × 2 × 1 "plates" (right) and "L"s or "hinges" (left).

In the following sections, we will devise the recursive formulae for the two sequences using a combinatorial method described in Proofs that Really Count. Additionally, as H_n is an existing $\{2, 1, 2\}$ sequence defined by a generating function on the OEIS, we will also discuss its relationship with other $\{2, 1, 2\}$ sequences, as well as some interesting identities on generating functions. It should be noted that while the first section of this paper concerning the H_n sequence was inspired by Qianyu's work, the B_n sequence is a further inves- tigation solely devised by the author, and the remaining sections are original findings.

Table 1. The first few values for H_n and B_n.

n	0	1	2	3	4	5	6	7	8	9	10
H_n	1	1	3	9	23	61	163	433	1151	3061	8139
B	4	1	9	19	53	131	357	939	2501	6643	17669

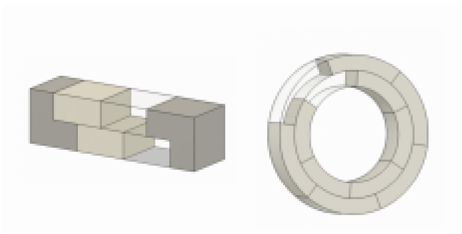

Figure 3. example of an $H_n$tiling (right) and a B_n tiling (left).

2. The recursion formula of H_n

In this section, we will devise the recursion formula of H_n. First, to describe the possible orientations of these plates and "L"s, we will use the notation given by Qianyu Guo in his paper [2] from June 2020. He called the three tan plates on the first row flat plates, side plates, and face plates, from left to right, respectively.

As for the grey "L"s, we will categorize the four orientations on the second row as flat L_s and the four on the third row as face L_s.

[1] H_n is actually an existing sequence in the Online Encyclopedia of Integer Sequences. It's right here: A077996.

[2] Note that we have defined H_0 to be 1 and B_0 to be 4 so that the recurrence formulae are satisfied.

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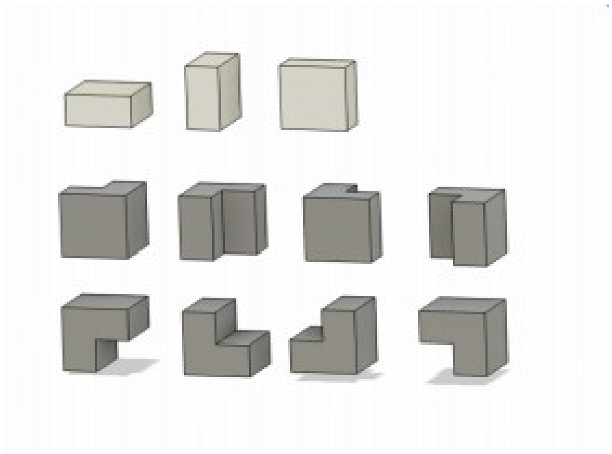

Figure 4. possible orientations of plates and "L"s.

Now, With everything defined, we can start finding the first few values of H_n . First, for a $2 \times 2 \times 1$ block, there is one way to tile it: a single side plate.

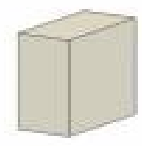

Figure 5. Tilings for a $2 \times 2 \times 1$ box.

For a $2 \times 2 \times 2$ block, we can cut it in the middle and treat it as two $2 \times 2 \times 1$ block, giving $1 \times 1 = 1$ possibility. Additionally, there are two extra unbreakable tilings for the block, which are the two tilings on the right shown below:

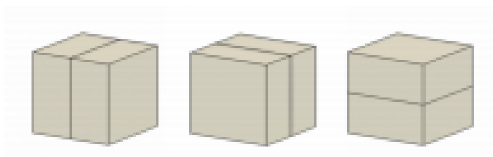

Figure 6. Tilings for a $2 \times 2 \times 2$ box.

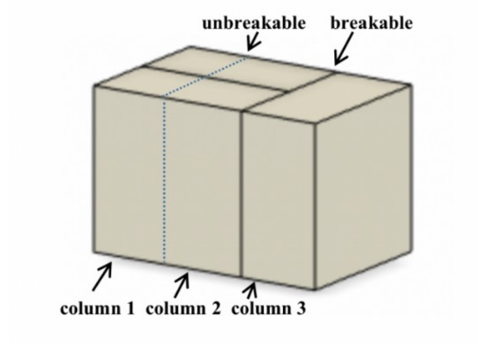

Figure 7. "breakability".

