

The Effect of Constraint on Portfolio Construction Using the Index Model

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Abstract. This study examines the performance and characteristics of the Single Index Model under different constraint regimes. Utilizing the S&P 500 index and 21 of its constituent stocks across five sectors, we construct and compare optimal portfolios and efficient frontiers for an unconstrained scenario and a "box" constraint scenario where the absolute weight of any asset is limited to 100%. The analysis confirms the theoretical prediction that the unconstrained model generates an efficient frontier that dominates the constrained one, offering a higher maximum Sharpe ratio (1.537 vs. 1.519) and a lower minimum variance (11.48% vs. 12.03%). The unconstrained model achieves this through more concentrated allocations, including significant short positions and leverage, which are restricted under the box constraint. However, the constraint effectively eliminates impractical, high-risk portfolios that rely on excessive short selling, resulting in a more parabolic and realistic efficient frontier. The findings validate core portfolio theories, demonstrating that while constraints limit optimization potential and reduce efficiency, they also curb firm-specific risk and produce more implementable portfolio strategies.

Keywords: Portfolio Construction, Index Model, "box" Constraint

1. Introduction

The index model developed by William Sharpe in 1963 remains one of the most widely utilized financial portfolio construction models until this day, despite the subsequent development of more sophisticated multi-factor frameworks. More advanced and accurate models such as the Three Factor Fama and French Model, and the Five Factor Fama and French Model were developed and ameliorated the explanatory power by incorporating factors such as market risk, size, value, profitability, and investment [1]. Empirical research has consistently validated the advancements of these multi-factor models. For example, Chiah, Chai, and Zhong verified the superiority of the Five Factor Fama French Model over the Three Factor model by demonstrating its ability to explain more pricing anomalies [2]. In response to the growing number of models, Fama and French also provided further guidance on using the maximum squared Sharpe ratio as a way of measuring model efficacy [3]. Research by Ye, Wang, and Raza has further explored the maximum attainable Sharpe ratio under different constraints [4]. The utility of the Index Model for optimization purposes has also been a focus of study, as examined by Mistry and Khatwani [5].

Despite these advancements, the Single Index Model retains significant relevance for educational and practical applications. This study leverages its streamlined framework to investigate how portfolio optimization behaves under different constraints. The portfolios to be examined will be constructed of the S&P 500 index and 21 constituent stocks, spanning five major sectors. Comparisons will be made on the results produced by the index model under the unconstrained ("Free") and constrained ("Box" constraint, where $|w_i| \leq 1$) scenarios, to simulate a realistic case.

The goal of this analysis is to empirically verify fundamental portfolio theories, such as the trade-off between risk and return and the impact of constraints on the efficient frontier; and to provide a hands-on, practical perspective that complements textbook treatments for investors and future researchers. By bridging theoretical models with real-world applications, this study aims to enhance understanding of how constraints shape investment outcomes and portfolio efficiency.

2. Data and model

2.1. Data

2.1.1. Stock selection

The portfolios constructed in this paper will be based on the S&P 500 Index and 21 constituent stocks (table 1) from five different sectors, including Financial, Industrial, Technology, and Basic Materials. To mimic a realistic investor portfolio, this study selected industry leaders among the most popular stocks.

Table 1. Sample stocks

JPM	MS	USB	WFC	EMR	HON
CAT	DE	MMM	FDX	UNP	UPS
AAPL	ACN	IBM	AMD	INTC	NVDA
QCOM	TXN	LIN			

2.1.2. Data processing

This study used 20 years of daily data (September 2004 to September 2024) obtained from OEX Bloomberg for each of the selected securities and performed a series of processing to extract useful information for portfolio construction.

As the Central Limit Theorem suggests, data of aggregated financial returns depict normality due to aggregational Gaussianity [6]. For verification, the distribution graphs of daily returns and monthly returns have been plotted, and the results are consistent with the theory. Thus, for the sake of accuracy, assumptions of the model used, and simplicity of calculations, the data will be aggregated into monthly data. This process is done through Excel by taking out the securities' prices at the end of each month, utilizing the filter function.

