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Validity Testing of Classical Asset Allocation Models: An Empirical Study

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Abstract: This paper uses a 20 years of daily total return data for the S&P 500 Index (ticker symbol "SPX") and 10 stocks, make all the necessary calculations to plot a portfolio area that gather an efficient boundary, a minimum risk or variance boundary, and a minimum return boundary together for a given set of constraints. Analyze all the outcomes in order to compare the various restrictions for each optimization issue (MM and IM), as well as the two solutions to the same optimization problem. The Excel solver was the primary tool utilized during calculation to resolve optimization issues for each point on the minimal risk or variance border. Also, this paper use an Excel solution table to calculate a large number of multipoints on any desired boundary. Through calculation and research, we found that, Markowitz model makes full use of covariance matrix to generate excellent portfolio. However, the results are numerically unstable. At the same time, the hypothesis of normality, stationarity and mean square error are verified. The exponential model simplifies the Markowitz model and produces more robust results. However, it introduces additional assumptions about the independence, normality, and homoscedasticity of the regression residuals, which are also invalid. The reduction (CDaR) model is hypothesis-free. The numerical stability can be obtained by transforming the nonlinear optimization problem into a linear programming problem.

Keywords: Sharpe ratio, minimal risk, efficient frontier, minimal return frontier, Markowitz model

1. Introduction

The basic idea of this theory that Markowitz find out is the time when he was reading John Burr Williams, "The Theory of Investment Value" [1]. Markowitz addressed these limitations by introducing a formula that allowed investors to balance risk tolerance and reward expectations, leading to the creation of an optimal portfolio. The Modern Portfolio Theory (MPT) he developed was grounded in two key principless [2]:

(1) The primary goal for every investor is to maximize returns while managing risk.

(2) Diversification across unrelated securities can effectively reduce portfolio risk.

The Markowitz model has been widely employed in portfolio optimization processes and has been shown to be successful in real-world applications. Markowitz portfolio theory may be applied to consistently outperform the market in the Chinese stock market. [3]. Although the current securities

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market in China is not standardized, the system risk still occupies a large proportion of investment risks, the Markowitz model can be used to obtain a better performance than the average securities market portfolio.

An Index Model is a statistical model used to analyze security returns, differing from economic models that are based on market equilibrium principles. The Single Index Model (SIM) delineates two distinct factors contributing to the uncertainty of a security's return[4]:

(1) Systematic (macroeconomic) uncertainty, which is presumed to be effectively represented by a singular index reflecting stock returns.

(2) Unique (microeconomic) uncertainty, represented by a random component specific to the individual security.

This research details the operational and financial performance of 11 organizations. Based on raw data, it analyzes the annualized data and correlation coefficient. Therefore, investors may select one restriction and discover the optimum portfolio for their circumstances and preferred level of risk and return. The daily data are combined in this study to create monthly observations, from which we derive the necessary optimization inputs for the complete Markowitz Model ("MM") and the Index Model ("IM"). We identify the areas of acceptable portfolios (efficient frontier, minimum risk portfolio, optimum portfolio, and minimal return portfolios frontier) for the extra limitations using these optimization inputs for MM and IM. [5].

First of all, familiarize the market and select the required representative company data. Secondly, optimizing the data for problem solving (from a daily to monthly frequency summary), computing each new set of optimization constraints as well as all the inputs required for the MM and IM optimization problems. Then, two key boundary points (maximum Sharpe ratio and minimum risk) are calculated, as well as two boundaries: the efficiency boundary and the minimum return boundary. To compare the various constraints for each optimization problem (MM and IM), as well as two distinct solutions to the same optimization issue, is the final step in the analysis of all the findings.

2. Data Preprocessing

The data preprocessing part mainly focuses on generating the inputs for portfolio optimization models and check if the assumptions of the models are reasonable.

2.1. Inputs for Markowitz Model

Based on monthly excess returns, the author computes the annualized average, annualized standard deviation, and correlation matrix. The return Rt of a portfolio at time t can be defined to be the total value Tt of the portfolio divided by the total value at an earlier time t−1, i.e [6]. The results are shown in table 1. There is also a visualization of correlation matrix in figure 1. These are inputs for Markowitz model.

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Table 1: Inputs for Markowitz model.