Now, for H_3, we condition on the last breakable column, which is a method introduced in Proofs that Really Count. If it is last breakable at column 2, there are $2 \times 1 = 2$ ways to tile the block, as shown in the first row in the figure below. If it's last breakable at the column 1, there are 1 (the number of ways to tile a $2 \times 2 \times 1$ block) ×2 (the number of unbreakable ways to tile a $2 \times 2 \times 2$ block) = 2 ways.

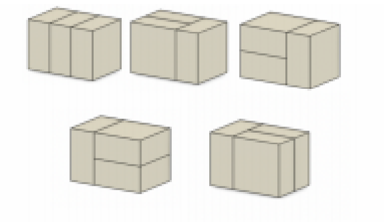

Figure 8. Breakable tilings for a $2 \times 2 \times 3$ box (top row breakable at column 2, bottom row on column 1).

Furthermore, there are four unbreakable tilings as shown below, two of which consist of face "L"s and the other two flat "L"s.

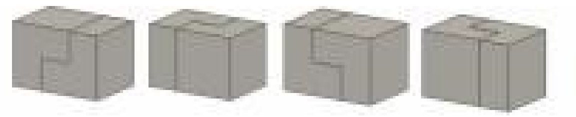

Figure 9. Unbreakable tilings for a 2 × 2 × 3 box.

At this point, we can see that for $n \geq 3$, there will always be 4 unbreakable tilings for a $2 \times 2 \times n$ block, as shown in figure 10, two of which consists of face "L"s and the other two flat "L"s.

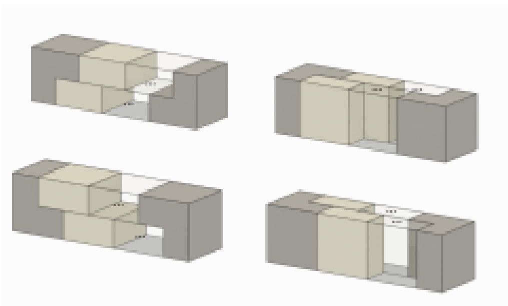

Figure 10. Generalized unbreakable tilings.

We can now find the recursion formula for Hn.

Theorem 1. For every n ≥ 3, we have

$$H_n = 2H_{n-1} + H_{n-2} + 2H_{n-3} .$$

Proof. Condition on the last breakable column. If the tiling is last breakable at the $(n-1)th$ column, there are $1 \times H_{n-1} = H_{n-1}$ possibilities. If it is the $(n-2)th$ column, there are $2 \times H_{n-2} = 2H_{n-2}$ possibilities. If it is the $(n-i)th$ column, where $3 \leq i \leq n$, there are $4H_{n-i}$ possibilities. (when $i = n$, the block is breakable at column 0, which essentially means the block is unbreakable.) Thus, we will have

$$H_n = H_{n-1} + 2H_{n-2} + 4\sum_{i=3}^{n} H_{n-i}$$

Now, to get rid of the sum, we write

$$H_n = H_{n-1} + 2H_{n-2} + 4\sum_{i=3}^{n} H_{n-i}$$

$$H_{n-1} = H_{n-2} + 2H_{n-3} + 4\sum_{i=4}^{n} H_{n-i}$$

Subtracting the two lines and simplifying

$$H_n - H_{n-1} = H_{n-1} + H_{n-2} + 2H_{n-3}$$

$$H_n = 2H_{n-1} + H_{n-2} + 2H_{n-3}$$

As a result of Theorem 1, we have the following corollary.

Corollary 1. For $n \geq 3$, we have

$$4\sum_{i=3}^{n} H_{n-1} = H_n - H_{n-1} - 2H_{n-2}$$

3. B_n 's recursive relation with H_n

In Proofs that Really Count, we "rolled up" the board tilings of the Fibonacci numbers and obtained the bracelet tilings for the Lucas sequence. In this section, we will do the same with tilings of H_n to devise the recursion for B_n .

Before going into the B_n numbers, however, let us first look at the 2D interpretations of unbreakable length n tiles so our visualization will be easier. A single side plate will be represented by a single square. The two unbreakable length n blocks as shown in the upper right section below will be represented by a blue and green domino. Finally, the four unbreakable length n blocks for all $n \geq 3$ will be represented by four colors of n-minos.