The next step is to find the relevant values required to perform calculations of the index model, given the monthly prices of the selected securities, which include the annualized average excess return of the S&P 500 index $E(R_M)$, the S&P 500 Index standard deviation σ_M , Beta (β_i), Annualized Alpha (α_i), and Annualized firm-specific risk $\sigma^2(e_i)$.

It is worth noting that the equation for the Index model is based on excess return; thus, the data must be adjusted accordingly. To achieve this, this study relies on estimating a set of values of the

notional risk-free rate (NRFR) based on the Fed Funds Effective Rate (FEDL01) using the following function, with the first period of FEDL01 set to be 1000 for simplicity of calculations. Where the 100 is used to convert the value of FEDL01 to a percentage, and 261 is the number of trading days within a year.

$$NRFR_{n+1} = NRFR_n \star (1 + FEDL01)/100/261 \quad (1)$$

The monthly return for each security was calculated first by dividing the security's price in the current month by its price in the previous month, minus one. The monthly excess return for each of the securities is then computed as the monthly return minus NRFR. Annualized Average Excess Return of the S&P 500 Index $E(R_M)$

The computed Annualized Excess Return of the S&P 500 Index is 9.53% based on the following formula. Where the 12 is used to annualize the monthly returns.

$$E(R_M) = \frac{\sum_{i=1}^n R_i}{n} \star 12 \quad (2)$$

σ_M S&P 500 Index standard deviation

The computed standard deviation of the S&P 500 Index excess return is 14.95%, with the Excel function `stdev(data of excess returns) * sqrt(12)`, where the `sqrt(12)` is used to annualize the monthly standard deviations.

Beta (β_i). The beta of each individual security is defined to be the sensitivity to changes in the market index, and is therefore computed as the slope of the linear regression of the index model and the security.

Alpha (α_i). The annualized alpha of each individual security is defined as the intercept of the linear regression of the index model and the security multiplied by 12.

$\sigma(e_i)$ Annualized firm-specific risk / Residual Standard Deviation

The firm-specific risk is defined as the standard deviation squared of the residual values of the linear regression; thus, to compute this value, the firm-specific surprises or the residual values must be computed first through the following formula.

$$e_i(t) = R_i(t) - \alpha_i - \beta_i \bullet R_M(t) \quad (3)$$

Following this process, the residual standard deviations for each individual security may be computed using the formula `StDev(data of residual values) * sqrt(12)`. It is worth noting that the annualized average of the firm-specific surprises residual values should be equal to 0, verifying the assumptions of the index model.

2.2. Index model

2.2.1. Model

The Single Index model, also known as the Diagonal model, is one of the simplest and yet useful models. The model's simplicity comes from its essential assumption that a single underlying factor, often a market index, may be used to capture the return of all securities, which also granted it the name index model. The return function of a security will then be composed of "random factors and this single outside element [7]." The function of return used in this paper will be formatted as follows.

$$R_i(t) = \alpha_i + \beta_i \bullet R_M(t) + e_i(t) \quad (4)$$

In which, $e_i(t)$ is the firm-specific surprise or residuals with an expected value of zero. The α_i (alpha) is the expected return of a security when the market excess return is zero. $R_M(t)$ resembles the market excess return of the market Index (M), and β_i (Beta) measures the security's sensitivity to changes in the market Index.

It then follows that the expected value of return of the security can be expressed as follows, where the expected value of e_i is zero.

$$E(R_i) = \alpha_i + \beta_i \bullet E(R_M) \quad (5)$$

The total risk of the security will be described as follows.

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i) \quad (6)$$

Where $\beta_i^2 \sigma_M^2$ represents the risk associated with the market Index, also known as systematic risk, and $\sigma^2(e_i)$ stands for the risk of firm-specific risk or random factor.