Panel A: Annualized average and annualized standard deviation of monthly excess return											
	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
Average	0.338	0.340	0.156	0.119	0.090	0.154	0.098	0.130	0.225	0.174	0.075
Std	0.414	0.345	0.415	0.290	0.162	0.211	0.214	0.267	0.307	0.239	0.149
Panel B: Correlation matrix of monthly excess return											
	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
AMZN	1.000										
AAPL	0.377	1.000									
CTXS	0.217	0.332	1.000								
JPM	0.252	0.244	0.324	1.000							
BRK/A	0.118	0.173	0.181	0.452	1.000						
PGR	0.200	0.240	0.271	0.393	0.264	1.000					
UPS	0.296	0.231	0.264	0.361	0.404	0.392	1.000				
FDX	0.280	0.330	0.331	0.440	0.385	0.365	0.675	1.000			
JBHT	0.308	0.268	0.290	0.442	0.239	0.280	0.459	0.537	1.000		
LSTR	0.256	0.287	0.252	0.375	0.234	0.289	0.441	0.482	0.590	1.000	
SPX	0.485	0.542	0.437	0.697	0.523	0.502	0.575	0.614	0.521	0.495	1.000

Readers can see from panel A of table 1 that in general, stocks with higher average excess return tend to have higher standard deviation. And from panel B of table 1 and figure 1 readers can see that all the stocks are highly correlated with the market and the four industrial stocks are highly correlated with each others.

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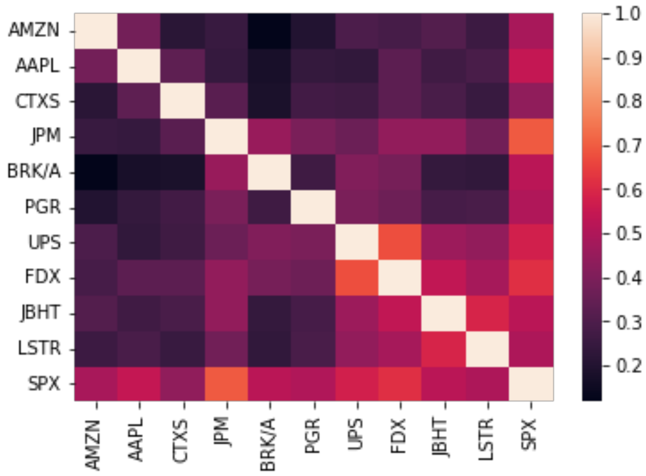

Figure 1: Heatmap of correlation matrix of monthly excess return.

Photo credit: Original

2.2. Inputs for Index Model

The author runs linear regressions with the monthly excess returns of S&P 500 as the explanatory variable, and monthly excess return of each of the ten stocks, one at a time, as dependent variable. The results, including beta, annualized alpha and annualized residual standard deviation are shown in table 2.

Table 2: Inputs for index model.

	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR
Beta	1.3513	1.2569	1.2206	1.3608	0.5721	0.7120	0.8296	1.1036	1.0763	0.7975
Annualized Apla	0.2360	0.2454	0.0643	0.0160	0.0469	0.1004	0.0359	0.0463	0.1441	0.1137
Annualized residual std	0.362	0.290	0.373	0.208	0.138	0.182	0.175	0.211	0.262	0.208

2.3. Model Diagnostics

First of all, the research tests for normality of monthly excess return in three ways: (i) Q-Q plots [7]; (ii) Skewness and kurtosis [8]; (iii) Formal statistical tests including Kolmogorov-Smirnov test [9], Cramervon Mises test [10], Shapiro-Wilks test, and Jarque-Bera test [11]. The results are shown in figure 2, table 3 and table 4, respectively.

Table 3: Skewness and kurtosis of monthly excess return.

	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
Skewness	0.574	-0.249	0.530	-0.282	0.140	-0.248	0.041	0.040	-0.010	-0.022	-0.570
Kurtosis	3.558	0.855	3.842	1.449	0.647	0.145	2.887	1.951	3.143	0.006	1.351

Note: Kurtosis are computed with 3.0 subtracted to give 0.0 for normal distribution.

