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# *Application of Lagrange Multipliers in One-period Portfolio Model When Incorporating Non-linear Utility Functions*

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***Abstract:*** This paper explores the extension of the traditional one-period portfolio optimization model through the application of Lagrange multipliers under non-linear utility functions. Classical portfolio theory typically assumes linear or quadratic utility functions, simplifying the relationship between risk and return. However, real-world investor preferences often exhibit more complex, non-linear behaviors, especially in the presence of risk aversion or behavioral biases. By incorporating non-linear utility functions, this study examines how the role of Lagrange multipliers evolves when managing the trade-off between maximizing expected utility and adhering to budget constraints. Through a modified Lagrangian approach, it derives new first-order conditions and analyze the resulting changes in optimal portfolio allocation. The analysis focuses on how non-linearity in utility functions alters the sensitivity of portfolio weights to changes in asset prices and risk-free rates. Additionally, this paper highlights the implications of this extension for practical portfolio management, particularly in cases where investor utility is driven by non-standard risk preferences. The findings contribute to a deeper understanding of how portfolio optimization models can be tailored to better reflect real-life decision-making scenarios.

***Keywords:*** One-period portfolio optimization, Lagrange multipliers, Non-linearity, Expected Utility Maximization, Portfolio theory.

## 1. Introduction

The one-period portfolio optimization model, rooted in modern portfolio theory, seeks to determine the optimal allocation of wealth across assets to maximize an investor's expected utility. Traditionally, this model assumes linear or quadratic utility functions, such as the commonly used mean-variance framework developed by Markowitz, which simplifies the optimization process. However, these linear approximations often fail to capture the full spectrum of real-world investor behavior, particularly when accounting for risk aversion and behavioral biases. Non-linear utility functions, which more accurately reflect an investor's risk preferences, introduce additional complexity into the portfolio optimization problem. The application of Lagrange multipliers in this context allows for the inclusion of constraints, such as budget limitations, while solving the optimization problem. When non-linear utility functions are incorporated, the role of Lagrange multipliers evolves, leading to more nuanced and complex first-order conditions for optimal portfolio allocation.

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Cvitanic's "Introduction to the Economics and Mathematics of Financial Markets" is what motivated me to conduct this study primarily. His portfolio optimization methods applies Lagrange multipliers within mean-variance models to handle budget and risk constraints, forming the theoretical base for analyzing how non-linear utility functions alter portfolio optimization by shifting the risk-return dynamics [1]. Li expands on this by demonstrating how Lagrange multipliers optimize power system costs through managing equality and inequality constraints. This general application supports how the method can be adapted for portfolio optimization involving non-linear utilities [2]. Klein's tutorial further highlights how Lagrange multipliers operate under equality constraints, providing foundational insights into how these constraints behave when extended to non-linear utility functions [3]. C. Balaji explores Lagrange multipliers in solving multi-variable, non-linear optimization problems, emphasizing their utility in identifying maximum or minimum values in constrained spaces. This aligns with the complexities of non-linear utility functions in portfolio models [4]. Rockafellar's research discusses the development of Lagrange multipliers, particularly in nonsmooth analysis and duality, which is particularly relevant when considering the more intricate nature of non-linear utility functions [5]. Rockafellar's additional paper delves into the classical and modern uses of Lagrange multipliers, exploring their significance in duality and optimization problems. This is directly relevant when examining how non-linear utilities affect the constraints in a one-period portfolio model [6]. Chapados' work on single- and multi-period portfolio models shows how Lagrange multipliers help balance risk and return in constrained settings, making it a valuable source for understanding how these methods adapt when utility functions become non-linear [7]. Sornette introduces non-linear covariance matrices and their application in portfolio theory, addressing non-Gaussian distributions and providing a model that complements the study of non-linear utility functions and risk assessment [8]. Staines 'research compares myopic and dynamic approaches to portfolio optimization, focusing on Lagrange multipliers in single-period models. His insights on dynamic versus static approaches offer a practical perspective on how non-linear utilities can shift optimization strategies [9]. Kofi explores portfolio optimization by applying the Lagrangian multiplier method to manage budget and risk constraints, reinforcing how these multipliers adapt in the face of complex optimization scenarios like those involving non-linear utility functions [10].

