Proceedings of the 2nd International Conference on Business and Policy Studies
DOI: 10.54254/2754-1169/7/20230250

Review and Practice of Option Pricing Research

Yuxuan Li[1,a,*]

[1]*University of Bristol, Bristol BS8 1TH, UK*
a. le9820783@gmail.com
**corresponding author*

Abstract: This paper reviews the past research on option pricing, including the Bachelier option pricing formula, Black-Scholes model, and the Cox-Ross-Rubinstein (CRR) binomial option pricing model. Then some practical calculations are performed on the Black-Scholes model and CRR binomial model and the differences between the results of these two models are analyzed for investors to choose the appropriate model for option prices. From the practical calculation results, it is verified that the Black-Scholes model can give out the analytical solutions of option prices and decrease the calculation of solution, while it can only be used for European options with stock prices regarded as a continuous process. For the CRR binomial model, although it can only give numerical solutions and requires lots of calculation because of too many branches, but it can solve the American option prices with early exercises and the European option prices with stock prices regarded as a discrete process.

Keywords: option pricing, Black-Scholes Model, CRR binomial model

1. Introduction and Literature Review

The option is a financial derivative. Compared with other derivatives, the option has its functions and roles in risk management and measurement, asset allocation, and price discovery.

In the trade between ancient Greece and ancient Phoenicia around 1200 BC, the embryonic form of options trading has already appeared. Modern options trading began in the U.S., as well as European markets, in the 18th century. However, options are valuable, which brings up the question of valuing options, which is the theory of option pricing.

The Black–Scholes model, which is a method to determine the value of options, plays an important role in modern option pricing, and it helped Scholes and Merton get a Nobel prize in 1997. Louis Bachelier, the French mathematician, is regarded as the originator of option pricing, who found the European option pricing formula [1]. Black and Scholes found the original Black-Scholes Model for pricing European options without dividends, and in the same year, Merton further promoted and edited it, which becomes the well-known Black-Scholes Model now [2]. Cox, Ross, and Rubinstein came up with the CRR binomial option pricing model, which is a looking-back algorithm to calculate the option prices, and this model allows to calculate the prices of American options [3]. Since then, plenty of options pricing methods have been developed, such as Monte Carlo simulations on pricing European options, finite differencing methods, and methods for option pricing in incomplete markets [4].

DOI: 10.54254/2754-1169/7/20230250

2. The Role of Options

In financial markets, returns are often accompanied by risks. Some traders believed that risks could be avoided through personal experience and judgment, while some financial scientists believed that prices are completely random and unpredictable, and they tried to deal with risk in a more quantitative way. Louis Bachelier suggested that market prices were random and unpredictable, and options could be used to control risk [1]. If people worry that the price will fall and bring losses, the put option allows them to sell at an agreed price. If people believe that the price of an asset will go up and the call option allows them to buy the asset at a certain agreed price in the future.

The option has an important role in the financial market for its properties.

1)Hedge market risk: as mentioned before, if investors short assets and are worried about rising asset prices, they can buy call options to hedge the upside risks in the market. On the contrary, if investors long assets and are afraid that assets price will fall, they can do the opposite.

2)Minimize risk: For option buyers, the biggest deficit is losing the value of the options themselves.

3)Build a portfolio: variables of options, such as expirations, strike prices, call and put options, and leverages can be combined with the same underlying asset in various ways to create different investment strategies to adjust the expected return and risks of the portfolio, which makes the option, the emerging financial derivative, be full of flexibility.

4)Provide leverage: Investors only need to pay a small amount, instead of buying the asset, and can gain from the price rising of the asset, while the deficit is no more than the option price. Therefore, the leverage is very high.

Options are delivered in the future and can complement the current stock market. It is precise because of the nature of its future delivery that hedging investment strategies for options and stocks have emerged in the capital market, and the following question is how to price the options. Since the pricing of options involves the subjective judgment of each investor on the market, which remains to be a difficult problem.

