Proceedings of the 2nd International Conference on Financial Technology and Business Analysis
DOI: 10.54254/2754-1169/82/20230995

Early Exercise of American Options in the Two-Period Binomial Model

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Abstract: This paper is mainly about the research object of early exercise of American options by two-period binomial models. In this work, we use the two-period binomial model to analyze the price of American options and European options, decide whether to exercise early or not, and compare the differences between American options and European options. What is more, we use Python to help us with our calculations. Options are a kind of financial derivatives that help one to arbitrage, speculate, and hedge. To make more profits, people can use the binomial model to calculate the price of options. Through this analysis, we got to know each price of options in different situations, and we know that the price of American options is always equal to or larger than the price of European options. This work can help us to calculate the price of the option and decide if it is exercised early, and make sure to get more profits and apply it to daily life.

Keywords: The price of options, American options, European options, binomial model, exercise price

1. Introduction

The options are a kind of financial derivatives. They can help the holders to make the speculating, arbitrage, and hedging to make more profit or manage the risks. It is classified into call options and put options. The call options give the holders the right to buy the assets at a specific time in the future. When the market price increases, the holders can buy them at the strike price to make more profit. The put options give the holders the right to sell the assets at a specific time in the future. When the market price decreases, the holders can sell them to make more profit. The options include American options and European options. The main difference between them is that unlike the European option, which holders can only exercise it at the maturity date, with the American option, holders can exercise it at any time before or at the maturity date. Because of the particularity of the American options and the uncertainty of the financial environment [1], it is complicated to analyze the price of the American option. Therefore, one will use a kind of model called the binomial model to help us to work it out.

The binomial model is one of the most frequently used models used to compute the price of the option in different periods and different situations [2,3]. The binomial model can help one to compute the price of the option clearly [4], and to analyze if it needs to be exercised early in different situations.

DOI: 10.54254/2754-1169/82/20230995

1.1. Related Work

We take the American put option as an example. We analysis the binomial tree model, the probability and got some formulas to calculate the price of options.

1.1.1. Binomial Models

When analyzing the price of American options, one often uses binomial tree models like figure 1, which is two-period binomial models.

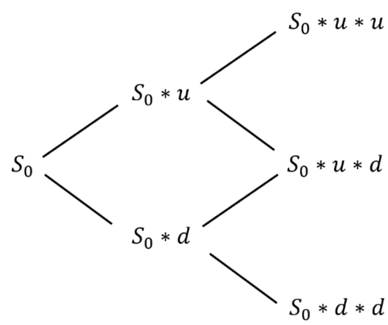

Figure 1: Two-period binomial tree model

In the figure, S_0 is the price of the options at time 0, and u means the price going up, d means the price going down. When the price goes up, the probability is computed as $q = \frac{1+r-d}{u-d}$, and when the price goes down, the probability is computed as $1 - q = \frac{u-1-r}{u-d}$.

1.1.2. Decide Whether to Exercise Early

In the American put option, if the strike price is larger than the market price, holders will choose to exercise it. Because for instance, holders can sell the options at the strike price K, and then buy the options at the market price, which can help the holders to make a profit. However, if the strike price is less than the market price, holders will not to exercise it early. Therefore, the profit holders will get is $Max(0, K - S_0u)$ or $Max(0, K - S_0d)$.This is the basic condition of the research. And the formula to compute the S_0 is

$$S_0 = \frac{1}{r+1}[q * max(0, K - S_0u) +$$

$$(1 - q) * max(0, K - S_0d)].$$

This formula will help to compute the price of options.

1.2. Our Contribution

This paper presents [5] that under the guidance of the financial market, the issued options and stock options have been highly concerned by investors, funds, and some financial institutions, but there are also some problems. If investors, funds, and some financial institutions cannot get the option price timely or correctly, they may not be able to correctly judge the cost of hedging [6]. If the arbitrage opportunity is not seized or the portfolio is not well balanced, the whole market, whether from the perspective of investors or financial institutions, may fail because of the failure to accurately and timely offer the bid price and offer price, so there is no corresponding order, and finally lose a good

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deal. Therefore, the approach suggested in this paper involves deriving the pricing formula for American options based on the concept of expectation and comparing the price with that of European options. To get the corresponding results and get the benefits of Early Exercise of American Options in the Two-Period Binomial Model.

1.3. Paper Structure

The subsequent sections of this article are structured as follows. The methodologies employed in prior research are discussed in this paper. This includes the Python codes and variables used, which will be introduced in 2.1. 2.2 to 2.5, American option and European option models are established, namely Case A, Case B, Case C and Case D, and the algorithm and results are introduced in detail. In conclusion, this article provides a comprehensive overview of the entire text and proposes potential avenues for future research.

