Proceedings of the 3rd International Conference on Business and Policy Studies
DOI: 10.54254/2754-1169/78/20241738

Feasibility Study on Investment Portfolio under Five Additional Constraints Based on Markowitz Model and Index Model

Haonan Liu[1,a,*]

[1]*Department of Polymer Science and Engineering, Sichuan University, Chengdu, Sichuan, China*
a. 201010130415@stu.swmu.edu.cn
**Corresponding author*

Abstract: This article simulates investment situations that are closer to reality. Under five additional constraints, construct the optimal investment portfolio based on MM (Markowitz Model) and IM (Index Model). A significant conclusion was drawn by comparing the data obtained: enterprising investors are more suitable for using the Markowitz Model to construct the optimal investment portfolio, while conservative investors are more suitable for using the Index Model. In addition, there were some additional gains, such as discovering that investors can use the threshold of constraint strength to simplify the construction process of their optimal investment portfolio by comparing the first three constraints. Furthermore, including a broad index can diversify risk, but the effect is significantly different for different investors: conservative investors are not suitable for inclusion in a broad index, while enterprising investors are suitable for inclusion in a broad index. Although such results are strange, the prerequisite for such results is that one can hold a short position, which makes sense and leads to another conclusion of this study. Comparing the data obtained from constraint 4 with other constraints, it can be found that short-selling behavior can, to some extent, diversify risks and significantly increase potential maximum returns. The result of this research can provide investment advice suitable for different types of investors under different investment restrictions.

Keywords: Markowitz Model, Index Model, Constraints, Enterprising Investors, Conservative Investors

1. Introduction

Given the ongoing growth of the financial sector, more and more quantitative investment methods are being applied to actual investment. These quantitative investment methods are usually based on certain models, and through quantitative analysis, investment behavior can be more rational rather than relying solely on intuition to invest. In the investment process, it is usually necessary to find the optimal investment portfolio for the selected stocks, and the Markowitz Model and Index Model have been used in the past few years to determine the optimal investment portfolio and have been continuously studied and compared by scholars. However, scholars often conduct research based on ideal conditions and do not take into account various factors that may affect the final outcome in the real world, such as investor risk preferences, national policy factors, potential risk factors for

DOI: 10.54254/2754-1169/78/20241738

companies, and investment cycles for special products. These factors directly affect the return and volatility of investment portfolios, leading to previous research results not being applicable to reality.

Modern investment theory has a development history of several decades, among which the most important theoretical basis is the Markowitz model and index model applied in this article. Markowitz proposed the Markowitz model in 1952 [1]. This model is also known as the mean-variance model, which believes that investors' investment desire is to pursue high expected returns and avoid risks as much as possible [2]. Therefore, for an investment portfolio, it is important to focus on both expected returns and the risks involved. The Markowitz model focuses on optimizing investment portfolios through diversification, aiming to maximize the expected return of securities based on a given level of risk [3]. In 1958, economist James Tobin derived the concepts of "capital market line" and "efficient boundary" based on Markowitz's work [4]. Tobin's model proposes that as long as the expected returns are the same, the proportion of their chosen investment portfolio will be the same, which overlooks the existence of risk preference [5]. In 1964, William Sharpe et al. proposed the Capital Asset Pricing Model (CAPM) based on Markowitz's work, which continues to serve as the theoretical foundation for investment today [6]. In 1965, Lintner further derived the CAPM [7]. The index model is an innovation based on previous efforts, which assumes that the return of any stock can be decomposed into systematic parts (related to market returns) and non-systematic parts (specific to the stock itself). Compared to the Markowitz model, the index model provides a simplified method for evaluating and constructing investment portfolios [8]. Modern portfolio theory is derived from the above theoretical foundations, and the rapid development of mutual funds and index funds also benefits from the theoretical basis of modern investment portfolios [9]. Diversified investment has always been the core concept of modern portfolio theory; as Markowitz once said, "Diversified investment is the only free lunch in the world" [10].

Very few studies take real-life factors into account by reviewing academic materials. Therefore, in order to obtain conclusions that are closer to real-life applications, this article proposes five constraints related to real life in this article, and specifically studies and compares the feasibility of obtaining the optimal investment portfolio through the Markowitz model and index model under these constraints in the hope of providing investors with better quantitative investment methods.