3. Option Pricing Models

3.1. Bachelier Option Pricing Formula

Loius Bachelier is the earliest researcher who focuses on the method of option pricing, and in his doctoral dissertation, " Théorie de la spéculation," which was the first time to give the Brownian motion a strict mathematical analysis. Bachelier presumed that the process of changes of stock price was a drift-free standard Brownian motion, where the variance is σ^2/unit time, and he yield that the price at expiration T of an option is expected to be

$$C(S,T) = S \cdot \Phi\left(\frac{S-K}{\sigma\sqrt{T}}\right) - K \cdot \Phi\left(\frac{S-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T} \cdot \varphi\left(\frac{S-K}{\sigma\sqrt{T}}\right) \tag{1}$$

$$\mathrm{P}(\mathrm{S},\mathrm{T}) = \mathrm{K} \cdot \Phi\left(\frac{\mathrm{K}-\mathrm{S}}{\sigma\sqrt{\mathrm{T}}}\right) - \mathrm{S} \cdot \Phi\left(\frac{\mathrm{K}-\mathrm{S}}{\sigma\sqrt{\mathrm{T}}}\right) + \sigma\sqrt{\mathrm{T}} \cdot \varphi\left(\frac{\mathrm{S}-\mathrm{K}}{\sigma\sqrt{\mathrm{T}}}\right) \tag{2}$$

where $\mathrm{C}(\mathrm{S},\mathrm{T})$ is European call option price, $\mathrm{P}(\mathrm{S},\mathrm{T})$ is European put option price, S is stock price, K is strike price.

Louis Bachelier proposed the pricing formula of options, but the main flaw is he assumed that the stock prices obey normal distribution, as the Brownian motion allows the assumption that stock prices can be negative. Moreover, he did not consider the monetary value after adding the value of time, which is the interest rate. Of course, we can derive a version of this model that considers risk-free interest rate r:

$$C(S,T) = S \cdot \Phi\left(\frac{S-K}{\sigma\sqrt{T}}\right) - K \cdot e^{-rT} \cdot \Phi\left(\frac{S-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T} \cdot \varphi\left(\frac{S-K}{\sigma\sqrt{T}}\right) \tag{3}$$

$$P(S,T) = K \cdot e^{-rT} \cdot \Phi\left(\frac{K-S}{\sigma\sqrt{T}}\right) - S \cdot \Phi\left(\frac{K-S}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T} \cdot \varphi\left(\frac{S-K}{\sigma\sqrt{T}}\right) \tag{4}$$

Although Bachelier remained some problems, he had identified the most important factors in option pricing: the stock price and its volatility. And it was the first time that the random process was introduced to describe the market dynamics.

3.2. Black-Scholes Model for Pricing European Options

Fisher Black and Myron Scholes derived a model to evaluate the price of options by using the idea of delta hedging, which is the original Black-Scholes Model [2]. They discovered that if a stock is combined with some options (delta = stock/option), the stock price's volatility will disappear, and it can result in risk-free returns, then this option can be replaced by holding a portfolio which contains stocks and risk-free assets, and it has no variables relevant to its own risk.

The disadvantage of the original Black-Scholes model is that Black and Scholes assume the market is always in equilibrium, which is not possible to use in realistic. The market is always changing dynamically, and they need continuous hedging to keep the stock and option value fluctuations in the portfolio offsetting each other, rather than maintaining static hedging. Robert C. Merton helped edit the original Black-Scholes model and published his research results. He adopted Ito's lemma and researched continuous hedging more accurately, by using continuous time. Then it became the Black–Scholes theory of options pricing what we know now (Black-Scholes model).

By building a portfolio of options and stocks, the profits and losses of options and stocks can offset each other, and the portfolio thus is risk-free. In a market where it is impossible to make arbitrage, the portfolio's return should be the same with the interest rate under risk-free, and the differential equation of the option price in the Black-Scholes model is

$$\frac{\partial V_t}{\partial t} + rS_t\frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma^2\frac{\partial^2 V_t}{\partial S_t^2} - rV_t = 0 \tag{5}$$

The European option's formula of pricing is given by

$$Call(S,T) = S_0 \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2) \tag{6}$$

$$Put(S,T) = K \cdot e^{-rT} \cdot N(-d_2) - S_0 \cdot N(-d_1) \tag{7}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{8}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{9}$$

where S_t is the price of the stock at time t, V_t is the price of the option at time t, r is the riskless interest rate, σ is the volatility of stock price, T is the maturity time of the option, t is the current time, N is the cdf of standard normal distribution.

Although the Black-Scholes model is nearly perfect in theory, it must be admitted that any financial market, in reality, cannot meet the strict assumptions required by the model. Therefore, the Black-Scholes model is limited in practical applications. Moreover, the Black-Scholes model can only give out the price of European options, since it cannot consider the early exercise of American options.