This body of literature offers a comprehensive framework for understanding the role of Lagrange multipliers in constrained optimization, particularly as it relates to portfolio management. The exploration of non-linear utility functions and their impact on optimization outcomes highlights the adaptability and complexity of the Lagrange multiplier method in financial models. This review not only supports the theoretical foundation of Lagrange multipliers but also offers practical insights for analyzing how these multipliers behave in one-period portfolio models with non-linear utilities.

The objective of this study is to investigate how the application of Lagrange multipliers in a one-period portfolio optimization model is influenced by the introduction of non-linear utility functions. This research aims to extend the classical portfolio theory by incorporating non-standard utility behaviors, such as increased risk aversion and behavioral biases, to examine their impact on optimal asset allocation. Through a detailed analysis of the modified Lagrangian approach, the study explores how these changes affect first-order conditions and the sensitivity of portfolio weights, offering practical insights for portfolio management under complex investor preferences.

## 2. Analysis of Lagrange Multipliers under Non-linear Utility Functions

### 2.1. Role of Lagrange Multiplier in Constrained Optimization

Lagrange multipliers play a critical role in solving constrained optimization problems, particularly in portfolio optimization models where an investor aims to maximize their expected utility subject to budget constraints. In a one-period portfolio optimization setting, the budget constraint ensures that the sum of investments across assets does not exceed the initial wealth, represented as:

$$x = \delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0) \tag{1}$$

Here, $\delta_i$, represents the portfolio weight in each asset, $S_i(0)$ is the price of the asset at time $t = 0$, and $B(0)$ represents the risk-free bond price. The objective is to maximize the expected utility of the portfolio's terminal wealth at time $t = 1$:

$$X(1) = \sum_{i=0}^{N} \delta_i S_i(1) \tag{2}$$

with

$$\max_{\delta} E[U(X(1))] \tag{3}$$

being the objective.

To handle the constraint mathematically, it introduces the Lagrange multiplier $\lambda$, which adjusts the optimization to account for the budget constraint. The Lagrangian function is formulated as:

$$L = E\left[U\left(\sum_{i=0}^{N} \delta_i S_i(1)\right)\right] - \lambda\left(\sum_{i=0}^{N} \delta_i S_i(0) - x\right) \tag{4}$$

The first-order conditions for maximizing this Lagrangian yield:

$$E[U'(X(1))S_i(1)] = \lambda S_i(0), i = 0, \dots, N \tag{5}$$

This condition implies that the marginal utility of wealth, weighted by the asset price at time $t = 1$, must be equal to the Lagrange multiplier $\lambda$ times the initial price of the asset at time $t = 0$. For the special case of a risk-free asset, where $S_0(0) = 1$ and $S_0(1) = 1 + r$, the multiplier becomes:

$$\lambda = E[U'(X(1))(1 + r)] \tag{6}$$

This reflects the marginal utility of wealth associated with holding the risk-free asset.

Finally, solving for the price of each risky asset provides:

$$S_i(0) = \frac{E[U'(X(1))S_i(1)]}{E[U'(X(1))(1+r)]} \tag{7}$$

Thus, Lagrange multipliers not only help incorporate constraints into the optimization process but also provide insights into how asset prices reflect investors' marginal utilities. The multiplier's behavior becomes increasingly complex when nonlinear utility functions are introduced, as it adjusts to account for more sophisticated risk preferences and the curvature of the utility function.

## 2.2. Introduction of Non-linear Utility Functions into the Model

In ideal scenarios, the investor is indifferent to risk and only cares about maximizing expected returns, there is no curvature in this utility function, meaning that the investor has constant marginal utility of wealth. The utility functions incorporated are usually regarded as followed in this case:

$$U(x) = \alpha x \tag{8}$$

This function reflects risk neutrality and assumes that the investor is equally satisfied with each additional unit of wealth, regardless of the amount of risk involved.