From the Q-Q plots readers can see that the points deviate from the line in the top-right and bottom-left corner. This form of deviation implies heavy tail of monthly excess return. From the sample skewness and kurtosis readers can see those monthly excess returns of almost all the stocks have heavy tail. And only monthly excess return of BRK/A, PGR and LSTR seems to be normal. As for the p-values of formal test, Shapiro-Wilks test and Jarque-Bera test are much more sensitive to outliers, and only the monthly excess return of PGR and LSTR are not rejected by any of the four tests to be normal. Nonnormality has one negative impact on Markowitz and index model: Both the two models presume that investors only care about expected return and standard deviation of portfolios. This may be with normal distribution of excess return true since expected return and standard deviation are enough to determine a normal distribution. However, with nonnormal distribution, investors may take higher moments into account.

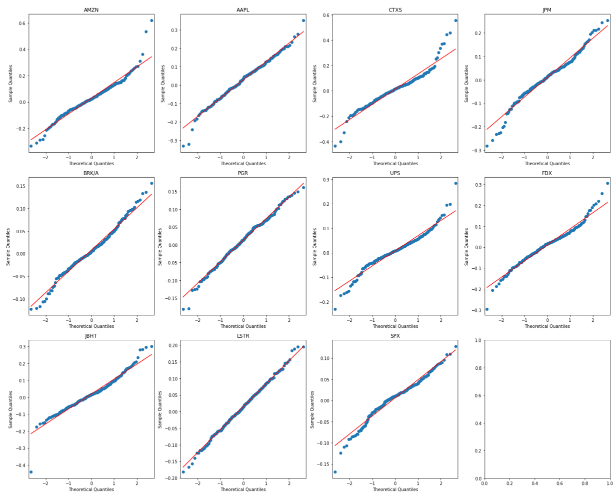

Figure 2: Q-Q plots of monthly excess returns.

Photo credit: Original

The author then tests the autocorrelation, nonstationarity and heteroskedasticity of monthly excess return. It turns out that the time series of monthly excess return does not exhibit property of autocorrelation. This is consistent with the assumption of the models. However, monthly excess return of UPS exhibits strong nonstationarity with a p-value of augmented Dickey-Fuller test of 0.198. This is extremely weird that the return itself has a unit root.

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Table 4: P-values of formal normality tests for monthly excess return.

	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
Kolmogorov-Simirnov	0.296	0.493	0.073	0.221	0.474	0.615	0.105	0.275	0.189	1.000	0.033
Cramer-von Mises	0.214	0.792	0.050	0.174	0.303	0.747	0.070	0.201	0.179	1.000	0.056
Shapiro-Wilks	0.000 (***)	0.104	0.000 (***)	0.000 (***)	0.025 (*)	0.285	0.000 (***)	0.000 (***)	0.000 (***)	0.973	0.000
Jarque-Bera	0.000 (***)	0.007 (**)	0.000 (***)	0.000 (***)	0.083	0.262	0.000 (***)	0.000 (***)	0.000 (***)	0.990	0.000 (***)

And it is consistent with the historical price of UPS but inconsistent with the models. Moreover, heteroskedasticity is common among monthly excess returns, as shown in table 5, where the p-values of autoregressive conditional heteroskedasticity tests are shown. This is another violation of the assumptions, which invalidates the use of standard deviation. In fact, heavy tail can be derived from heteroskedasticity.

Table 5: P-values of ARCH test for monthly excess return.

	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
ARCH	0.555	0.013 (*)	0.000 (***)	0.000 (***)	0.001 (**)	0.003 (**)	0.055	0.147	0.008 (**)	0.110	0.000 (*)

The index model enlists has additional assumptions: (i) Residuals are uncorrelated with the market; (ii) Residuals of different stocks are uncorrelated; (iii) Residuals are normally distributed; (iv) Residuals do not exhibit autocorrelation, nonstationarity and heteroskedasticity. For assumption (i), the author computes the correlation coefficients between the residuals and the market, and it turns out that the macroeconomic factor and firm-specific factors are indeed uncorrelated. For assumption (ii), the author computes the correlation matrix of residuals of different stocks, the results are shown in table 6, and there is also a visualization in figure 3.

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Table 6: Correlation matrix of residuals.