DOI: 10.54254/2754-1169/7/20230250

3.3. CRR Binomial Option Pricing Model

The binomial option pricing model was proposed by Cox, Ross, and Rubinstein [3]. The binomial model is essentially a numerical representation of the model of Black-Scholes. The way it obtains option prices is to use a discrete-time binomial distribution to approximate the normal distribution. The method it adopts is to divide the validity period of the option enough, and within each divided interval, the stock price is ought to move from S at the beginning to S_u or S_d at the end, where $S_u > S$, $S_d < S$. The probability of moving from S to S_u is p, and the probability of moving from S to S_d is $q = 1 - p$. Since the rate of change in stock price has a normal distribution, using the principle of risk-neutral pricing, the up factor and down factor can be obtained:

$$u = e^{\sigma\sqrt{\Delta t}} \tag{10}$$

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}} \tag{11}$$

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{12}$$

$$q = \frac{u - e^{r\Delta t}}{u - d} \tag{13}$$

where $u = \frac{S_u}{S}$ is the factor of upward and $d = \frac{S_d}{S}$ is the factor of downward.

Consider a stock with a price S_0 at time 0, and there are two possible prices for this stock at time 1, which are denoted by $S_1(u)$ and $S_1(d)$. So, for an N-period binomial model, there are $\frac{N(N+1)}{2}$ prices to be calculated, and 2^N processes to look backwards.

S_0

$S_1(u) = S_0 u$ $\quad$ $S_1(d) = S_0 d$

$S_2(uu) = S_0 u^2$ $\quad$ $S_2(ud = du) = S_0 ud$ $\quad$ $S_2(dd) = S_0 d^2$

$S_3(uuu) = S_0 u^3$ $\quad$ $S_3(uud = udu = duu) = S_0 u^2 d$ $\quad$ $S_3(udd = dud = ddu) = S_0 ud^2$ $\quad$ $S_1(ddd) = S_0 d^3$

In the CRR model, the option price is calculated by discounting the interest rate under risk-free r from the expected value in the risk-neutral measure at the next time, which is backtracking the present value of the option from its value in the future, so it can also fit for pricing the American options. But the limitation of the CRR model is that the binomial model assumes that the stock price is only able to go upward or downward in each divided interval, but in fact, the stock price can be constant. And as the binomial option pricing model has too many branches, it is difficult to guarantee its accuracy and computing speed at the same time.

4. The Practice of Option Pricing

Consider an option that satisfies the following conditions:

1)The initial stock price is $S_0 = 32$
2)The strike price is $K = 50$
3)The interest rate for risk free is $r = 25\%$
4)The up factor $u = 2$, and the down factor $d = 0.5$
5)The maturity period $T = 5$

The CRR binomial stock price process can be obtained in Table 1.

Table 1: Stock prices S_t at time 0, 1, 2, 3, 4, 5.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					$S_5 = 1024$
				$S_4 = 512$	
			$S_3 = 256$		$S_5 = 256$
		$S_2 = 128$		$S_4 = 128$	
	$S_1 = 64$		$S_3 = 64$		$S_5 = 64$
$S_0 = 32$		$S_2 = 32$		$S_4 = 32$	
	$S_1 = 16$		$S_3 = 16$		$S_5 = 16$
		$S_3 = 8$		$S_4 = 8$	
			$S_3 = 4$		$S_5 = 4$
				$S_4 = 2$	
					$S_5 = 1$

4.1. Price of European Options

If the option is European, the Black-Scholes model is able to price for option at time 0:

$$\sigma = \frac{\ln(u)}{\sqrt{\Delta t}} = \frac{\ln 2}{1} = \ln 2 \tag{14}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 1.7407$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 0.1908$$

$$Call\_price = S_0 \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2) = 23.1535$$

$$Put\_price = K \cdot e^{-rT} \cdot N(-d_2) - S_0 \cdot N(-d_1) = 5.4788$$

Similarly, the values of European options can be obtained in Table 2 and Table 3.

Table 2: European call option prices $Call_{Et}$ at time 0, 1, 2, 3, 4, 5 using B-S model.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					$Call_{E5} = 974$
				$Call_{E4} = 473.06$	
			$Call_{E3} = 226.08$		$Call_{E5} = 206$
		$Call_{E2} = 106.48$		$Call_{E4} = 89.88$	

	$Call_{E1} = 49.74$		$Call_{E3} = 38.85$		$Call_{E5} = 14$

Table 2: (continued).