2. Overview

2.1. Objective Overview

Calculate the required data for the study using the raw data of the selected samples. Then, based on two models, create five optimal investment portfolios under different constraint conditions. Further analysis reveals whether the investment portfolio regions obtained through the two models are effective under real-world constraints.

2.2. Companies Overview

In order to simulate the real investment situation as much as possible, the following ten companies were chosen as research subjects (Table 1).

Table 1: Companies introduction

#	Stock	Full Name	Sector (Yahoo!finance)
1	NVDA	NVIDIA Corporation	Technology
2	CSCO	Cisco Systems, Inc.	Technology
3	INTC	Intel Corporation	Technology
4	GS	The Goldman Sachs Group, Inc.	Financial Services

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Table 1: (continued).

5	USB	U.S. Bancorp	Financial Services
6	TD CN	The Toronto-Dominion Bank	Financial Services
7	ALL	The Allstate Corporation	Financial Services
8	PG	The Procter & Gamble Company	Consumer Defensive
9	JNJ	Johnson & Johnson	Healthcare
10	CL	Colgate-Palmolive Company	Consumer Defensive

From the above table, it can be seen that the stocks which have been selected come from four different industries. This diversified sample can prevent the covariance obtained from being too high, thereby avoiding excessive bias in the results obtained by the Markowitz model (which can be better compared with the index model).

2.3. Data Overview

In order to better obtain conclusions that are close to reality, the following data has been chosen as the research basis.

Selection of raw data: 20 years of historical daily total return data for ten stocks.

Equity index: S&P 500.

A proxy for risk-free rate: 1-month Fed Funds rate.

2.4. Constraints Overview

Constraints 1: Regulation T by FINRA

In order to simulate the restrictions on leverage in real life, this constraint prohibits investors from using more than twice the amount of leverage: $\sum_{i=1}^{11}|w_i| \leq 2$;

Constraints 2: Arbitrary "box" constraints

Used to simulate situations where excessive leverage is prohibited for a single asset, with less constraint than constraint 1: $|w_i| \leq 1, for\ \forall i$;

Constraints 3: A free problem

Simulate a rational situation: there are no restrictions in this world and investors can use leverage at will. Suitable as a control group;

Constraints 4: Not allowed to have any short positions

Short selling behavior is strictly regulated in many countries, so this constraint simulates an investment environment where one cannot hold short positions: $w_i \geq 0, for\ \forall i$;

Constraints 5: Exclusion of the broad index

In order to investigate whether incorporating a generalized index has an impact on investment returns, this limitation simulates the absence of a generalized index in an investment portfolio: $w_i = 0$

2.5. Overview of Relevant Theories and Formulas

In this study, the article mainly used the Markowitz model and index model to construct possible investment portfolio regions based on this foundation. In addition to providing a brief description of the two models in the introduction section, the article also needs a lot of relevant theoretical foundations and mathematical formulas to assist in completing the research. Therefore, some related theories and formulas will be simplified and introduced.

2.5.1. Theories Overview

Efficient Frontier: The efficient frontier is the optimal portfolio of risky assets the market can offer. The entire frontier is called the minimum variance frontier because a rational investor would choose a portfolio with higher expected returns for the same risk. The part of the minimal-variance frontier above the Global Min-Variance Portfolio is called the Efficient Frontier of risky assets.

Utility Indifference Curve: On this curve, the returns obtained by any investment portfolio of investors are the same (each investment portfolio is at a certain point on the curve).

Capital Allocation Line: The points on this line describe the relationship between the expected returns and risks of all investment portfolios.