2. Analysis of Early Exercise of American Options

2.1. Variables and Python Code

This article will use $\boldsymbol{S_0, t, u, d, K, r, C, E, P^A, P^E}$

$\boldsymbol{S_0}$ represent price of the purchase option on the first day,
$\boldsymbol{t}$ represents time,
$\boldsymbol{u}$ represents **S0** going up ,
$\boldsymbol{d}$ represents **S0** going down (we set $\boldsymbol{d} = \frac{1}{u}$),
$\boldsymbol{K}$ represents strike price of the option,
$\boldsymbol{r}$ represents interest rates,
$\boldsymbol{C}$ represents continuation price,
$\boldsymbol{E}$ represents exercise price,
$\boldsymbol{P^A}$ represents the price of American option,
$\boldsymbol{P^E}$ represents the price of European option.

This python code is used to calculate the range of r in Case B,Case C below to compare when $E_I(u)$ and $E_I(d)$ are maximized.

```
from sympy import symbols, solve, Interval, And
k, s0, u, d, r, kin= symbols('K S0 u d r kin', real=True)
d = 1/u
KInterval = Interval(s0 * d, s0)
expr1 =formula1
expr2 =formula2
sol = solve(expr1 - expr2, r, domain = {k:kInterval, r:Interval(-1, 1)})
print(f"if r > {sol} ,then exp1 > exp2")
print(f"if r < {sol} ,then xp1 < exp2")
```

This python code is used to simplify some complex sentence expressions in the following Case B,Case C, such as calculating the expressions of $\boldsymbol{P^A, P^E}$ and $\boldsymbol{P^A - P^E}$, so as to calculate the simplest expressions more quickly and concisely

```
import sympy as sp
var_str="r,u,k,S0"
y_str="formula"
sp.var(var_str)
```

DOI: 10.54254/2754-1169/82/20230995

```
y2=sp.simplify(y_str)
print(y2)
```

2.2. Case A

The range of $\boldsymbol{K}$ is $\boldsymbol{K < S_0 u^2}$.

The continuation value at time 1 if the stock goes up is $C_1(u) = \frac{K}{1+r} - S_0 u$, And the continuation value at time 1 if the stock goes down is $C_1(d) = \frac{K}{1+r} - \frac{S_0}{u}$

If the option holder decided to exercise at time 1 in state u then they would receive $E_1(u) = K - S_0 u$, If the option holder decided to exercise at time 1 in state d, they would receive

$E_1(d) = K - S_0$d

To see whether it is larger or less than 0. Therefore, we got that the range of r, if r is larger than or equal to 0, the $C_1(u)$ is larger than the $E_1(u)$, if r is larger than or equal to 0, the $C_1(d)$ is larger than the $E_1(d)$which means that holders will not exercise early. however, if r is less than 0, American option prices and European option prices are the same, which is

$$P^A = P^E = \frac{K}{(1+r)^2}$$

However, if the interest rates are strictly positive, then the prices are not the same:

$$P^E = \frac{K}{(1+r)^2} - S_0 < \frac{K}{1+r} - S_0 = P^A$$

$$P^A - P^E = \frac{Kr}{(1+r)^2}$$

2.3. Case B

The range of $\boldsymbol{K}$ is $\boldsymbol{S_0 < K < S_0 u^2}$.

The continuation value at time 1 if the stock goes up is $C_1(u) = \frac{u-1-r}{1+r} * \frac{u(K-S_0)}{u^2-1}$. And the continuation value at time 1 if the stock goes down is

$$C_1(d) = \frac{K}{1+r} - S_0 u$$

If the option holder decided to exercise at time 1 in state u then they would receive $E_1(u) = \max(0, K - S_0 u)$.If the option holder decided to exercise at time 1 in state d, they would receive

$$E_1(d) = K - S_0 \frac{1}{u}$$

To see whether it is larger or less than 0. Therefore, we got that the range of r, if r is less than or equal to $\frac{(S_0 u^2 - K)(u-1)}{u(K-S_0)}$, the $C_1(u)$ is larger than the $E_1(u)$, if r is less than or equal to 0 , the $C_1(d)$ is larger than the $E_1(d)$which means that holders will not exercise early. However, if r is less than 0, American option prices and European option prices are the same, which is

$$P^A = P^E = \frac{u(S_0-K)(u(r+1)-1)(r-u+1)-(r+1)(u^2-1)(S_0(r+1)-Ku)(-r+u-1)}{(r+1)^2(u^2-1)^2}$$