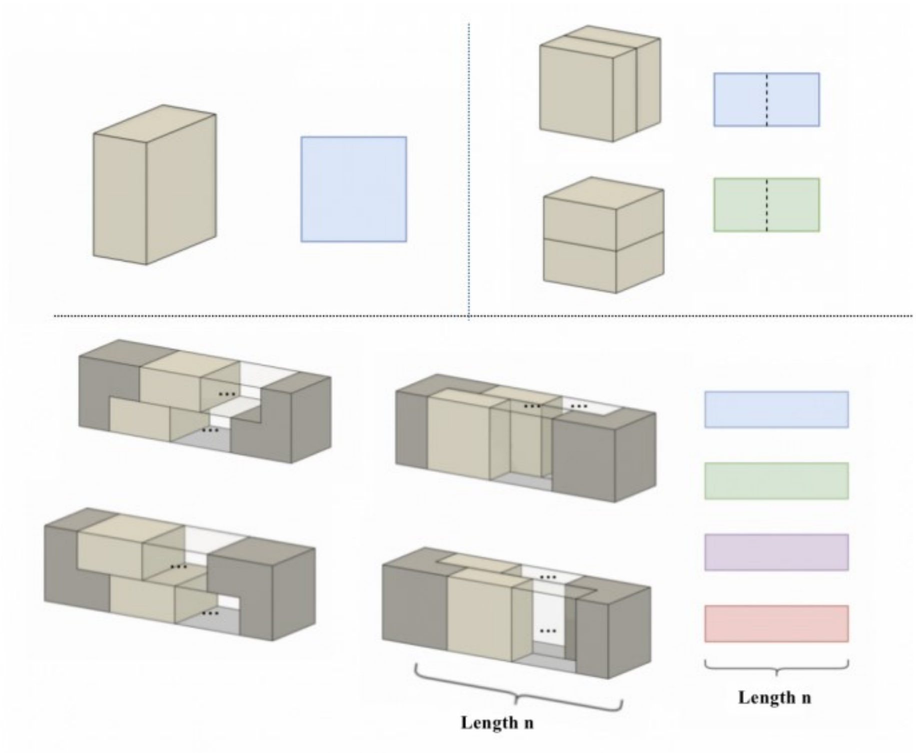

Figure 11. 2-D interpretation.

It is easy to see that tiling a $1 \times n$ board with these tiles will give the same recursive formula as the H_n numbers, since when tiling in the three dimension, we are essentially putting together different lengths of unbreakable "blocks" . Hence, our Bn sequence is the same sequence given by tiling a $1 \times n$ 2D bracelet using these tiles.

Before we continue, we need to introduce one last concept: the phase. A tiled bracelet is in phase if it is breakable between cell n and cell 1, otherwise, it is out of phase. We can see that there are 2 possibilities of orientation for an out of phase bracelet in which a 3-mino occupies cells 1 and n (since the third cell it occupies can be either cell 2 or cell $(n - 1)$,as shown on the first row below). For 4-minos, there are 3 possibilities, and for an n-mino there are $(n - 1)$.

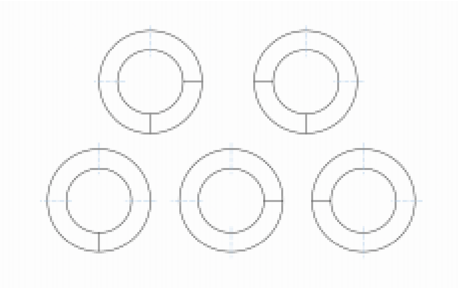

Figure 12: out of phase tilings of a 1×4 bracelet with 3-minos (up) and 4-minos (down)

Finally, with everything defined, we can find the recurrence formula for B_n .

Theorem 2. For $n \geq 2$, we have

$$B_n = B_{n-1} + 3H_n - 4H_{n-1} - H_{n-2} + 4(-1)^n .$$

Proof. For an in-phase length n bracelet, the number of ways to tile it is the same as a length n board since we can "roll" the two ends of the board together. As for out-of-phase bracelets, we will condition on the length of the tile that occupies cell 1. A domino only has one orientation possibility, which is occupying cell 1 and cell n. Taking away the domino and "spreading" the bracelet, we have a length $(n - 2)$ board, in which there are H_{n-2} ways to tile it. Similarly, for a k-mino, there are $(k - 1)$ possible out-of-phase orientations, and the rest of the bracelet will have H_{n-k} possible tilings. Thus, in total, there are 4 (the number of k-minos where $k \geq 3$) $\times(k - 1) \times H_{n-k}$ possible tilings, and we have the following:

$$B_n = H_n + 1 \cdot 2H_{n-2} + 2 \cdot 4H_{n-3} + 3 \cdot 4H_{n-4} + \ldots + (n - 1) \cdot 4H_0$$

$$B_{n-1} = H_{n-1} \cdot 2H_{n-3} + 2 \cdot 4H_{n-4} + 3 \cdot 4H_{n-5} + \ldots + (n - 2) \cdot 4H_0.$$