Relatively, the return of a portfolio constructed using the index model will simply be the weighted average of its constituent securities.

$$E(R_p) = \alpha_p + \beta_p \bullet E(R_M) \quad (7)$$

$\alpha_p = \sum (w_i \bullet \alpha_i)$; $\beta_p = \sum (w_i \bullet \beta_i)$. Similarly, the risk of the portfolio will be represented by the following equation.

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \sigma^2(e_p) \quad (8)$$

Where β_p (portfolio beta) is the weighted average of each individual security's beta, and $\sigma^2(e_p)$ is the weighted average of the firm-specific risks or residual variance. $\beta_p = \sum (w_i \bullet \beta_i)$, $\sigma^2(e_p) = \sum [W_i^2 \bullet \sigma^2(e_i)]$.

2.2.2. Research hypothesis

Based on theoretical reasoning, the “Box” constraint should limit some potential portfolios available to the investors in the unconstrained scenarios. Portfolios subjected to constraints produce frontiers that are less efficient compared to the classic unconstrained portfolios, as constraints limit the optimization potential [8]. Similarly, a tracking error constraint on a portfolio creates a tracking error frontier, and the portfolios within such a frontier are less efficient compared to unconstrained portfolios [9]. The “Box” constraint also curbs the risk of a portfolio by limiting the weight that investors may allocate to a single stock, which reduces the firm-specific risk of the portfolio. Therefore, the hypothesis for this study is that the efficient frontier generated using the index model under the unconstrained scenario should dominate the efficient frontier of the “box” constraint. Hence, the unconstrained scenario enables more potential portfolios and covers a greater area compared to the “box” constraint.

3. Results

3.1. Unconstrained scenario

The Unconstrained Portfolio/ Free Problem provides no constraints on the weights of securities, and it is assumed that investors may use every technique that exists to construct their portfolios (table 2-3, and figure 1).

Under the unconstrained scenario, the minimal variance portfolio has a return of 2.57% and a standard deviation of 11.48%, with a Sharpe ratio of 0.224. The optimal portfolio, a portfolio with the maximum Sharpe ratio, has a return of 121.12 and % standard deviation of 78.78%, with a Sharpe ratio of 1.537.

Table 2. Weights for minimum variance portfolio (unconstrained)

Stock	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX
Weight	181.87%	-6.44%	-10.50%	0.11%	-3.84%	-13.25%	-13.39%	-12.07%	-6.42%	2.77%	-7.49%
Stock	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
Weight	-3.34%	1.24%	-4.42%	-2.38%	8.53%	-6.68%	-0.60%	-6.20%	-3.68%	-5.18%	11.34%

Table 3. Weights for optimal portfolio (unconstrained)

Stock	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX
Weight	-510.4%	20.88%	-31.11%	-30.51%	-20.17%	-34.88%	18.71%	32.00%	57.22%	-39.34%	-36.05%
Stock	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
Weight	120.04%	-40.20%	159.31%	112.82%	12.68%	14.39%	-40.98%	88.67%	10.02%	65.81%	171.12%

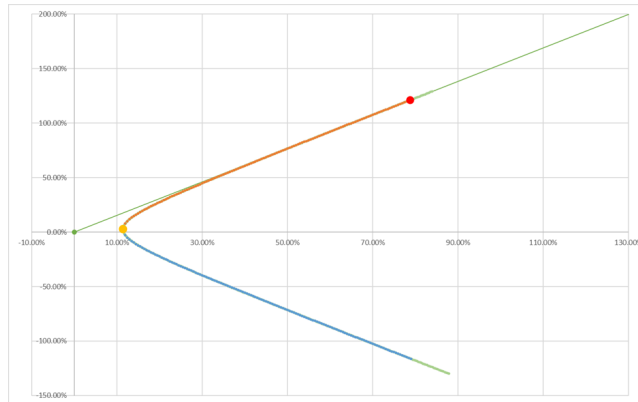


Figure 1. Index model (unconstrained)

3.2. “Box” constraint $|w_i| \leq 1$

The “box” constraint used for comparison in this paper limits the weight investors may allocate to each of the securities within the portfolio. Particularly in this scenario, it is limited to $-1 \leq w_i \leq 1$ or $|w_i| \leq 1$. Meaning that the absolute weight of any individual security may not surpass 100% of the portfolio.