	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR
AMZN	1.000									
AAPL	0.156	1.000								
CTXS	0.006	0.126	1.000							
JPM	-0.137	-0220	0.031	1.000						
BRK/A	-0.182	-0.153	-0.062	0.144	1.000					
PGR	-0.058	-0.045	0.067	0.070	0.001	1.000				
UPS	0.025	-0.116	0.018	-0.0698	0.148	0.146	1.000			
FDX	-0.025	-0.005	0.088	0.021	0.095	0.082	0.498	1.000		
JBHT	0.074	-0.020	0.081	0.128	-0.046	0.025	0.228	0.322	1.000	
LSTR	0.022	0.026	0.046	0.048	-0.034	0.054	0.220	0.260	0.447	1.000

The correlation coefficients between the firm-specific factors of different firms are low, expect for those industrial firms. This will lead to inferior portfolios because index model presumes that firms are related to each others only via the macroeconomic factor. For assumption (iii), the author uses the same method as when testing the normality of monthly excess returns. The results are similar: The Q-Q plots imply heavy tail, only monthly excess return of BRK/A, PGR and LSTR seems to be normal when skewness and kurtosis are considered, and only the monthly excess return of PGR is not rejected by any of the four tests to be normal. Nonnormality of residuals can exert negative impact on the estimation of beta, alpha, and residual standard deviation. For assumption (iv), the time series of residuals do not exhibit autocorrelation and nonstationarity, but heteroskedasticity is common among residuals. Residual heteroskedasticity invalidates the use of residual standard deviation.

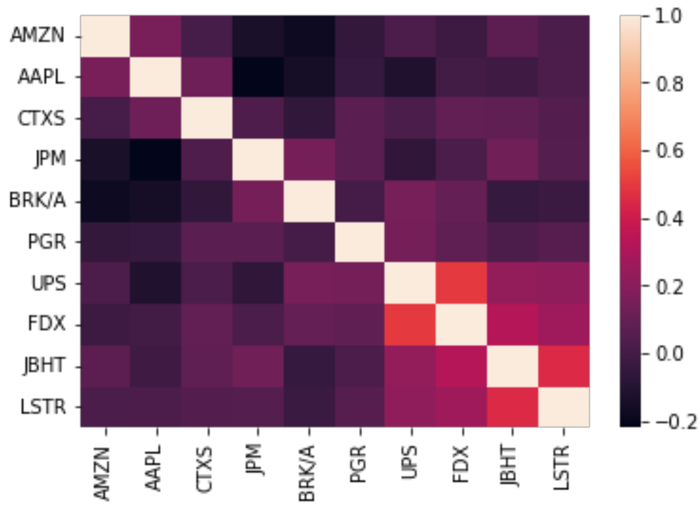

Figure 3: Heatmap of correlation matrix of residuals.

Photo credit: Original

3. Portfolio Optimization

Based on the inputs, the author uses sequential least square programming algorithm implemented in Python to generate critical portfolios and frontiers for Markowitz and index model. The results are not necessarily the same as general reduced gradient implemented in Excel, but the differences are negligible.

3.1. Markowitz Model

Table 7 shows the weights of the maximal Sharpe portfolios and the minimal variance portfolios under the following five different constraints:

(1) $\sum_{i=1}^{11}|\omega_i| \leq 2$;

(2) $|\omega_i| \leq 1$, for $\forall i$;

(3) No constraints;

(4) $\omega_i \geq 0$, for $\forall i$;

(5) $\omega_1 = 0$.

Table 7: Weights in critical portfolios with Markowitz model.

Panel A: Maximal Sharpe ratio portfolios											
Constraint	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
1	0.164	0.300	0.000	0.000	0.413	0.330	0.000	-0.014	0.125	0.167	-0.486
2	0.223	0.397	-0.012	-0.005	0.625	0.460	-0.031	-0.106	0.209	0.239	-1.000
3	0.370	0.654	0.004	0.173	0.915	0.682	0.012	-0.086	0.310	0.340	-2.375
4	0.130	0.252	0.000	0.000	0.192	0.227	0.000	0.000	0.088	0.110	0.000
5	0.147	0.267	-0.034	-0.156	0.363	0.320	-0.121	-0.132	0.179	0.168	0.000
Panel B: Minimal variance portfolios											
Constraint	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
1	-0.022	-0.037	-0.010	-0.186	0.364	0.142	0.028	-0.098	-0.005	0.109	0.714
2	-0.022	-0.037	-0.010	-0.186	0.364	0.142	0.028	-0.098	-0.005	0.109	0.714
3	-0.022	-0.037	-0.010	-0.186	0.364	0.142	0.028	-0.098	-0.005	0.109	0.714

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Table 7: (continued).