Central Limit Theorem(CLT): The more sample size, the closer its distribution is to a normal distribution. (Explain why aggregation from daily to monthly works)

2.5.2. Formulas Overview

Utility Indifference Curve: $\mathrm{U} = \mathrm{E(r)} * \frac{\sigma^2}{2}$;

Capital Allocation Line: $\mathrm{E}(r_c) = r_f + \sigma_c * \frac{E(r_p)-r_f}{\sigma_p}$;

Central Limit Theorem(CLT): $\overline{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$;

Normal distribution: $\mathrm{f(x)} = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$;

Monthly return: $R_m = \left(\frac{p_t}{p_{t-1}} - 1\right) * 100\%$;

Excess return: $R_e = p_t - NRTR_T$;

Residual returns: $R_r = R_e - \beta * SPX - \frac{\alpha}{12}$;

Annual average return: $\mathrm{AAR} = \frac{1}{n}(\sum_{t=1}^{n} R_e) \times 12$

Annual standard deviation: $\mathrm{ASD} = \sqrt{\frac{1}{n}[\sum_{t=1}^{n}(R_E - ARR)^2] \times 12}$

Correlation coefficient: $\mathrm{Corr(X,Y)} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2 \sum(y-\overline{y})^2}}$

α = Portfolios Actual Return-(Risk Free Rate+β×(Benchmark Return-Risk Free Rate))

$\beta = \frac{Covariance(Assest\ Returns, Market\ Returns)}{Variance(Market\ Returns)}$

3. Data Processing

3.1. Price from Daily to Monthly

As mentioned earlier, in order to reduce non-Gaussian effects, daily data needs to be aggregated into monthly observations, which means that the article needs to present daily prices(Table 2), prices for each trading week (i.e., five days) (Table 3) and monthly prices (Table 4).

Table 2: Part of the daily prices in the sample

SPX		NVDA		CSCO		INTC	
2001/5/11	1245.6700	2001/5/11	13.0233	2001/5/11	19.05	2001/5/11	27.94
2001/5/14	1249.0158	2001/5/14	13.1350	2001/5/14	18.57	2001/5/14	27.41
2001/5/15	1249.5442	2001/5/15	13.6067	2001/5/15	18.74	2001/5/15	27.2
2001/5/16	1285.2952	2001/5/16	14.2483	2001/5/16	20.00	2001/5/16	28.36

Table 2: (continued).

2001/5/17	1288.8082	2001/5/17	14.5000	2001/5/17	19.86	2001/5/17	28.6
2001/5/18	1292.5248	2001/5/18	14.3833	2001/5/18	20.20	2001/5/18	28.76
2001/5/21	1313.4039	2001/5/21	15.0333	2001/5/21	22.87	2001/5/21	29.9

Table 3: Part of the weekly prices in the sample

SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	NRFR
1,245.67	13.02	19.05	27.94	92.85	20.66	20.05	41.83	32.75	48.63	29.07	1,000.00
1,292.52	14.38	20.20	28.76	97.25	21.54	20.00	43.87	33.74	50.68	28.70	1,000.82
1,278.54	15.28	22.05	29.10	97.96	21.92	20.18	44.47	31.75	48.80	28.39	1,001.61
1,261.79	15.06	18.85	28.74	96.70	22.19	20.36	43.28	31.76	49.17	28.63	1,002.42
1,266.32	16.13	20.49	30.67	95.36	22.42	20.72	44.05	32.14	51.19	29.44	1,003.21

Table 4: Part of the monthly prices in the sample

Date	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	NRFR
5/11/01	1,245.67	13.02	19.05	27.94	92.85	20.66	20.05	41.83	32.75	48.63	29.07	1,000.00
5/31/01	1,256.94	14.27	19.26	27.01	95.10	21.99	19.98	45.21	32.12	48.65	28.32	1,002.26
6/29/01	1,226.34	15.46	18.20	29.25	85.80	22.66	19.45	44.18	31.90	50.18	29.50	1,005.57
7/31/01	1,214.25	13.48	19.22	29.81	83.28	23.61	19.80	35.11	35.71	54.29	27.19	1,008.89
8/31/01	1,138.24	14.12	16.33	27.98	80.21	24.10	20.92	34.26	37.28	53.08	27.17	1,012.26
9/28/01	1,046.31	9.16	12.18	20.45	71.45	22.25	19.69	37.72	36.59	55.78	29.22	1,014.67

3.2. Different Types of Returns

Calculate the raw data above using Excel and obtain different types of returns: monthly returns (Table 5), excess returns (Table 6), and residual returns (Table 7).