The results is shown in table 4-5 and figure 2. Under the “box” constraint, where the absolute value of the weight of a single security must not exceed 1, the minimal variance portfolio had a

return of 4.72%, a standard deviation of 12.03%, and a Sharpe ratio of 0.392. The optimal portfolio has a return of 66.23%, a standard deviation of 43.61%, and a Sharpe ratio of 1.519.

Table 4. Weights for minimum variance portfolio (box constraint)

Stock	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX
Weight	100.00%	-3.14%	-8.91%	4.06%	-1.48%	-8.43%	-5.37%	-10.19%	-3.68%	9.04%	-4.88%
Stock	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
Weight	1.71%	6.18%	-2.62%	4.15%	14.78%	-6.67%	2.24%	-5.83%	-1.91%	-0.41%	21.37%

Table 5. Weights for optimal portfolio (box constraint)

Stock	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX
Weight	-100.0%	4.59%	-24.12%	-22.13%	-15.92%	-31.64%	-5.79%	10.17%	25.72%	-28.99%	-26.90%
Stock	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
Weight	59.34%	-28.41%	85.00%	53.70%	2.76%	5.12%	-26.96%	46.48%	1.68%	28.47%	87.82%

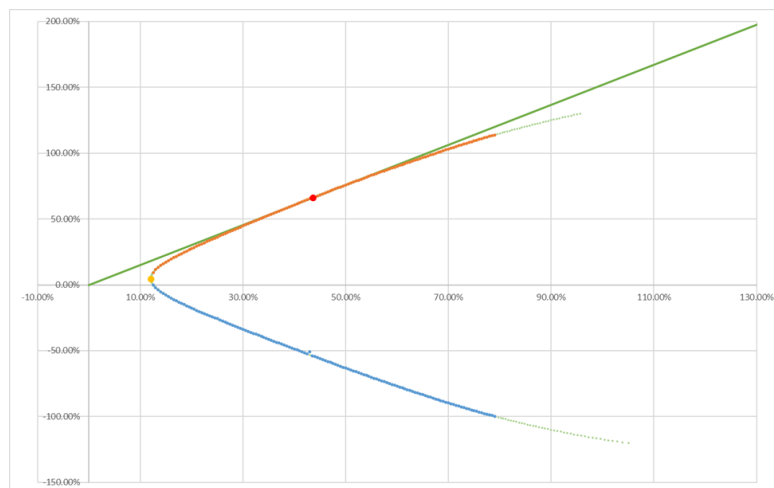


Figure 2. Index model (“box” constraint)

3.3. Comparison

3.3.1. Minimal variance

It is observed that the weights for the minimal variance portfolios in both scenarios are well diversified, with no single security’s weight exceeding 25%, except for the S&P index. This result is consistent with the assumptions of the Index Model, where diversification can eliminate firm-specific risk; thus, the minimal variance portfolio should be more diversified [10]. The major difference is that the unconstrained scenario allocated 181.87% of the weight to the S&P index, whereas the “box” constraint limited the weight of the S&P index to 100%.

It is also worth noting that under the unconstrained scenario, the index model was able to construct a portfolio with a standard deviation of 11.48%, whereas under the “box” constraint, the minimal variance portfolio has a standard deviation of 12.03%. This complies with the research hypothesis that an unconstrained scenario enables more combinations, and therefore, its efficient

frontier contains a greater area than the “box” constraint. The “box” constraint’s minimum variance portfolio has a higher Sharpe ratio compared to the unconstrained scenario, making it more efficient.