If we subtract those two lines, we get

$$B_n - B_{n-1} = H_n - H_{n-1} + 2H_{n-2} + 2H_{n-3} + 4\sum_{i=3}^{n} H_{n-i}$$

Using corollary 1, we have

$= H_n - H_{n-1} + 2H_{n-2} + 2H_{n-3} + H_n - H_{n-1} - 2H_{n-2}$

$= 2H_n - 2H_{n-1} + 2H_{n-3}$

$B_n = B_{n-1} + 2H_n - 2H_{n-1} + 2H_{n-3}.$

Using theorem 1, we get

$= B_{n-1} + 2H_n + (2H_{n-1} + H_{n-2} + 2H_{n-3}) - 4H_{n-1} - H_{n-2}$

$= B_{n-1} + 3H_n - 4H_{n-1} - H_{n-2}$

However, for even-length bracelets, there are 4 extra completely unbreakable tilings, as shown below, making the above formula always off by $4(-1)n$. Adding this to our formula gives$B_{n-1} + 3H_n - 4H_{n-1} - H_{n-2} + (-1)n$.

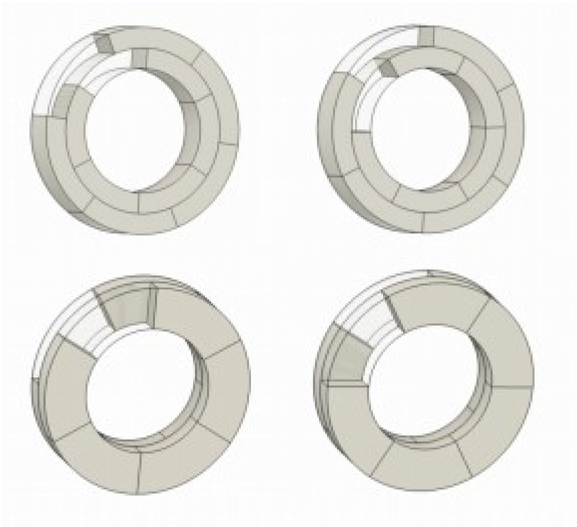

Figure 13. 4 unbreakable 2n-bracelets with flat plates (up) and face plates (down).

Theorem 3. For n ≥ 5, we have

$$B_n = 2B_{n-1} + 2B_{n-2} - B_{n-4} - 2B_{n-5} .$$

Proof. Using theorem 2, we see that $B_{n-1} = B_n + 3H_n - 4H_{n-1} - H_{n-2}$. We can use this to transform the left hand side to the right. We will omit the calculation process in this paper.

4. A lucas-style sequence related to H_n

In this section, we will introduce another Lucas-Style sequence that is closely related to H_n .

First, let us clarify what we mean by Lucas-style. There are two ways to think of a Lucas sequence (L_n):

1. The Bracelet interpretation: L_n counts the number of ways to tile a bracelet of length n with squares and dominos

2. The Binet interpretation: $L_n = \phi_1^n + \phi_2^n$, ϕ_1, $\phi_2$the two roots of $x^2 - x - 1$.

We've looked at the bracelet interpretation, B_n , in the last section. Now, let's look at the Lucas-style sequence corresponding to the Binet interpretation; we will call this new sequence P_n , and they will be defined as:

$$P_n = \theta_1^n + \theta_2^n + \theta_3^n,$$

where θ_1 , θ_2 , and θ_3 are three roots of $x^3 - 2x^2 - x - 2$.

Theorem 4. For $n \geq 3$,

$$P_n = 2P_{n-1} + P_{n-2} + 2P_{n-3} .$$

Proof. Consider a polynomial $P(x)$ of degree n,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 .$$

Let $P(x) = 0$ have roots $x_1, x_2, . . . ,x_n$. Define the sum:

$$P_k = x_1^k + x_2^k + \cdots + x_n^k$$

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Newton's sums tell us that,

$$a_n P_k + a_{n-1}P_{k-1} + a_{n-2}P_{k-2} + \ldots + a_0 P_{k-n} = 0$$

for all k ≥ n. Now, consider the polynomial $P(x) = x^3 - 2x^2 - x - 2$. We know that

$$P_k - 2P_{k-1} - P_{k-2} - 2P_{k-3} = 0.$$

Hence,

$$P_k = 2P_{k-1} + P_{k-2} + 2P_{k-3}$$

for all $k \geq 3$. We can calculate the first few values of P_n .