2.3.1. The Range of K is $S_0u < K < S_0u^2$

When $r > \frac{(S_0u^2-K)(u-1)}{u(K-S_0)}$,

$$P^A = \frac{1}{1+r}(q(K-S_0u)+(1-q)(K-S_0d) = \frac{Ku-s}{(1+r)u} - \frac{S_0-u^2}{u^2-1}$$

In case B, since $C_1(d)$ and $E_1(d)$ are unique, P^E is always equal to the following formula:

$$P^E = \frac{1}{1+r}(q(\frac{K}{1+r}-\frac{S_0}{u})+(1-q)\left(\frac{K}{1+r}-\frac{S_0}{u}\right) = \frac{u(S_0-K)(u(r+1)-1)(r-u+1)-(r+1)(u^2-1)(S_0(r+1)-Ku)(-r+u-1)}{(r+1)^2(u^2-1)^2}$$

$$P^A < P^E$$

The difference between American option and European option is

$$P^A - P^E = \frac{\begin{array}{c}(u(-S_0+u^2)(r+1)^2(u^2-1)\\ -u(u(S_0-K)(u(r+1)-1)\\ +(r+1)(u^2-1)(S_0(r+1)-Ku)(r-u+1)\\ +(r+1)(u^2-1)^2(Ku-s))\end{array}}{u(r+1)^2(u^2-1)^2}$$

When $\frac{(S_0u^2-K)(u-1)}{u(K-S_0)} > r > 0$

$$P^A = \frac{1}{1+r}(q\frac{1}{1+r}(K-S_0)+(1-q)(K-S_0d) = -u(S_0-K) * \frac{(u(r+1)-1)-(S_0-Ku)(r+1)(u^2-1)(-r+u-1)}{(r+1)^2(u^2-1)^2}$$

$$P^E = \frac{1}{1+r}(q(\frac{K}{1+r}-\frac{S_0}{u})+(1-q)\left(\frac{K}{1+r}-\frac{S_0}{u}\right) = \frac{u(S_0-K)(u(r+1)-1)(r-u+1)-(r+1)(u^2-1)(S_0(r+1)-Ku)(-r+u-1)}{(r+1)^2(u^2-1)^2}$$

$$P^A < P^E$$

The difference between American option and European option is

$$P^A - P^E = \frac{\begin{array}{c}(-u^2(S_0 - Ku^2)(r+1)(-r+u-1)\\ +u(u(S_0-K)(u(r+1)-1)\\ +(S_0-Ku)(r+1)(u^2-1)(r-u+1)\\ +(-S_0+Ku^2)(r+1)(u^2-1)\end{array}}{u(r+1)^2(u^2-1)^2}$$

2.3.2. The Range of K is $S_0 < K < S_0 u$

When $r > 0$

$$P^A = \frac{1}{1+r}(q\frac{1}{1+r}(K - S_0) + (1-q)(K - S_0 \mathrm{d})$$

$$= -u(S_0 - K) * \frac{\begin{array}{c}(u(r+1)-1)\\ -(S_0-Ku)(r+1)(u^2-1)(-r+u-1)\end{array}}{(r+1)^2(u^2-1)^2}$$

$$P^E = \frac{1}{1+r}(q(\frac{K}{1+r} - \frac{S_0}{u}) + (1-q)\left(\frac{K}{1+r} - \frac{S_0}{u}\right)$$

$$= \frac{\begin{array}{c}u(S_0-K)(u(r+1)-1)(r-u+1) -\\ (r+1)(u^2-1)(S_0(r+1)-Ku)(-r+u-1)\end{array}}{(r+1)^2(u^2-1)^2}$$

$$P^A < P^E$$

The difference between American option and European option is

$$\boldsymbol{P^A - P^E = \frac{\begin{array}{c}(-u^2(S_0 - Ku^2)(r+1)(-r+u-1)\\ +u(u(S_0-K)(u(r+1)-1)\\ +(S_0-Ku)(r+1)(u^2-1)(r-u+1)\\ +(-S_0+Ku^2)(r+1)(u^2-1)\end{array}}{u(r+1)^2(u^2-1)^2}}$$

2.4. Case C

The range of $\boldsymbol{K}$ is less than $\boldsymbol{\frac{S_0}{u} < K < S_0}$.