3.3.2. Optimal portfolio

The effect of the “Box” Constraint on the Index Model is more conspicuous when observing the optimal portfolios of the two scenarios. The unconstrained scenario has more concentrated weights with multiple securities’ absolute weights above 100%, meaning that more weight has been allocated to the few stocks that have extraordinary returns. In contrast, the absolute weights allocated to each security in the “box” constraint never exceed 100%. To illustrate the difference, five of the securities with the greatest absolute weight are listed in Table 6.

Table 6. Greatest absolute weights

	SPX	LIN	AAPL	UNP	ACN	Return	StDev	Sharpe
“Free”	-510.42%	171.12%	159.31%	120.04%	112.82%	121.12%	78.78%	1.537
“Box”	-100.00%	87.82%	85.00%	59.34%	53.70%	66.23%	43.61%	1.519

It is worth noting that LIN, AAPL, and UNP are stocks with the highest return-to-standard-deviation ratios within their own industry. Each was respectively 0.7623, 1.0445, and 0.7296. This combination guarantees a high return while pertaining diversification into various industries, providing a valid reason for the Index Model to prefer a more concentrated weight in these securities.

However, ACN’s high concentration was due to another factor, as it had a return-to-standard-deviation ratio of 0.6913, only the third-highest in the Technology sector. A potential explanation for why ACN was among the most concentrated stocks is due to its second-highest correlation of 71.3% with the S&P index, slightly lower than EMR’s 76.04%. A higher concentration of ACN would be able to hedge against the large negative weight allocated to the index. Though why ACN was preferred over EMR is not completely clear, this can be partly explained by its higher return-to-standard-deviation ratio compared to EMR’s 0.4138, making it a more attractive choice when searching for a combination with the highest Sharpe Ratio.

A cross-comparison reveals that the optimal portfolio for the “box” constraint generates a slightly lower Sharpe ratio than the unconstrained scenario. Meaning that it eliminated potential combinations with higher concentration and also higher risk-adjusted returns. Such an observation is consistent with the research hypothesis. Although the Sharpe ratio remained close in both scenarios, the return and standard deviation of the unconstrained scenario were around 1.8 times the values of the “box” constraint. This implies the Index Model under the unconstrained scenario tends to construct high-risk and high-return portfolios. This phenomenon is because the unconstrained scenario allows unlimited short selling and thus, theoretically, is capable of constructing portfolios with returns as high as desired at the cost of increasingly high risk [11].

The “box” constraint’s less concentrated portfolio limits the firm-specific risk a portfolio may take through diversification. The standard deviation, $\sigma(e_p)$, of the unconstrained scenario and “box” constraint scenarios are respectively 71.68% and 38.19%. The risk and return trade-off is hence also limited, and the “box” constraint eliminates some portfolios with high risk and return. It is worth noting that under the box constraint, the Index model is unable to generate a portfolio with a return less than -120%.

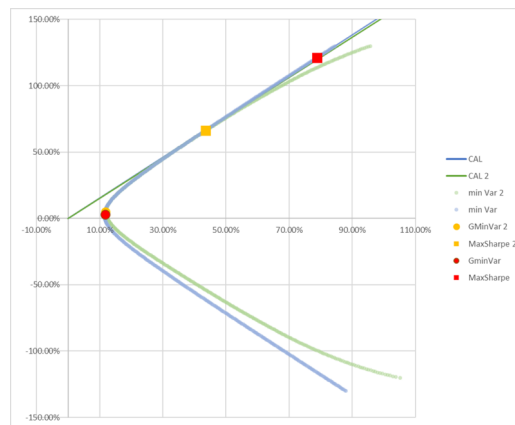


Figure 3. Index model unconstrained vs “box” constraint

The graphical demonstration of the observations mentioned earlier also reveals that the unconstrained scenario has a slightly steeper CAL, which is consistent with the higher Max Sharpe ratio (figure 3). The graph of unconstrained efficient frontiers is almost linear toward higher standard deviations; in contrast, the efficient frontier for the “box” constraint is parabolic. A result of restricting unlimited short selling mentioned earlier. As the hypothesis predicted, the unconstrained frontiers cover a greater area; however, the differences occur mostly on the inefficient frontier and portfolios with high standard deviations. Suggesting that the constraint effectively excludes portfolios that take on unnecessary risk and rely on excessive short-selling, which are impractical in reality. Rational investors will not take on portfolios that use excessive short selling.