Table 2. The first few values for P_n and H_n .

n	0	1	2	3	4	5
P_n	3	2	6	20	50	132
H_n	1	1	3	9	23	61

Since P_n 's recursion formula is the same as H_n , we are confident that they have a relationship.

Theorem 5. For $n \geq 0$,

$$7H_n - H_{n+1} = 2P_n = 2(\theta_1^n + \theta_2^n + \theta_3^n)$$

Proof. Let P_n = A · H_n +B · H_{n+1} +C · H_{k+2} . Solving for A,B, and C, we have

$$A = \frac{7}{2}$$

$$B = -\frac{1}{2}$$

$$C = 0.$$

Hence, we can deduce that $P_n = \frac{7}{2}H_n - \frac{1}{2}H_{n+1}$, and we can prove this with strong induction. We first see that the theorem is valid for $0 \leq n \leq 2$. Then, we can say that the theorem is valid for all $0 \leq n \leq k$. Now, we have

$$P_{k+1} = 2P_k + P_{k-1} + 2P_{k-2}$$

and our theorem is true for all $0 \leq n \leq k$, so

$$\begin{aligned} P_{k+1} &= 2\left(\frac{7}{2}H_k - \frac{1}{2}H_{k+1}\right) + \frac{7}{2}H_{k-1} - \frac{1}{2}H_k + 2\left(\frac{7}{2}H_{k-2} - \frac{1}{2}H_{k-1}\right) \\ &= -H_{k+1} + \frac{13}{2}H_k + \frac{5}{2}H_{k-1} + 7H_{k-2} \\ &= 7H_k + \frac{7}{2}H_{k-1} + 7H_{k-2} - H_{k+1} - \frac{1}{2}H_k - H_{k-1} \\ &= \frac{7}{2}H_{k+1} - \frac{1}{2}H_{k+2}, \end{aligned}$$

Thus, our theorem stands now for all $0 \leq n \leq k + 1$, completing the strong induction. Finally, according to our definition of P_n and multiplication on both sides, we have

$$H_{n+1} = 2(\theta_1^n + \theta_2^n + \theta_3^n)$$

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5. Hn 's relation with existing {1,2,1} sequences

After looking at two newly derived sequences in the previous sections, we will now look at H_n 's relation with some existing $\{1,2,1\}$ sequences. Let's first clarify that $\{1,2,1\}$ sequences are those that can be expressed in the form $a_n = 2a_{n-1} + a_{n-2} + 2a_{n-3}$. It's apparent that H_n belongs in this category. Additionally, there are two other existing $\{1,2,1\}$ sequences[3] on the OEIS. They begin with $(1, 2, 5)$, which we will call the T_n sequence, and $(1, 2, 6)$, which we will call the S_n sequence. We will now look at the correlations between the three sequences.

5.1. H_n and T_n

Table 3. The first few values for T_n and H_n .

n	0	1	2	3	4	5	6	7	8	9	10
t_n	1	2	5	14	37	98	261	694	1845	4906	13045
H_n	1	1	3	9	23	61	163	433	1151	3061	8139

The original definition of T_n is the coefficients of the Taylor expansion of $\frac{1}{1-2x-x^2-2x^3}$. Additionally, it can also be interpreted in a combinatorial way using the method given in Proofs that Really Count [1].

Theorem 6. T_n counts the number of ways to tile a $1 \times n$ board with squares of two colors, dominos of one color, and trominos of two colors .

Proof. We just need to prove that this tiling sequence is a $\{1,2,1\}$ sequence starting with 1,2, and 5. As is typical for tiling sequences, we define t_0 to be 1 because there is just one way to tile a board of length 0, and that is to use no dominos, squares, or trominos. Next, there are two ways to tile a 1×1 board: with either color of the squares. Thus, $t_1 = 2$. Then, for a 1×2 board, we can either use 2 squares, giving $2 \times 2 = 4$ ways, or use one domino, giving a total of 5 ways. For a 1×3 board, there are 2^3 (all square)+2×2(one square and one domino)+2(one tromino) = 14 ways. Finally, for $n > 3$, to tile a length n board, we can either add a square to all $(n-1)$ tilings, which has two possibilities due to the square's two colors, or a domino to all $(n-2)$ tilings, giving one possibility, or a tromino to all $(n - 3)$ tilings, giving three possibilities. Hence, for $n > 3, t_n = 2t_{n-1} + t_{n-2} + 2t_{n-3}$, making this tiling sequence identical to T_n.