4	0.000	0.000	0.000	0.000	0.386	0.135	0.015	0.000	0.000	0.078	0.386
5	0.024	0.045	0.008	-0.069	0.561	0.232	0.120	-0.086	0.007	0.158	0.000

Table 8 summaries the annualized average return, annualized standard deviation and Sharpe ratio of these critical portfolios.

Table 8: Summary statistics of critical portfolios with Markowitz mode.

Panel A: Maximal Sharpe ratio portfolios			
Constraint	Annualized portfolio average return	Annualized portfolio std	Portfolio Sharpe ratio
1	0.264	0.187	1.414
2	0.332	0.221	1.501
3	0.496	0.332	1.539
4	0.221	0.176	1.254
5	0.239	0.180	1.326
Panel B: Minimal variance portfolios			
Constraint	Annualized portfolio average return	Annualized portfolio std	Portfolio Sharpe ratio
1	0.073	0.122	0.595
2	0.073	0.122	0.595
3	0.073	0.122	0.595
4	0.100	0.131	0.761
5	0.132	0.134	0.989

Figure 4 shows the efficient frontiers, inefficient frontiers, and minimal variance frontiers of these critical portfolios, as well as the capital allocation line. Note that the capital allocation line is constructed using the latest annualized notional risk-free return.

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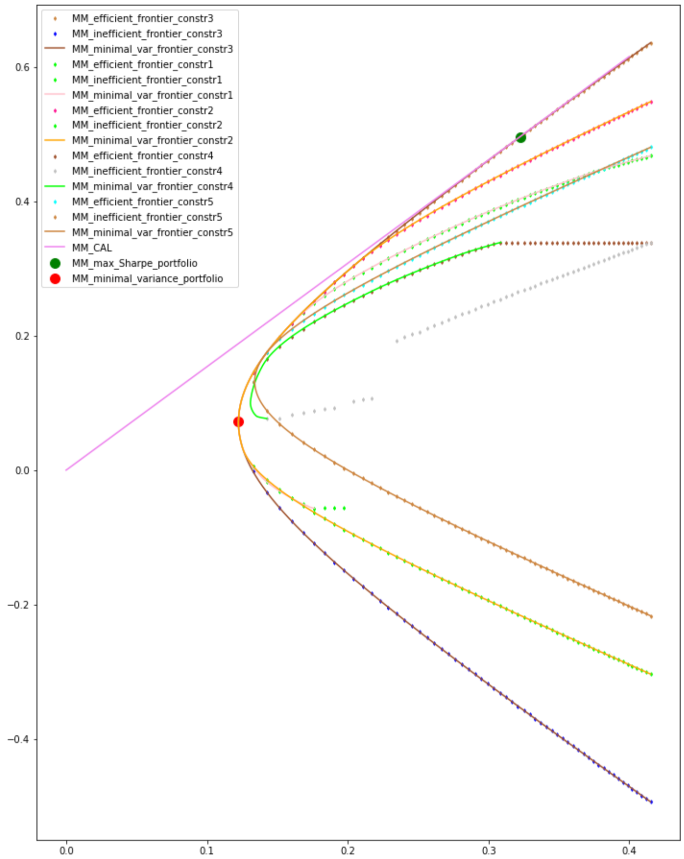

Figure 4: Frontiers with Markowitz model.

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3.2. Index Model

Table 9 shows the weights in maximal Sharpe portfolios and minimal variance portfolios with different constraints.

Table 9: Weights in critical portfolios with index model.

Panel A: Maximal Sharpe ratio portfolios											
	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
Constraint 1	0.178	0.306	0.004	-0.027	0.225	0.316	0.012	0.000	0.191	0.266	-0.473
Constraint 2	0.218	0.366	0.031	-0.088	0.333	0.400	0.095	0.056	0.250	0.338	-1.000
Constraint 3	0.431	0.701	0.110	0.088	0.585	0.725	0.279	0.249	0.504	0.629	-3.302
Constraint 4	0.155	0.271	0.000	0.000	0.032	0.214	0.000	0.000	0.140	0.189	0.000

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Table 9: (continued).