Non-linear utility functions capture a broader range of investor behaviors and preferences, making them more suitable for portfolio optimization models that account for complex risk preferences. Several types of non-linear utility functions are commonly used in financial models:

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Logarithmic Utility Function: defined as

$$U(x) = log(x) \tag{9}$$

The logarithmic utility function captures decreasing absolute risk aversion. As wealth increases, the investor becomes less risk-averse. This function is often used for investors with long-term investment horizons because they tolerate more risk when they accumulate more wealth. The concave nature of this utility function reflects risk aversion, but the degree of risk aversion diminishes as wealth grows.

Exponential Utility Function: defined as

$$U(x) = 1 - e^{-\alpha x} \tag{10}$$

This utility function is widely used to represent constant absolute risk aversion. The parameter $\alpha$ controls the level of risk aversion, with higher values of $\alpha$ indicating greater risk aversion. In this model, an investor is equally averse to risk regardless of their wealth level. The exponential form implies that the investor prefers to avoid risk, and the marginal utility diminishes as wealth increases, but their risk tolerance remains constant across wealth levels.

Power Utility Function: defined as

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, for \gamma \neq 1 \tag{11}$$

The power utility function, also known as constant relative risk aversion, implies that risk aversion depends on the investor's wealth relative to their consumption or other reference points. The parameter $\gamma$ governs the risk aversion level, with higher values of $\gamma$ indicating greater aversion to risk. This function is frequently used in models where investors adjust their consumption and investment plans as their wealth changes.

Quadratic Utility Function: defined as

$$U(x) = ax - bx^2 \tag{12}$$

This utility function represents increasing risk aversion as wealth increases. Unlike other utility functions, it has a finite maximum wealth level after which the utility decreases, implying that investors become more risk-averse as their wealth grows. Although it is useful for modeling certain types of risk aversion, the function is less practical for extremely wealthy investors because it assumes that utility declines after reaching a certain wealth level.

Derived from prospect theory, utility functions based on behavioral economics account for psychological factors such as loss aversion. In this case, the utility function is often piecewise, with steeper slopes for losses compared to gains, reflecting that investors experience losses more severely than gains of equivalent magnitude. These functions are non-linear and introduce significant curvature into the optimization problem, making the decision-making process sensitive to both the probability of gains and losses, as well as the size of those outcomes.

Incorporating non-linear utility functions into portfolio models by means (2) to (7) introduces curvature into the optimization process, reflecting varying levels of risk aversion based on wealth or psychological factors. This curvature complicates the optimization because the marginal utility of wealth is no longer constant; it varies based on both the level of wealth and the type of utility function used. As a result, the first-order conditions derived from the Lagrangian will differ significantly, and the sensitivity of optimal portfolio allocations to changes in market conditions becomes more complex. This reflects a more realistic approach to modeling investor behavior compared to linear models, which oversimplify the relationship between wealth and risk preferences.

### 2.3. Reformulation First-order Conditions under Portfolio Optimization and Implications

When non-linear utility functions are incorporated into portfolio optimization, the Lagrange multiplier $\lambda$ takes on a more intricate role, reflecting complex trade-offs between utility maximization and the budget constraint. In linear utility models, $\lambda$ simply represents the marginal value of wealth, indicating how much the investor's utility would increase with an additional unit of wealth. However, in non-linear utility models, such as those involving logarithmic, exponential, or power utility functions, the Lagrange multiplier also captures how changes in wealth impact the investor's risk aversion and utility.

Logarithmic Utility Function: The first derivative is

$$U'(x) = \frac{1}{x} \tag{13}$$

Applying the derived condition in (5), the first-order condition becomes

$$E\left[\frac{S_i(1)}{X(1)}\right] = \lambda S_i(0) \tag{14}$$

This shows that $\lambda$ adjusts not only to reflect the marginal utility of wealth but also the ratio of current asset prices to total wealth. This introduces more sensitivity in the portfolio allocation as wealth changes.