As a matter of fact, all sequences defined by coefficients of the Taylor ex- pansions of fractions in the form of $\frac{1}{1-ax-bx^2-cx^3}$ have a combinatorial repre - sentation.

Theorem 7. For $\frac{1}{1-ax-bx^2-cx^3} = f_0 x^0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + \ldots, f_n$ counts the number of ways to tile a $1 \times n$ board with squares of a colors, dominos of b colors, and trominos of c colors .

Proof. We can prove this using a method introduced in the book Generating- functionology [4]. First, using the same method as the proving the theorem above, if fn counts the ways to tile a $1 \times n$ board with squares of a colors, dominos of b colors, and trominos of c colors, f_n will satisfy $f_n = af_{n-1} + bf_{n-2} + cf_{n-3}$ for all $n \geq 3$ (we are defining f_0 to be 1). We can also see that $f_1 = a$ and $f_2 = a^2 + b,$ since we can tile a length 2 board with 2 squares or a domino. Now, let there be a function $f(x)$, in which its Taylor expansion is $f(x) = f_0 x^0 + f_1 x^1 + f_2x^2 + f_3x^3 + \ldots$, so now what we have to prove is that this function has the form $\frac{1}{1-ax-bx^2-cx^3}$. To do this, we re-write the recursive formula as $f_{n+3} = af_{n+2} + bf_{n+1} + cf_n$, and multiply each term by x^n . Finally, we sum over the values of n for which the recurrence is valid, which is for $n \geq 0$.

We will then have

$$f_{n+3} = af_{n+2} + bf_{n+1} + cf_n$$

[3] A077938 and A224232

$$\sum_{n\geq 0} f_n + 3x^n = \sum_{n\geq 0} af_{n+2}x^n + \sum_{n\geq 0} bf_{n+1}x^n + \sum_{n\geq 0} cf_nx^n$$

$$\frac{f(x) - f_0x^0 - f_1x^1 - f_2x^2}{x^3} = a\frac{f(x) - f_0x^0 - f_1x^1}{x^2} + b\frac{f(x) - f_0x^0}{x} + cf(x)$$

$$\frac{f(x) - 1 - ax - (a^2 + b)x^2}{x^3} = a\frac{f(x) - 1 - ax}{x^2} + b\frac{f(x) - 1}{x} + cf(x)$$

multiplying each term by x^3 , we have

$$f(x) - 1 - ax - a^2x^2 - bx^2 = axf(x) - ax - a^2x^2 + bx^2f(x) - bx^2 + cx^3f(x)$$

$$(1 - ax - bx^2 - cx^3)f(x) = 1 + ax + a^2x^2 + bx^2 - ax - a^2x^2 - bx^2$$

$$f(x) = \frac{1}{1 - ax - bx^2 - cx^3}$$

Now, going back to our H_n s and T_n s. As said before, they have the same recursive formula, so we are confident that they are related. Indeed, using the same method as theorem 5, which is deducing a relationship with linear equations and then proving it by strong induction, we have the following

Theorem 8. For $n \geq 1$,

$$H_n = T_n - T_{n-1}$$

This theorem is basically saying that H_n is the amount we need to add to obtain the next T_n. Inspired by this, we have

Theorem 9. For $n \geq 0$,

Proof.

$$\sum_{k=0}^{n} H_k = H_0 + H_1 + H_2 + H_3 + \cdots + H_n$$

$$= 1 + (t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) + \cdots + (t_n - t_{n-1})$$

$$= 1 - t_0 + t_1 - t_1 + t_2 - t_2 + \cdots + t_{n-1} - t_{n-1} + t_n$$

$$= 1 - 1 + t_n$$

$$= t_n$$

So what is the value of this sum? Inspired by a method in Proofs that Really Count [1], we have

Theorem 10. For $n \geq 1$,

$$\sum_{k=0}^{n} H_k = T_n = \sum_{j\geq 0}^{n/3} \sum_{k\geq 0}^{(n-3j)/2} \binom{n-k-2j}{k}\binom{n-2k-2j}{j} 2^{n-2k-2j}$$