Constraint 5	0.186	0.317	-0.005	-0.212	0.117	0.277	-0.053	-0.054	0.186	0.242	0.000
Panel B: Minimal Varience Portfolios											
	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	SPX
Constraint 1	-0.042	-0.048	-0.024	-0.130	0.346	0.134	0.085	-0.036	-0.016	0.075	0.655
Constraint 2	-0.042	-0.048	-0.024	-0.130	0.346	0.134	0.085	-0.036	-0.016	0.075	0.655
Constraint 3	-0.042	-0.048	-0.024	-0.130	0.346	0.134	0.085	-0.036	-0.016	0.075	0.655
Constraint 4	0.000	0.000	0.000	0.000	0.377	0.145	0.096	0.000	0.000	0.078	0.304
Constraint 5	-0.016	-0.009	-0.001	-0.050	0.480	0.218	0.180	0.034	0.030	0.136	0.000

Table 10 summaries the annualized average return, annualized standard deviation and Sharpe ratio of these critical portfolios.

Table 10: Summary statistics of critical portfolios with index model.

Panel A: Maximal Sharpe ratio portfolios			
	Annualized portfolio average return	Annualized portfolio std	Portfolio Sharpe ratio
Constraint1	0.286	0.200	1.430
Constraint2	0.341	0.224	1.521
Constraint3	0.610	0.382	1.596
Constraint4	0.245	0.192	1.277
Constraint5	0.269	0.202	1.331
Panel B: Minimal varience portfolios			
	Annualized portfolio average return	Annualized portfolio std	Portfolio Sharpe ratio
Constraint1	0.065	0.124	0.521
Constraint2	0.065	0.124	0.521
Constraint3	0.065	0.124	0.521
Constraint4	0.102	0.130	0.787
Constraint5	0.114	0.132	0.866

Figure 5 shows the efficient frontiers, inefficient frontiers and minimal variance frontiers of these critical portfolios, as well as the capital allocation line. Note that the capital allocation line is constructed using the latest annualized notional risk-free return.

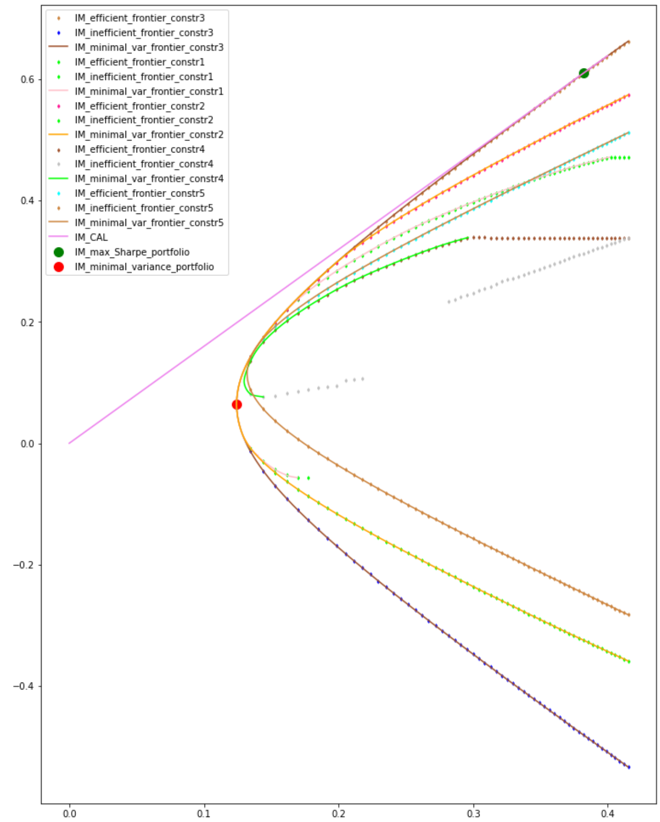

Figure 5: Frontiers with index model.

Photo credit: Original

3.3. Drawdown Model

The paper adopts the same linear programming approach as in professor's paper Portfolio optimization with drawdown constraints [12] to generate optimal portfolios including ten stocks and S&P 500. Note that in consistence with professor's original paper, the author uses daily cumulative uncompounded returns. Moreover, the technological constraints are xmin =1/22 and xmax =1/11. The model implicitly recommends investing half of the available money if all positions are equal to the lesser bound. The model implicitly recommends that the leverage should be two if all of the positions are equal to the higher bound. The Python-based gurobipy software handles the linear programming issues.