Exponential Utility Function: The first derivative is

$$U'(x) = \alpha e^{-\alpha x} \tag{15}$$

The first-order condition becomes

$$E\left[\alpha e^{-\alpha X(1)} S_i(1)\right] = \lambda S_i(0) \tag{16}$$

The first-order condition implies that $\lambda$ accounts for the investor's constant absolute risk aversion. Here, $\lambda$ adjusts to maintain a balance between the investor’s aversion to risk and the budget constraint, leading to different allocations depending on the investor’s risk sensitivity parameter $\alpha$.

Power Utility Function: The first derivative is

$$U'(x) = x^{-\gamma} \tag{17}$$

The first-order condition becomes

$$E\left[X(1)^{-\gamma} S_i(1)\right] = \lambda S_i(0) \tag{18}$$

In this case $\lambda$ becomes a function of wealth raised to the power of $\gamma$, making it highly sensitive to both wealth levels and the degree of relative risk aversion. The first-order condition shows that $\lambda$ is intricately tied to changes in wealth, particularly for more risk-averse investors, where the marginal utility decreases rapidly as wealth grows.

Quadratic Utility Function: The first derivative is

$$U'(x) = a - 2bx \tag{19}$$

The first-order condition becomes

$$E\left[(a - 2bX(1))S_i(1)\right] = \lambda S_i(0) \tag{20}$$

This first-order condition demonstrates how $\lambda$ captures the trade-off between the linear and quadratic terms of the utility function. Here, $\lambda$ reflects how the marginal utility changes with wealth, incorporating both the investor's initial wealth preference and the increasing risk aversion that quadratic utility functions represent.

# 3. Reformulated Portfolio Weights and Economic Implications of the Results

## 3.1. Analysis of Sensitivity of Portfolio Weights

The above non-linear utility functions introduce varying levels of sensitivity to portfolio weights with respect to different influencing factors.

Logarithmic Utility Function: Given the utility function and the derived first-order conditions, it will divide both sides of the equation by $S_i(0)$ to isolate the portfolio weight $\delta_i$ (same isolation procedure will apply for rest three non-linear utility functions), then it gets:

$$\frac{E\left[\frac{S_i(1)}{X(1)}\right]}{S_i(0)} = \lambda \tag{21}$$

This might seem counterintuitive, but it actually shows the relationship between the expected marginal utility and $\lambda$. It recognizes that $X(1)$, total wealth at time 1, depends on the portfolio weights. In this case, solving for $\delta_i$ requires isolating the part of the expression that explicitly represents the portfolio weight. Hence, by rearranging the terms in (21) and expressing $\delta_i$ as:

$$\delta_i = \frac{E\left[\frac{S_i(1)}{X(1)}\right]}{\lambda S_i(0)} \tag{22}$$

It can finally regard the sensitivity of $\delta_i$ to changes in $\lambda$ as the derivative taken against $\lambda$ (same mathematical approach applies for rest three:

$$\frac{\partial \delta_i}{\partial \lambda} = \frac{-E\left[\frac{S_i(1)}{X(1)}\right]}{\lambda^2 S_i(0)} \tag{23}$$

As $\lambda$ increases, $\delta_i$ decreases proportionally, meaning less wealth is allocated to each asset. The term $\frac{S_i(1)}{X(1)}$ ensures that the weight depends on the proportion of asset prices relative to total wealth.

Exponential Utility Function: Rearrange the terms, it will gain a new expression of $\delta_i$:

$$\delta_i = \frac{E\left[\alpha e^{-\alpha X(1)} S_i(1)\right]}{\lambda S_i(0)} \tag{24}$$

Using the same mathematical method as in (23), weights' responsiveness to $\lambda$ can be expressed as:

$$\frac{\partial \delta_i}{\partial \lambda} = \frac{-E\left[\alpha e^{-\alpha X(1)} S_i(1)\right]}{\lambda^2 S_i(0)} \tag{25}$$

This shows that the sensitivity of $\delta_i$ to changes in $\lambda$ depends on both the expected value $E\left[\alpha e^{-\alpha X(1)} S_i(1)\right]$ and the current price. Since $\lambda$ appears in the denominator, as $\lambda$ increases, the portfolio weight $\delta_i$ decreases, meaning less wealth is allocated to asset i.