Proof. What we actually have to prove here is that the number of ways to tile a $1 \times n$ board with squares of two colors, dominos of one color, and trominos of two colors is $\sum_{j\geq 0}^{n/3}\sum_{k\geq 0}^{(n-3j)/2}\binom{n-k-2j}{k}\binom{n-2k-2j}{j}2^{n-2k-2j}$. To do this, let a length n tiling use k dominos and j trominos. This means using $n - 2k - 3j$ squares, and $n - k - 2j$ tiles in total. Now, we can first pick k of the $n - k - 2j$ tiles to be dominos, giving $\binom{n-k-2j}{k}$ choices. Then, we pick j of the remaining $n - 2k - 2j$ tiles to be trominos, which gives $\binom{n-2k-2j}{j}$ choices. We also need to note the two possible colors for trominos and squares, which means we need to multiply

$\binom{n-k-2j}{k}\binom{n-2k-2j}{j}$ by $2^j \cdot 2^{n-2k-3j} = 2^{n-2k-2j}$. Finally, for the limits of our sum, we should have $j \le \frac{n}{3}$. As for n, since we've let j tiles be trominos, we will have $k \le \frac{n-3j}{2}$. Note that we don't need to worry about using floor division since we're using binomials. Hence, summing everything together, we have $T_n = \sum_{j\ge 0}^{n/3}\sum_{k\ge 0}^{(n-3j)/2}\binom{n-k-2j}{k}\binom{n-2k-2j}{j}2^{n-2k-2j}$.

As we're talking about the sum of H_n s, here's another interesting theorem that we can prove using the generating function of H_n , which is given on the OEIS as $\frac{1-x}{1-2x-x^2-2x^3}$.

Theorem 11.

$$\sum_{n\ge 0}\frac{H_n}{k^n} = \frac{1-\frac{1}{k}}{1-2\cdot\frac{1}{k}-\left(\frac{1}{k}\right)^2-2(\frac{1}{k})^3}$$

for all $k > \theta_1$, where θ_1 is the real root of $x^3 - 2x^2 - x - 2$, as mentioned in section 4.

Proof. Since $\frac{1-x}{1-2x-x^2-2x^3} = H_0 + H_1x + H_2x^2 + H_3x^3 + \cdots$, replacing the x with a constant $\frac{1}{k}$ will give $\frac{1-\frac{1}{k}}{1-2\frac{1}{k}-\left(\frac{1}{k}\right)^2-2(\frac{1}{k})^3} = H_0 + \frac{H_1}{k} + \frac{H_2}{k^2} + \frac{H_3}{k^3} + \ldots$ However, we take $k = 2$, the result will be negative. This means that our series grows unbounded. Now, we will prove that the radius of convergence is $k > \theta_1$ using the ratio test. To do this, we can use the method from theorem 5 again to find and prove a relation beweeen H_n and P_n , that is, $H_n = aP_n + bP_{n+1} + cP_{n+2}$for some real constants a, b, and c (but in reality the three constants are complicated fractions so we will stick with a, b, and c). Then, we will have

$$\begin{aligned}H_n &= a(\theta_1^n + \theta_2^n + \theta_3^n) + b(\theta_1^{n+1} + \theta_2^{n+1} + \theta_3^{n+1}) + c(\theta_1^{n+2} + \theta_2^{n+2} + \theta_3^{n+2})\\ &= (a + b\theta_1 + c\theta_1^2)\theta_1^n + (a + b\theta_2 + c\theta_2^2)\theta_2^n + (a + b\theta_3 + c\theta_3^2)\theta_3^n\\ &= A\theta_1^n + B\theta_2^n + C\theta_3^n\end{aligned}$$

for some real constants A, B, and C. So, applying the ratio test, we now want

$$\lim_{n\to\infty}\frac{(A\theta_1^{n+1} + B\theta_2^{n+1} + C\theta_3^{n+1})/(k^{n+1})}{(A\theta_1^n + B\theta_2^n + C\theta_3^n)/(k^n)} < 1$$

$$\lim_{n\to\infty}\frac{A\theta_1^{n+1} + B\theta_2^{n+1} + C\theta_3^{n+1}}{A\theta_1^n + B\theta_2^n + C\theta_3^n} < k$$

So we basically want to prove that

$$\lim_{n\to\infty}\frac{A\theta_1^{n+1} + B\theta_2^{n+1} + C\theta_3^{n+1}}{A\theta_1^n + B\theta_2^n + C\theta_3^n} = \theta_1$$

To prove this, let there be a function $f(x) = 1 - 2x - x^2 - 2x^3$. It has one real root, so let the roots be p, $re^{i\theta}$, and $re^{-i\theta}$. Note that the coefficients of $f(x)$ are just $x^3 - 2x^2 - x - 2$ backwards, so the roots of the two functions are reciprocals: $\theta_1 = \frac{1}{p}$, $\theta_2 = \frac{1}{re^{i\theta}}$ and $\theta_3 = \frac{1}{re^{-i\theta}}$.