4. Conclusion

This study applied the Single Index Model to a portfolio of the S&P 500 index and 21 stocks to analyze the impact of constraints on portfolio construction. The comparison between an unconstrained "free" scenario and a "box" constrained scenario yielded clear and theoretically consistent results.

The analysis is consistent with the hypothesis that the unconstrained efficient frontier dominates the constrained frontier, covering a greater area in the risk-return space. This is evidenced by the unconstrained model's ability to achieve a lower global minimum variance and a higher maximum Sharpe ratio, facilitated by its freedom to employ significant short selling to achieve any level of return required. In contrast, the box constraint limited concentration and leverage, resulting in a more diversified optimal portfolio with lower absolute risk and return.

While the unconstrained model is theoretically superior, its optimal portfolio requires extreme short positions and leverage that are often infeasible for real-world investors. The box constraint, though slightly less efficient, produced a more realistic and implementable set of portfolios by eliminating these impractical, high-risk combinations. This effectively demonstrates a key trade-off in portfolio management: constraints necessarily reduce theoretical efficiency but simultaneously enhance practical applicability by enforcing diversification and limiting exposure to excessive firm-specific risk.

Ultimately, this exercise successfully verifies fundamental portfolio theories and provides a practical supplement to textbook finance, illustrating how the Index Model behaves under realistic limitations and highlighting the critical balance between theoretical optimization and practical implementation.

References

- [1] Fama, E. F., & French, K. R. (2015). A five-factor asset pricing model. *Journal of Financial Economics*, 116(1), 1–22.
- [2] Chiah, M., Chai, D., & Zhong, A. (2015). A better model? an empirical investigation of the Fama-French five-factor model in Australia. *SSRN Electronic Journal*.
- [3] Fama, E. F., & French, K. R. (2018). Choosing factors. *Journal of Financial Economics*, 128(2), 234–252.
- [4] Ye, J., Wang, Y., & Raza, M. W. (2022). The Sharpe ratio's upper bound of the portfolios in the presence of a benchmark: Application to the US financial market. *Journal of Mathematical Finance*, 12(03), 566–581.
- [5] Mistry, J., & Khatwani, R. A. (2023). Examining the superiority of the Sharpe single-index model of portfolio selection: A study of the Indian mid-cap sector. *Humanities and Social Sciences Communications*, 10(1).
- [6] Antypas, A., Koundouri, P., & Kourrogenis, N. (2013). Aggregational Gaussianity and barely infinite variance in financial returns. *Journal of Empirical Finance*, 20, 102–108.
- [7] Sharpe, W. F. (1963). A simplified model for portfolio analysis. *Management Science*, 9(2), 277–293.
- [8] Abate, G., Bonafini, T., & Ferrari, P. (2022). Portfolio constraints: An empirical analysis. *International Journal of Financial Studies*, 10(1), 9.
- [9] Jorion, P. (2003). Portfolio optimization with tracking-error constraints. *Financial Analysts Journal*, 59(5), 70–82.
- [10] Napon, J. C. (2023). Brief Review on Asset Selection and portfolio construction: Diversification, risk and return. *SSRN Electronic Journal*.
- [11] Alemanni, B., Maggi, M., & Uberti, P. (2021). Unleveraged portfolios and Pure Allocation Return. *Journal of Risk and Financial Management*, 14(11), 550.