Table 5: Part of the monthly returns in the sample

SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	NRFR	Date
0.0090	0.0956	0.0110	-0.0333	0.0242	0.0644	-0.0037	0.0808	-0.01924	0.0004	-0.0258	0.0023	5/31/01
-0.0243	0.0834	-0.0550	0.0829	-0.0978	0.0306	-0.0263	-0.0229	-0.0068	0.0315	0.0415	0.0033	6/29/01
-0.0099	-0.1278	0.0560	0.0191	-0.0294	0.0417	0.0181	-0.2053	0.1193	0.0820	-0.0781	0.0033	7/31/01
-0.0626	0.0471	-0.1503	-0.0615	-0.0368	0.0211	0.0562	-0.0241	0.0441	-0.0224	-0.0009	0.0033	8/31/01
-0.0808	-0.3514	-0.2541	-0.2690	-0.1092	-0.0769	-0.0588	0.1008	-0.0183	0.0510	0.0757	0.0023	9/28/01

Table 6: Part of the excess returns in the sample

SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Date
0.0068	0.0933	0.0088	-0.0356	0.0220	0.0622	-0.0060	0.0786	-0.0215	-0.0019	-0.0281	5/31/01
-0.0276	0.0801	-0.0583	0.0796	-0.1011	0.0273	-0.0296	-0.0262	-0.0102	0.0282	0.0382	6/29/01
-0.0132	-0.1311	0.0527	0.0158	-0.0327	0.0384	0.0148	-0.2086	0.1160	0.0787	-0.0815	7/31/01
-0.0659	0.0438	-0.1537	-0.0648	-0.0402	0.0177	0.0528	-0.0274	0.0407	-0.0258	-0.0043	8/31/01
-0.0831	-0.3538	-0.2565	-0.2713	-0.1116	-0.0793	-0.0612	0.0984	-0.0207	0.0487	0.0733	9/28/01

Table 7: Part of the residual returns in the sample

SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Date
0.000%	6.501%	0.001%	-4.356%	1.225%	5.346%	-1.557%	6.963%	-2.956%	-0.922%	-3.421%	5/31/01
0.000%	11.991%	-2.162%	11.250%	-6.227%	5.200%	-1.204%	0.126%	-0.427%	3.942%	4.769%	6/29/01
0.000%	-11.993%	7.032%	3.151%	-1.430%	4.903%	2.094%	-19.644%	11.599%	8.214%	-7.853%	7/31/01
0.000%	15.934%	-6.642%	1.354%	5.268%	7.963%	10.050%	4.047%	6.213%	0.616%	2.264%	8/31/01
0.000%	-20.418%	-14.651%	-17.256%	0.547%	-0.067%	0.005%	18.448%	0.765%	8.988%	10.805%	9/28/01

3.3. Data Related to the Model

After obtaining the above data, further Excel calculations can obtain data related to the Markowitz and exponential models (Table 8).

Table 8: AAR, ASD, α, β and Annualized residual standard deviation

	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL
Annualized Average Return	7.542%	32.802%	9.714%	8.905%	10.825%	9.878%	11.010%	10.080%	9.437%	8.464%	7.105%
Annualized StDev	14.850%	55.774%	30.809%	30.503%	29.572%	23.680%	18.134%	24.884%	14.587%	14.785%	15.350%
beta	1	1.978752378	1.320579818	1.187512188	1.410043841	0.971191485	0.787000103	1.056243491	0.405117984	0.539836975	0.454430762
Annualized Alpha	0	0.178772585	-0.002459727	-0.000519865	0.001899888	0.025529424	0.050736483	0.021133854	0.063814442	0.043922488	0.036773305
Annualized residual stDev	0.000%	47.405%	23.762%	24.889%	20.881%	18.781%	13.865%	19.317%	13.288%	12.423%	13.787%

3.4. Correlation Matrix

As shown in Table 9, the correlation of ten samples can be observed. As shown in Table 9, the correlation of ten samples can be seen. Mark the maximum and minimum values in green and blue, respectively, based on the data in the table.

From the data in the table, it can be seen that the correlation between The Goldman Sachs Group and the SPX index is the highest (0.708091709), while the correlation between The Procter & Gamble Company and NVIDIA Corporation is the lowest (0.059557816).