Now, $f(\frac{1}{2}) > 0$ and $f(0) < 0$, so $0 < p < \frac{1}{2}$. Using Vieta's formula, $pr^2 = \frac{1}{2} < \frac{1}{2}r^2$, which means $r > 1$. Hence, $0 < \frac{p}{r} < \frac{1}{2}$. Now, if we multiply the top and bottom of the limit we want by p^{n+1}, we will have

$$\lim_{n\to\infty}\frac{A\frac{1}{\theta_1}+B\frac{p}{re^{i\theta}}^{n+1}+C\frac{p}{re^{-i\theta}}^{n+1}}{A+B\frac{p}{re^{i\theta}}^{n}+C\frac{p}{re^{-i\theta}}^{n}}=\lim_{n\to\infty}\frac{A+B(\frac{p}{r}e^{-i\theta})^{n+1}+C(\frac{p}{r}e^{i\theta})^{n+1}}{A\frac{1}{\theta_1}+B(\frac{p}{r}e^{-i\theta})^{n}+C(\frac{p}{r}e^{i\theta})^{n}}$$

and since $0 < \frac{p}{r} < \frac{1}{2}$, $\lim_{n\to\infty}(\frac{p}{r})^n = 0$. Thus,

$$\lim_{n\to\infty}\frac{A+B(\frac{p}{r}e^{-i\theta})^{n+1}+C(\frac{p}{r}e^{i\theta})^{n+1}}{A\frac{1}{\theta_1}+B(\frac{p}{r}e^{-i\theta})^{n}+C(\frac{p}{r}e^{i\theta})^{n}}=\theta_1$$

So as long as $k > \theta_1$, the series will converge.

5.2. H_n and S_n

Table 4. The first few values for s_n and H_n .

n	0	1	2	3	4	5	6	7	8	9	10
S_n	1	1	2	6	16	42	112	298	792	2106	5600
H_n	1	1	3	9	23	61	163	433	1151	3061	8139

Finally, we come to S_n. The formal definition of the S_n sequence is $a_n = n!$ if $n \leq 3$, otherwise $a_n = 2a_{n-1} + a_{n-2} + 2a_{n-3}$. It also known to count the number of permutations that are sortable after two passes through a pop stack. Going back to its relation with Hn , we have

Theorem 12. For $n \geq 1$,

$$S_n = \frac{1}{2}(H_n + H_{n-1}).$$

which can be proved using the same method as theorem 5. Then, taking ex- perience from our self-cancelling sum in the previous subsection, we can discover the following:

Theorem 13. for $n \geq 0$,

$$\sum_{k=0}^{2n}(-1)^k S_k = \frac{1}{2}(H_{2n}+1)$$

Proof.

$$\sum_{k=0}^{2n}(-1)^k S_k = S_0 - S_1 + S_2 - S_3 + \cdots - S_{2n-1} + S_{2n}$$

using theorem 12, we have

$$= 1 - \frac{1}{2}(H_0 + H_1) + \cdots - \frac{1}{2}(H_{2n-2} + H_{2n-1}) + \frac{1}{2}(H_{2n-1} + H_{2n})$$

$$= 1 + \frac{1}{2}(-H_0 - H_1 + H_1 + H_2 - H_2 - \cdots - H_{2n-1} + H_{2n-1} + H_{2n})$$

$$= 1 + \frac{1}{2}(-1 + H_{2n})$$

$$= \frac{1}{2}(H_{2n} + 1)$$

Additionally, odd numbers have a similar identity that can be proved in the same way.

Theorem 14. $For\ n \geq 0$,

$$\sum_{k=0}^{2n+1}(-1)^k S_k = \frac{1}{2}(1 - H_{2n+1})$$

6. Conclusion

We have explored the H_n sequence with both a combinatorial and algebraic interpretation, in which we incorporated mathematical knowledge concerning recursion, counting, and induction. We then derived two closely related sequences, B_n and P_n. Their relation with H_n is rather similar to the relation between the Fibonacci and Lucas numbers. We have found the recursive equation for each of the three sequences, and also the relationship between the three sequences. This paper also discussed combinatorial interpretations of generating functions, which is a newly proposed generalization, as well as identities of H_n with existing {*2,1,2*} sequences. Still, many identities used the more conventional algebraic proofs, and their combinatorial proofs remain an exciting field to explore. The generalizing property of three-dimensional tiling sequences is also an area of interest.

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