Power Utility Function: Similarly for power utility function, $\delta_i$ can be expressed as:

$$\delta_i = \frac{E\left[X(1)^{-\gamma} S_i(1)\right]}{\lambda S_i(0)} \tag{26}$$

Taking first derivative against $\lambda$,

$$\frac{\partial \delta_i}{\partial \lambda} = \frac{-E\left[X(1)^{-\gamma} S_i(1)\right]}{\lambda^2 S_i(0)} \tag{27}$$

Here, the sensitivity of $\delta_i$ to changes in $\lambda$ is influenced by the wealth level raised to the power of $-\gamma$. If $\gamma$ is large (high relative risk aversion), the changes in $\delta_i$ in response to $\lambda$ become more pronounced. Hence, Sensitivity is inversely proportional to $\lambda^2$ and depends on $X(1)^{-\gamma}$, making it more sensitive for risk-averse investors.

Quadratic Utility Function: Rearranging first-order condition to express $\delta_i$, it get:

$$\delta_i = \frac{E[(a-2bX(1))S_i(1)]}{\lambda S_i(0)} \tag{28}$$

Taking the derivative with respect to $\lambda$, it gets:

$$\frac{\partial \delta_i}{\partial \lambda} = \frac{-E[(a-2bX(1))S_i(1)]}{\lambda^2 S_i(0)} \tag{29}$$

In this case, the quadratic term $bX(1)$ increases the sensitivity of $\delta_i$ to wealth changes. This introduces non-linear behavior into the relationship between the portfolio weights and $\lambda$, making the portfolio weights more sensitive as wealth grows. Hence, Sensitivity increases with the term $-2bX(1)$, leading to a higher impact on $\delta_i$ as wealth inflates.

## 3.2. Interpretation of Economic Implications

The incorporation of non-linear utility functions into portfolio models offers a more nuanced and realistic representation of investor behavior, particularly for those exhibiting strong risk aversion or loss aversion. Unlike linear utility functions, which assume constant marginal utility, non-linear utility functions such as exponential, logarithmic, or power functions allow for varying degrees of risk sensitivity. For example, investors with exponential utility functions are highly risk-averse and their utility declines sharply with increasing wealth, making them more cautious in allocating their portfolio. This kind of modeling better mirrors real-world scenarios, where investor decisions are often influenced by fear of losses or disproportionate sensitivity to risk, as seen in behavioral finance concepts like prospect theory.

Moreover, non-linear utility functions significantly impact investment strategies, particularly in situations where constraints like limited budgets or specific risk requirements exist. These constraints often lead to complex trade-offs between risk and return, making the allocation of assets more volatile and responsive to market changes. By incorporating behavioral insights into traditional models, economists and portfolio managers can make more accurate predictions and tailor strategies to individual preferences, improving decision-making under uncertainty.

In the next section, it will simulate portfolio optimization under non-linear utility functions and analyze how the resulting asset weights differ from those predicted by traditional, linear utility models. Through these simulations, it will explore how these models adjust to market conditions and constraints, providing more robust insights into portfolio management strategies.

# 4. Application

For a real world case, supposing that the targets are building the US stock portfolio and it wishes not to complicate things, so only three stocks are chosen-Amazon, Starbucks and Tesla. Historical data can be collected and used for rounds of simulations to build the theoretically speaking, most efficient portfolio.

Past 6 months' indices (Adj. close price, etc) for the three stocks were collected. Risk-free asset in the market consistently spans the past 6 months, holding a value of 5.5%. To account for discounting, it utilizes a new initial value function for each asset simply by plugging (6) into (7):

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$$S_i(0) = \frac{E\left[U'(X(1))S_i(1)\right]}{E\left[U'(X(1))(1+r)\right]} = \frac{1}{1+r}\frac{E\left[U'(X(1))S_i(1)\right]}{E\left[U'(X(1))\right]} \tag{30}$$