This means that the stock price of The Goldman Sachs Group is most similar to the changes in the SPX index, with the magnitude of the company's stock price fluctuations close to the SPX index. However, there is almost no contact between Procter & Gamble Company and NVIDIA Corporation, and the stock prices of these two companies are unrelated.

Table 9: Correlation Matrix

	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL
SPX	1	0.526864583	0.636530392	0.57812835	**0.708091709**	0.609066203	0.644502528	0.630358528	0.412444882	0.542222226	0.439644911
NVDA	0.526864583	1	0.487198499	0.523781339	0.343133691	0.159844503	0.338000553	0.156912243	**0.059557816**	0.165279328	0.069448486
CSCO	0.636530392	0.487198499	1	0.614181358	0.487495211	0.328141332	0.410048991	0.297265563	0.220243822	0.238794783	0.164964333
INTC	0.57812835	0.523781339	0.614181358	1	0.410737401	0.279632364	0.411502848	0.285681853	0.136363253	0.324896207	0.110063729
GS	**0.708091709**	0.343133691	0.487495211	0.410737401	1	0.471677541	0.493821807	0.417366912	0.17310842	0.295535064	0.203124511
USB	0.609066203	0.159844503	0.328141332	0.279632364	0.471677541	1	0.539160432	0.540137382	0.335849739	0.234128129	0.217802744
TD CN	0.644502528	0.338000553	0.410048991	0.411502848	0.493821807	0.539160432	1	0.416709104	0.230973847	0.272731908	0.211711116
ALL	0.630358528	0.156912243	0.297265563	0.285681853	0.417366912	0.540137382	0.416709104	1	0.346274996	0.451773107	0.406644845
PG	0.412444882	**0.059557816**	0.220243822	0.136363253	0.17310842	0.335849739	0.230973847	0.346274996	1	0.493742994	0.483308319
JNJ	0.542222226	0.165279328	0.238794783	0.324896207	0.295535064	0.234128129	0.272731908	0.451773107	0.493742994	1	0.526760864
CL	0.439644911	0.069448486	0.164964333	0.110063729	0.203124511	0.217802744	0.211711116	0.406644845	0.483308319	0.526760864	1

4. Model Analysis

4.1. Constraint 3: A Free Problem

This constraint is equivalent to having no constraints, and it is placed at the beginning because it is more suitable as a blank control group for further discussion.

Tables 10 and 11 below represent the weights that generate the minimum variance and maximum Sharpe ratio. This is derived from the data above, and it is worth noting that this proportion is an estimate rather than an accurate one.

Firstly, analyze the data under the Markowitz Model (Table 10). The minimum variance means the least risk and is the most robust investment portfolio approach for investors. As shown in Table 10, the minimum variance value is 10.95%. The maximum Sharpe ratio implies a balance between risk and return, and the larger its value, the greater the return and the better the investment effect under the same risk conditions. From Table 10, it can be seen that its potential maximum Sharpe ratio is 103.11%, indicating that investors will take on 1 risk while receiving a 1.0311 return. At this point, the standard deviation of the investment portfolio is 16.48%, indicating that investors need to take on more risk to achieve higher returns. The excess returns in Table 10 can prove this, with the Conservative Party accounting for 7.51% and the Radical Party accounting for 16.99%.

Secondly, analyze the data based on the Index Model (Table 11), which is similar to the Markowitz Model and reveals the same principles. However, there are differences in the values, and it can be seen that the minimum standard deviation here is 9.63%, while the maximum Sharpe ratio and its corresponding standard deviation are 99.57% and 12.92%, respectively. The most robust investment portfolio brings an excess return of 7.15%, while the most aggressive investment portfolio brings an excess return of 12.87%.

In conclusion, the Index Model performs better in predicting the minimum standard deviation, although its excess return is slightly smaller than the investment portfolio based on the Markowitz Model. Because the investment portfolio created based on the Index Model has a smaller standard

DOI: 10.54254/2754-1169/78/20241738

deviation and a larger Sharpe ratio, this indicates that risk-averse investors can apply index models to obtain better investment portfolios. The Markowitz Model performs better in predicting potential maximum returns. Under the premise of high-risk tolerance, the standard deviation (risk) data is not as important. Its Sharpe ratio and excess return can reach 103.11% and 16.99%, respectively, significantly higher than the 99.57% and 12.87% under the Index Model. This indicates that enterprising investors are more likely to use the Markowitz Model to find the optimal investment portfolio.