$Call_{E0} = 23.15$		$Call_{E2} = 17.10$		$Call_{E4} = 6.51$	
	$Call_{E1} = 7.64$		$Call_{E3} = 3.13$		$Call_{E5} = 0$
		$Call_{E2} = 1.48$		$Call_{E4} = 0.04$	
			$Call_{E3} = 0.07$		$Call_{E5} = 0$
				$Call_{E4} = 0.00$	
					$Call_{E5} = 0$

Table 3: European put option prices Put_{Et} at time 0, 1, 2, 3, 4, 5 using B-S model.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					$Put_{E5} = 0$
				$Put_{E4} = 0.00$	
			$Put_{E3} = 0.41$		$Put_{E5} = 0$
		$Put_{E2} = 2.10$		$Put_{E4} = 0.82$	
	$Put_{E1} = 4.13$		$Put_{E3} = 5.18$		$Put_{E5} = 0$
$Put_{E0} = 5.48$		$Put_{E2} = 8.72$		$Put_{E4} = 13.45$	
	$Put_{E1} = 10.04$		$Put_{E3} = 17.46$		$Put_{E5} = 34$
		$Put_{E2} = 17.10$		$Put_{E4} = 30.98$	
			$Put_{E3} = 26.40$		$Put_{E5} = 46$
				$Put_{E4} = 36.94$	
					$Put_{E5} = 49$

However, we can use the CRR binomial option pricing model to calculate the option prices at time 4:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.25} - 0.5}{2 - 0.5} = 0.5227$$

$$q = \frac{u - e^{r\Delta t}}{u - d} = \frac{2 - e^{0.25}}{2 - 0.5} = 0.4773$$

$$Call_{E4}(uuuu) = \frac{0.5227 * \max(1024 - 50, 0) + 0.4773 * \max(256 - 50, 0)}{e^{0.25}} = 473.0600$$

$$Put_{E4}(uuuu) = \frac{0.5227 * \max(50 - 1024, 0) + 0.4773 * \max(50 - 256, 0)}{e^{0.25}} = 0$$

DOI: 10.54254/2754-1169/7/20230250

Similarly, for European options, prices can be obtained in Table 4 and Table 5.

Table 4: European call option prices $Call_{Et}$ at time 0, 1, 2, 3, 4, 5 using CRR model.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					$Call_{E5} = 974$
				$Call_{E4} = 473.06$	
			$Call_{E3} = 225.67$		$Call_{E5} = 206$
		$Call_{E2} = 106.13$		$Call_{E4} = 89.06$	
	$Call_{E1} = 49.33$		$Call_{E3} = 38.37$		$Call_{E5} = 14$
$Call_{E0} = 22.71$		$Call_{E2} = 16.48$		$Call_{E4} = 5.70$	
	$Call_{E1} = 7.06$		$Call_{E3} = 2.32$		$Call_{E5} = 0$
		$Call_{E2} = 0.94$		$Call_{E4} = 0$	
			$Call_{E3} = 0$		$Call_{E5} = 0$
				$Call_{E4} = 0$	
					$Call_{E5} = 0$

Table 5: European put option prices Put_{Et} at time 0, 1, 2, 3, 4, 5 using CRR model.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					$Put_{E5} = 0$
				$Put_{E4} = 0$	
			$Put_{E3} = 0$		$Put_{E5} = 0$
		$Put_{E2} = 1.75$		$Put_{E4} = 0$	
	$Put_{E1} = 3.72$		$Put_{E3} = 4.70$		$Put_{E5} = 0$
$Put_{E0} = 5.03$		$Put_{E2} = 8.10$		$Put_{E4} = 12.64$	
	$Put_{E1} = 9.46$		$Put_{E3} = 16.65$		$Put_{E5} = 34$
		$Put_{E2} = 16.57$		$Put_{E4} = 30.94$	
			$Put_{E3} = 26.33$		$Put_{E5} = 46$
				$Put_{E4} = 36.94$	
					$Put_{E5} = 49$

4.2. Price of American Options

When it is an American option, the B-S model is not feasible for early exercise, and we can only use CRR binomial model.

For the American option price at time 4,

Proceedings of the 2nd International Conference on Business and Policy Studies
DOI: 10.54254/2754-1169/7/20230250

$$Call_{E4}(uuuu) = 473.06,$$
$$Put_{E4}(dddd) = 36.94,$$
$$Call\_early_{A4}(uuuu) = \max(512 - 50, 0) = 462,$$
$$Put\_early_{A4}(dddd) = \max(50 - 2, 0) = 48$$

so the American call option price

$$Call_{A4}(uuuu) = \max(473.06, 462) = 473.06$$
$$Put_{A4}(dddd) = \max(36.94, 48) = 48$$

Similarly, the values of American options can be obtained in Table 6 and Table 7.