Table 11: List of markets for AvDD problem.

Reward	0.099	0.159	0.211	0.260	0.304	0.336	0.361	0.382	0.396
MaxDD	0.267	0.375	0.478	0.588	0.713	0.797	0.869	0.986	1.061
AvDD	0.023	0.032	0.041	0.050	0.059	0.068	0.077	0.086	0.092
MaxDDRatio	0.372	0.424	0.441	0.443	0.426	0.422	0.416	0.388	0.373
AvdDDRatio	4.318	4.980	5.415	5.204	5.150	4.945	4.688	4.445	4.308

Table 11 and Table 12 present the list of markets and corresponding sets of optimal weights for AvDD problem.

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Table 12: Optimal weights for AvDD problem.

AMZN	0.045	0.108	0.147	0.182	0.182	0.182	0.182	0.182	0.182
AAPL	0.045	0.091	0.131	0.170	0.182	0.182	0.182	0.182	0.182
CTXS	0.045	0.045	0.045	0.045	0.045	0.052	0.166	0.182	0.182
JPM	0.045	0.045	0.045	0.045	0.068	0.182	0.182	0.182	0.182
BRK/A	0.045	0.045	0.057	0.110	0.182	0.182	0.182	0.182	0.182
PGR	0.045	0.148	0.182	0.182	0.182	0.182	0.182	0.182	0.182
UPS	0.045	0.045	0.045	0.045	0.045	0.156	0.182	0.182	0.182
FDX	0.045	0.045	0.045	0.110	0.045	0.045	0.045	0.101	0.182
JBHT	0.045	0.056	0.119	0.172	0.182	0.182	0.182	0.182	0.182
LSTR	0.045	0.045	0.045	0.060	0.182	0.182	0.182	0.182	0.182
SPX	0.045	0.045	0.045	0.045	0.045	0.045	0.071	0.174	0.182

Table 13 and Table 14 present the list of markets and corresponding sets of optimal weights for MaxDD problem.

Table 13: List of markets for MaxDD problem.

Reward	0.111	0.157	0.197	0.235	0.271	0.303	0.333	0.355	0.370	0.384	0.396
MaxDD	0.270	0.350	0.430	0.510	0.590	0.670	0.750	0.830	0.910	0.990	1.061
AvDD	0.025	0.034	0.040	0.047	0.055	0.064	0.071	0.075	0.081	0.087	0.092
MaxDDRatio	0.410	0.449	0.458	0.461	0.459	0.453	0.443	0.427	0.407	0.388	0.373
AvDDRatio	4.487	4.668	4.884	4.970	4.889	4.718	4.680	4.711	4.575	4.412	4.308

Table 14: Optimal weights for MaxDD problem.

AMZN	0.077	0.095	0.132	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.182
AAPL	0.045	0.073	0.135	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.182
CTXS	0.045	0.045	0.045	0.053	0.082	0.182	0.182	0.182	0.182	0.182	0.182
JPM	0.045	0.045	0.045	0.059	0.182	0.182	0.182	0.182	0.182	0.182	0.182
BRK/A	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.165	0.182	0.182	0.182
PGR	0.045	0.045	0.045	0.045	0.104	0.104	0.182	0.182	0.182	0.182	0.182
UPS	0.045	0.045	0.045	0.045	0.045	0.045	0.151	0.141	0.182	0.182	0.182
FDX	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.105	0.182	0.182
JBHT	0.047	0.165	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.182
LSTR	0.045	0.045	0.045	0.045	0.045	0.045	0.136	0.182	0.182	0.182	0.182
SPX	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.067	0.182

Table 15 and Table 16 present the list of markets and corresponding sets of optimal weights for CDaR problem with $(1 - \alpha) = 0.05$. The solutions having the highest Reward/AvDD ratio and Reward/MaxDD ratio are boldfaced in the tables. In order for the optimization issue to still have solutions, the minimal risk value is selected. The biggest risk value is only the smallest risk value that may bring all positions to the upper limit of technological limitations.