Table 10: Constraint 3—MM

Constrain3	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.383666	-0.029681	-0.028927	0.013271	-0.058993	-0.003048	0.194149	-0.114814	0.259307	0.188331	0.196738	7.51%	10.95%	68.54%
MaxSharpe	-1.099737	0.224573	0.008930	-0.081858	0.127252	0.132126	0.464597	0.078969	0.534975	0.427167	0.183006	16.99%	16.48%	103.11%

Table 11: Constraint 3—IM

Constrain3	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.256224	-0.040425	-0.052698	-0.028095	-0.087283	0.007581	0.102837	-0.013989	0.312696	0.276747	0.266405	7.15%	9.63%	74.23%
MaxSharpe	-0.701578	0.103244	-0.005654	-0.001089	0.005655	0.093933	0.342517	0.073501	0.469030	0.369358	0.251082	12.87%	12.92%	99.57%

4.2. Constraint 1: Regulation T by FINRA

In reality, investors are often restricted by various conditions. Investors are prohibited from using more than twice the leverage under constraint 1.

Tables 12 and 13 below represent the Markowitz and Index Model under constraint 1, respectively. The symbolic meaning of the values in the table is consistent with that of the control group (constraint 3). Analyzing the data in the table yields the same conclusion as Constraint 3: conservative investors are suitable for the index model, while radical investors are suitable for the Markowitz model.

Further, compare the similarities and differences in the values in the table under different constraints. Surprisingly, under the constraint of minimum variance, the weights of each stock and the three major indicators (excess return, standard deviation, and Sharpe ratio) of the optimal investment portfolio obtained are the same. This phenomenon indicates that if investors are very averse to risk, leverage restrictions in real life will not affect their investment portfolio allocation and risk-return ratio. In other words, even without restrictions, investors will not overuse leverage; such data results are very realistic. In addition, when pursuing the maximum Sharpe ratio, it can be seen that the weight of SPX has decreased (in absolute values) compared to constraint 3, indicating that under this constraint, investors will be limited in their short-selling behavior. SPX is included in the research scope because it is a fully diversified index. At the same time, it can be seen that when investors pursue the maximum Sharpe ratio, all three indicators of the two models have declined, which means that investors are limited by the use of leverage, resulting in a simultaneous decline in risk and return, and the Sharpe ratio has slightly declined.

DOI: 10.54254/2754-1169/78/20241738

Table 12: Constraint 1—MM

Constrain1	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.383666	-0.029681	-0.028927	0.013271	-0.058993	-0.003048	0.194149	-0.114813	0.259307	0.188331	0.196738	7.51%	10.95%	68.54%
MaxSharpe	-0.427359	0.157450	-0.011478	-0.061096	0.032511	0.064780	0.352891	0.010679	0.457135	0.300020	0.124466	14.01%	13.95%	100.42%

Table 13: Constraint 1—IM

Constrain1	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.256224	-0.040424	-0.052698	-0.028095	-0.087283	0.007581	0.102837	-0.013989	0.312696	0.276747	0.266405	7.15%	9.63%	74.23%
MaxSharpe	-0.476212	0.088811	-0.012392	-0.004904	-0.006366	0.066696	0.295484	0.045742	0.439508	0.333538	0.230096	12.07%	12.18%	99.05%

4.3. Constraint 2: Arbitrary "Box" Constraints

Constraint 2 is not as strict as Constraint 1 and tends to limit a single asset more. That is to say, Constraint 1 simulates various constraints that investors may encounter, while Constraint 2 only restricts investors from excessive leverage on a single asset.

Tables 14 and 15 below represent the Markowitz Model and Index Model under constraint 2; the data in the column of minimum variance remains unchanged. This is as expected, as its constraint strength is between constraint 1 and constraint 3 (control group). This indicates that if the constraint strength does not reach a threshold in the investment process, the optimal investment portfolio allocation for conservative investors is equal to the allocation without any restrictions. This is a surprising point, as investors can use this and the magnitude of constraints to make stable investments. The optimal investment portfolio is easier to calculate without any constraints, and if it does not exceed the threshold, the final result is the same as the result without constraints.