Table 6: American call option prices $Call_{At}$ at time 0, 1, 2, 3, 4, 5 using CRR model.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					$Call_{A5} = 974$
				$Call_{A4} = 473.06$	
			$Call_{A3} = 225.67$		$Call_{A5} = 206$
		$Call_{A2} = 106.13$		$Call_{A4} = 89.06$	
	$Call_{A1} = 49.33$		$Call_{A3} = 38.37$		$Call_{A5} = 14$
$Call_{A0} = 22.71$		$Call_{A2} = 16.48$		$Call_{A4} = 5.70$	
	$Call_{A1} = 7.06$		$Call_{A3} = 2.32$		$Call_{A5} = 0$
		$Call_{A2} = 0.94$		$Call_{A4} = 0$	
			$Call_{A3} = 0$		$Call_{A5} = 0$
				$Call_{A4} = 0$	
					$Call_{A5} = 0$

Table 7: American put option prices Put_{At} at time 0, 1, 2, 3, 4, 5 using CRR model

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					$Put_{E5} = 0$
				$Put_{E4} = 0$	
			$Put_{A3} = 0$		$Put_{E5} = 0$
		$Put_{A2} = 1.75$		$Put_{E4} = 0$	
	$Put_{A1} = 7.40$		$Put_{A3} = 4.70$		$Put_{E5} = 0$
$Put_{A0} = 18$		$Put_{A2} = 18$		$Put_{E4} = 18$	
	$Put_{A1} = 34$		$Put_{A3} = 34$		$Put_{E5} = 34$
		$Put_{A2} = 42$		$Put_{E4} = 42$	
			$Put_{A3} = 46$		$Put_{E5} = 46$
				$Put_{E4} = 48$	
					$Put_{E5} = 49$

327

The best exercise time is $t = 0$ for this American put option.

From this practical case for option pricing, we can find that there is a minor difference between the B-S model and the CRR model in European option pricing. This is because the CRR model regards the option price as a discrete stochastic process, while the B-S model regards the option price as a continuous process. That is why the B-S model needs to use Ito's formula to derive the analytical solutions of the option price, and the CRR binomial model can only give numerical solutions. When the number of the time interval is limited to infinity, that is, the discrete process approaches a continuous process, the solution of the B-S model and CRR binomial model should be identical. And for American option pricing, the early exercise influences the option price, so we can only use the CRR binomial model to calculate the option prices.

5. Conclusion and Prospect

In this research project, the significance and role of options in the financial industry are mentioned first, which is the reason why this research aims at option and option pricing. Then, several past research on option pricing is introduced and compared. The Bachelier's formula for option pricing is the earliest model on pricing options, but the impossibility of stock price being negative and ignorance of the time value remain to be some defects. The Black-Scholes model solves the defects in the Bachelier formula, however, it is not used for American option pricing as early exercise cannot be considered in the Black-Scholes model, and the stock price is regarded as a continuous process. For CRR binomial model, the American option price can be calculated as CRR binomial model assumes the stock price to be a stochastic process, which is a discrete backward process and early exercise can be considered during the backward process, and the main problem is that CRR binomial model cannot guarantee both accuracy and computing speed for too many branches in this binomial tree. Investors can choose appropriate methods to obtain the option prices based on their situations, such as the performances of their computing devices and the frequency of changes in the price of the investment targets. If the option price changes frequently, it can be regarded as a continuous process and thus the computation should not be pressured. And if the price of option changes regularly, it is a discrete process and CRR binomial is able to be used to calculate the option price for the compatibility of American options.

From a macro perspective, take the Chinese financial market as an example, the Shanghai Securities Index has risen from 1300 in 2000 to the present 3200 points, while the scale of Chinese public mutual funds has grown from 8.4 trillion yuan at the end of 2015 to 2.5 trillion yuan at the end of 2021. Many of these mutual funds invest in stocks, bonds, options, and other financial products and financial derivatives using quantitative methods like automatically option pricing. This supports that quantitative investing, including option pricing, will play a much more important role in the financial industry, and other research focused on pricing options will also be valuable in the future.

References

[1] Bachelier, L. (1900). Théorie de la spéculation. In Annales scientifiques de l'École normale supérieure (Vol. 17, pp. 21-86).

[2] Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637–654.

[3] Cox, J. C., Ross, S. A., & Rubinstein, M. (1979). Option pricing: A simplified approach. Journal of Financial Economics, 7(3), 229-263.

[4] Hull, J. C. (2003). Options futures and other derivatives. Pearson Education India.