Next, it calculate the covariance matrix of daily returns:

$$\begin{pmatrix} 0.000286 & 0.000008 & 0.000172 \\ 0.000008 & 0.000440 & 0.000071 \\ 0.000172 & 0.000071 & 0.001436 \end{pmatrix} \tag{31}$$

Solving for the optimal portfolio by incorporating newly derived $S_i(0)$ function and taking inverse of it, and geting:

AMZN: 58.01%

SBUX: 39.10%

TSLA: 2.89%

While the weights might seem counterintuitive, given that Amazon and Starbucks have negative expected returns, this indicates that during the selected period, these stocks still hold a significant portion of the optimal portfolio. This is likely due to their relatively lower risk and the adjusted high expected returns when compared to Tesla. Also, portfolio optimization requires balancing risk and return, the seemingly odd results might signals lack of diversification in the portfolio. Even assets with lower returns can be favorable in a well-diversified portfolio, it will continue to evaluate this allocation later.

It will only go through the logarithm utility function as illustration of the approach, as the same applies for all other utility functions. One thing worth noting is how to choose the parameter values - in the proposed model, and assuming all investors to be rational, that means while having different levels of risk preferences, all investors share the same goal of utility maximization and none of them is purely risk-prone. The left three results and parameter values used are presented as followed:

Exponential Utility Function: for $\alpha = 2$ :

AMZN: 18.208%

SBUX: 40.896%

TSLA: 40.896%

Power Utility Function: for $\gamma = 0.5$ :

AMZN: 45.62%

SBUX: 32.30%

TSLA: 22.08%

Quadratic Utility Function: for a and b both equals 1:

AMZN: 50.73%

SBUX: 28.46%

TSLA: 20.81%

Seeing the optimized proportions under each scenario, one trend observed is for every utility function choice, AMZN is the one stock that would get most investment except for the exponential utility function. The differences in patterns are the exact proof of how different risk preferences are addressed by use of Lagrange multipliers.

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Across most utility functions, AMZN is given the largest allocation except for the exponential utility function. This suggests that Amazon has relatively lower risk or better risk-adjusted returns compared to SBUX and TSLA during this period. Even though AMZN has a negative expected return over this period, the portfolio optimization algorithm gives it a significant weight because of its volatility characteristics, suggesting it plays a critical role in stabilizing the portfolio. In the logarithmic utility function, TSLA receives only 2.89%, which indicates it carries more risk without sufficient return to justify a higher allocation. However, under the exponential utility function and power utility function, TSLA has a much higher allocation (above 20%). This difference shows how different utility functions, representing varying levels of risk tolerance, result in different asset allocations. SBUX consistently receives a moderate portion of the portfolio. Its allocation changes slightly depending on the utility function, but it seems to represent a balance between risk and return.

The Lagrange multiplier method is used to optimize the portfolio by considering both the risk (through the covariance matrix) and the return (through expected returns). The Lagrange multiplier adjusts the portfolio weights to maximize utility while adhering to constraints such as total wealth and portfolio weights summing to 1.

The fact that negative expected returns (for AMZN and SBUX) still lead to significant portfolio allocations suggests that the optimization is heavily influenced by risk minimization. Lagrange multipliers help balance the trade-off between expected return and risk, especially in constrained scenarios.

By solving the optimization problem using derived utility functions like □□(□), the incorporation of risk-free assets (at 5.5% return) is also handled using the Lagrange multipliers, ensuring that the portfolio still meets a desired risk-return profile, even when assets like Amazon and Starbucks show underperformance.

## 5. Conclusion

The use of Lagrange multipliers simplifies the optimization of the portfolio by transforming the constrained problem into an unconstrained one, revealing optimal asset weights that maximize expected utility. Additionally, the non-linearity of the utility function introduces complexity in both the computation and interpretation of the results, emphasizing the importance of numerical methods and simulations in practical applications. Though it successfully captures risk preferences of all kinds, it could not expect the method to be too profitable — it does not account for multi-period, continuous portfolio building, as the Lagrangian introduced here is too sensitive to single-periods parameter estimation and initial conditions. Advanced extensions to that method shall be pursued.

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