Next, comparing the numerical changes of the three indicators when pursuing the potential maximum Sharpe ratio, it can be seen that it is slightly lower than constraint 3 and higher than constraint 1. This result is also very reasonable. After all, its constraint strength is not as strong.

Table 14: Constraint 2—MM

Constrain2	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.383666	-0.029681	-0.028927	0.013271	-0.058992	-0.003048	0.194149	-0.114814	0.259307	0.188331	0.196738	7.51%	10.95%	68.54%
MaxSharpe	-1.000000	0.215033	0.003089	-0.081477	0.114582	0.122532	0.449241	0.068741	0.523281	0.410175	0.174801	16.56%	16.06%	103.06%

Table 15: Constraint 2—MM

Constrain2	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.256224	-0.040425	-0.052698	-0.028095	-0.087283	0.007581	0.102837	-0.013989	0.312696	0.276747	0.266405	7.15%	9.63%	74.23%
MaxSharpe	-0.701578	0.103244	-0.005654	-0.001089	0.005655	0.093933	0.342517	0.073501	0.469030	0.369358	0.251082	12.87%	12.92%	99.57%

4.4. Constraint 4: Not allowed to have any short positions

Short selling is strictly restricted or even prohibited in many countries (such as open-end mutual funds in the United States), and this constraint is in order to simulate investment under such conditions. Constraint 4 prohibits any short-selling behavior, which means that asset allocation weights cannot be negative. The strength of this restriction is significantly greater than Constraints 1 and 2.

Tables 16 and Table 17 below represent the asset allocation results for the Markowitz Model and Index Model under this constraint, respectively. Firstly, it can be seen that there have been significant changes in asset allocation and final output outcomes when pursuing minimum variance (minimum risk). From constraint 1 to constraint 3, whether it is based on MM or IM, the proportion of SPX allocation is high: 0.383666 and 0.25624, respectively. However, under constraint 4, the proportion of SPX in MM(Markowitz) based investment portfolio is only 0.094906, while IM(Index Model) is 0. This means that the optimal investment portfolio is not suitable for tracking the SPX index when investors cannot short.

In addition, there is a zero-weighted asset allocation if investors pursue minimum variance (minimum risk) or maximum Sharpe ratio (maximum potential return). After comparing with the previous asset ratio, it was found that stocks with a weight of 0 under this constraint are stocks with negative weights under other constraints, which is in line with expectations. When investors are unable to short, not allocating stocks with poor market conditions can yield better returns.

At the same time, it can be observed that investors receive significantly better returns when pursuing minimum variance than the first three constraints (whether it is the Markowitz Model or the Index Model), but the standard deviation is slightly higher than the first three constraints, indicating that short selling behavior can to some extent diversify risk.

When pursuing the maximum Sharpe ratio, investors receive significantly lower returns than the first three constraints of being able to short, which means that short-selling behavior can greatly improve the potential maximum returns of the investment portfolio.

Table 16: Constraint 4—MM

Constrain4	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.094906	0.000000	0.000000	0.000000	0.000000	0.000000	0.198491	0.000000	0.289052	0.206215	0.211337	8.88%	11.27%	78.78%
MaxSharpe	0.000000	0.109475	0.000000	0.000000	0.000000	0.000000	0.237267	0.000000	0.425604	0.161687	0.065967	12.06%	13.12%	91.90%

Table 17: Constraint 4—IM

Constrain4	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.092106	0.000000	0.335510	0.288925	0.283460	8.64%	10.16%	85.00%
MaxSharpe	0.000000	0.067419	0.000000	0.000000	0.000000	0.000000	0.177521	0.000000	0.373410	0.227506	0.154145	10.71%	11.72%	91.37%

4.5. Constraint 5: Exclusion of the Broad Index

The generalized index used in this study is SPX, and this constraint is to investigate whether the generalized index in the investment portfolio affects the final result. In order to compare with the previous assumptions, the SPX weight under this constraint is 0, and there is no restriction on short positions.