When the stock goes up, the continuation value at time 1 is $C_1(u) = 0$, because the strike price is less than the S_0, holder will decide not to exercise it. And when the stock goes down, the continuation value at time 1 is

$$C_1(d) = \frac{1}{1+r} * \frac{u^2 - u(1+r)}{u^2-1} * \left(K - S_0\frac{1}{u^2}\right)$$

$$= \frac{S_0 - Ku^2}{u(u^2-1)} + \frac{Ku^2 - S_0}{(u^2-1)(1+r)}.$$

DOI: 10.54254/2754-1169/82/20230995

If the option holder decided to exercise at time 1 in state u then they would receive $E_1(u) = 0$, because the strike price is less than the S_0, holder decides not to exercise it. If the option holder decided to exercise at time 1 in state d, they would receive

$$E_1(d) = \max\left(0, K - S_0\frac{1}{u}\right)$$

To decide whether to exercise early, we can compare the continuation value and exercise value, which is

$$C_1(d) - E_1(d) =$$

$$\frac{S_0 - Ku^2}{u(u^2-1)} + \frac{Ku^2 - S_0}{(u^2-1)(1+r)} - \left(K - S_0\frac{1}{u}\right)$$

To see whether it is larger or less than 0. Therefore, we got the range of r. If r is larger than or equals to $\frac{Ku-K-S_0u+S_0}{-Ku^2-Ku+K+S_0u}$, the continuation value is larger than the exercise value, which means that holders will not exercise early. If r is equal to $\frac{Ku-K-S_0u+S_0}{-Ku^2-Ku+K+S_0u}$, American option prices and European option prices are the same, which is

$$P^A = P^E = \frac{1}{1+r} * \frac{u^2 - u(1+r)}{u^2-1} *$$

$$\left(\frac{S_0 - Ku^2}{u(u^2-1)} + \frac{Ku^2 - S_0}{(u^2-1)(1+r)}\right) = \frac{(u(S_0 - Ku^2) - (S_0 - Ku^2)(r+1))(r-u+1)}{(r+1)^2(u^2-1)^2}$$

However, if r is less than $\frac{Ku-K-S_0u+S_0}{-Ku^2-Ku+K+S_0u}$, the continuation value is less than the exercise value, which means that holder will exercise the options early. Then it will go into two ways.

2.4.1. The Range of K is $S_0\frac{1}{u} < K < S_0$.

In this situation, we can get that American option price is larger than European option price.

$$P^E = \frac{(u(S_0 - Ku^2) - (S_0 - Ku^2)(r+1))(r-u+1)}{(r+1)^2(u^2-1)^2}$$
$$< \frac{1}{1+r} * \frac{u^2 - u(1+r)}{u^2-1} * \left(k - \frac{S_0}{u}\right)$$
$$= \frac{(S_0 - ku)(r-u+1)}{(r+1)(u^2-1)} = P^A$$

The difference between American option and European option is

$$P^A - P^E = (S_0 - Ku)(r+1)(u^2-1) -$$
$$(S_0 - Ku^2)(-r+u+1)\frac{(r-u+1)}{(r+1)^2(u^2-1)^2}$$

2.4.2. The Range of K is $S_0\frac{1}{u^2} < K < S_0\frac{1}{u}$.

Because the strike price is less than $S_0\frac{1}{u}$, the option holder will not decide to exercise early at time 1. Therefore, American option prices and European option prices are the same.

$$P^A = P^E = \frac{(u(S_0 - Ku^2) - (S_0 - Ku^2)(r+1))(r-u+1)}{(r+1)^2(u^2-1)^2}$$

2.5. Case D

The range of $\boldsymbol{K}$ is $\boldsymbol{K < S_0\frac{1}{u^2}}$.

Because the strike price is less than $S_0\frac{1}{u^2}$, holders will decide not to exercise early. Therefore, the continuation value and exercise value will all equal to 0, no matter the stock going up or going down.

$$C_1(u) = 0$$

$$C_1(d) = 0$$

$$E_1(u) = 0$$

$$E_1(d) = 0$$

3. Conclusion

In conclusion, the holder's rights of American options are more flexible. Under the same conditions, the pricing of American options is either higher or equal to that of European options. By utilizing the binomial model, we can effectively analyze options across various time periods and different scenarios, and makes it clearer to compute the price of options with rigorous probability theory. This conclusion of exercising early of American options with binomial model certainly applies to the current economic phenomenon. It also plays an important role in the financial market, which gives holders more opportunities for speculation, hedging and arbitrage, and holders can exercise the options more flexibly and obtain greater returns in the current market. The holder can exercise the options at any time in the event of adverse fluctuations in the market to reduce losses. However, it should be noted that although American options are more flexible, the holder needs to spend more effort to monitor the market and make decisions. Therefore, the American option is a very important market tool, suitable for the current economic phenomenon, and can be used for making profit. Of course, it can also be used for risk management and portfolio protection. However, one needs to get a very good understanding of the market and options' trading before making decisions.

Acknowledgment

Shengye Xu and Jiaxun Ji contributed equally to this work and should be considered co-first authors.

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