Tables 18 and Table 19 below represent the asset allocation results for the Markowitz Model and Index Model under this constraint, respectively. Firstly, by comparing constraint 5 with the first three constraints, it can be seen that the Sharpe ratios when pursuing minimum variance are 77.89% and 80.18%, respectively. The results are all higher than the 68.54% and 74.23% of the first three constraints, but the standard deviation is slightly higher than the first three constraints. This indicates that incorporating a broad index can diversify an investment portfolio's risk while limiting its excess returns when pursuing minimum risk. When pursuing the maximum Sharpe ratio, the performance of both models is not as good as the first three constraints, possibly because the first three constraints are short of the generalized index. Not allowing short positions limits potential maximum returns.

Compare Constraint 5 and Constraint 4 again, as Constraint 4 can be ignored in the weight allocation of SPX, thus controlling variables to study the impact of short-selling behavior on investment returns. It can be seen that the investment return performance under constraint 5 is slightly worse than constraint 4 when pursuing minimum variance and slightly better than constraint 5 when pursuing maximum Sharpe ratio. This result indicates that if an investor pursues minimum risk without considering inclusion in a broad index, they are not suitable to hold short positions, but when investors want to maximize their potential returns, they are suitable for holding short positions.

Table 18: Constraint 5—MM

Constrain5	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.000000	-0.009713	0.000842	0.025124	-0.009928	0.034982	0.246957	-0.081676	0.289140	0.255760	0.248513	8.71%	11.18%	77.89%
MaxSharpe	0.000000	0.149271	-0.068518	-0.101449	-0.013032	0.024327	0.306465	-0.022716	0.433103	0.236122	0.056426	13.06%	13.69%	95.40%

Table 19: Constraint 5—IM

Constrain5	SPX	NVDA	CSCO	INTC	GS	USB	TDCN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVariance	0.000000	-0.033459	-0.032829	-0.011432	-0.060171	0.032713	0.142494	0.011302	0.341300	0.315355	0.294725	7.82%	9.75%	80.18%
MaxSharpe	0.000000	0.076804	-0.058355	-0.044505	-0.068055	0.025592	0.228205	0.004690	0.387615	0.266881	0.181127	10.84%	11.48%	94.38%

5. Conclusion

This article mainly studies the return performance of investment portfolios under different constraint conditions. The Markowitz and Index Model are two fundamental models for constructing investment portfolios. Based on these two models, many conclusions applicable to real investment have been obtained. During the research process, the differences between the two models were compared, and the differences of the same model under different constraints were compared.

Firstly, there are differences between models. Data conclusions found that the Markowitz Model is more suitable for use by enterprising investors, while the Index Model is more suitable for use by conservative investors. Because under the five constraints, the Markowitz model consistently performs better in pursuing maximum potential returns, while the Index Model consistently performs better in pursuing minimum risk.

Secondly, there are additional gains during the research process. When comparing constraints 1 to 3, the study found that if the investment goal is to construct the optimal investment portfolio with minimum risk, investors should consider the strength of the constraint. Because when the constraint is within a certain threshold range, the weight and return performance of the investment portfolio are exactly the same as those of an unconstrained investment. This means that investors can use this phenomenon to simplify the construction of their optimal investment portfolio. When the constraint is within the threshold range, assuming there are no constraints, the results of constructing the optimal investment portfolio are the same as the actual required results.

During the research process, it was also found that incorporating a generalized index can effectively diversify risk, but it will reduce the investment return of conservative investors (who pursue returns with minimal risk), because the broad index is equivalent to a fully diversified investment product, which certainly reduces risk. However, due to its small fluctuations, it is inevitable that it will affect the return rate. It is interesting that incorporating the broad index can significantly increase potential maximum returns, as it allows for short selling of the broad index. Therefore, when enterprising investors pursue maximum returns, incorporating the broad index into their investment portfolio can yield better returns (provided that they can hold short positions).

This leads to the final conclusion of this article: the impact of holding short positions on investment portfolios and returns. Comparing constraint 4 with other constraints, it can be found that when investors cannot short, the optimal investment portfolio is unsuitable for tracking the SPX index and not allocating stocks with poor market conditions can achieve better returns. Short selling behavior can diversify risks to a certain extent and significantly increase potential maximum returns.

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