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Table 15: List of markets for CDaR problem with $(1-\alpha) = 0.05$.

Reward	0.106	0.151	0.190	0.225	0.258	0.288	0.318	0.343	0.364	0.383	0.396
MaxDD	0.268	0.352	0.436	0.511	0.583	0.658	0.739	0.822	0.917	0.981	1.061
AvDD	0.024	0.033	0.039	0.046	0.052	0.058	0.063	0.070	0.079	0.087	0.092
MaxDD Ratio	0.395	0.429	0.436	0.441	0.442	0.438	0.431	0.418	0.397	0.390	0.373
AvDD Ratio	4.426	4.605	4.844	4.917	4.944	5.006	5.046	4.878	4.618	4.425	4.308

Table 16: Optimal weights for CDaR problem with $(1-\alpha) = 0.05$.

AMZN	0.064	0.181	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.182
AAPL	0.045	0.045	0.045	0.045	0.076	0.125	0.177	0.182	0.182	0.182	0.182
CTXS	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.070	0.104	0.182	0.182
JPM	0.045	0.060	0.122	0.153	0.182	0.182	0.182	0.182	0.182	0.182	0.182
BRK/A	0.045	0.045	0.045	0.097	0.091	0.094	0.162	0.182	0.182	0.182	0.182
PGR	0.045	0.045	0.161	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.182
UPS	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.182	0.182	0.182	0.182
FDX	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.051	0.147	0.182	0.182
JBHT	0.045	0.045	0.075	0.158	0.182	0.182	0.182	0.182	0.182	0.182	0.182
LSTR	0.045	0.045	0.045	0.045	0.102	0.164	0.182	0.182	0.182	0.182	0.182
SPX	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.053	0.182

The efficient frontiers for the Reward-AvDD and Reward-MaxDD problems are shown in Figure 6 and Figure 7, respectively.

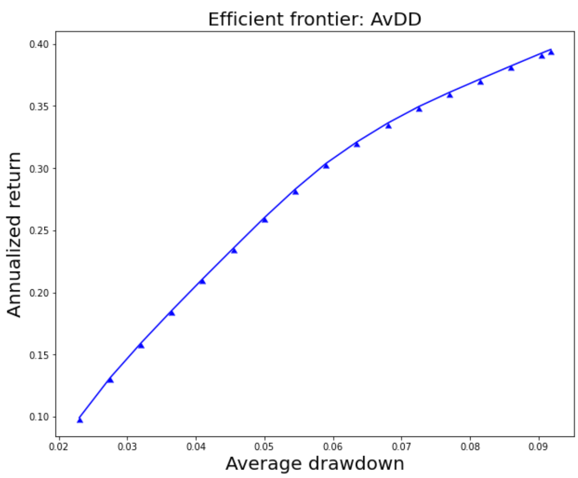

Figure 6: Efficient frontier for Rewad-AvDD problem.

Photo credit: Original

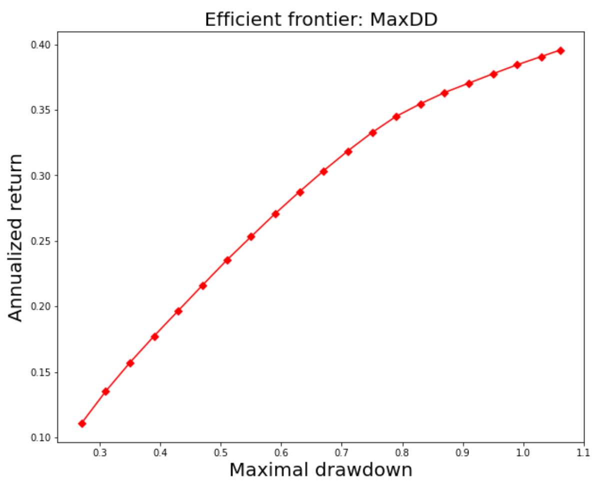

Figure 7: Efficient frontier for Rewad-MaxDD problem.

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Figure 8 shows the Reward-AvDD graphs for the portfolios optimal with $(1-\alpha) = 0$, 0.05, 0.4 and 1 CDaR constraints. As we thought, the scenario where (1) = 1 CDaR corresponds to AvDD has a concave efficient frontier that dominates other graphs, as one might anticipate.

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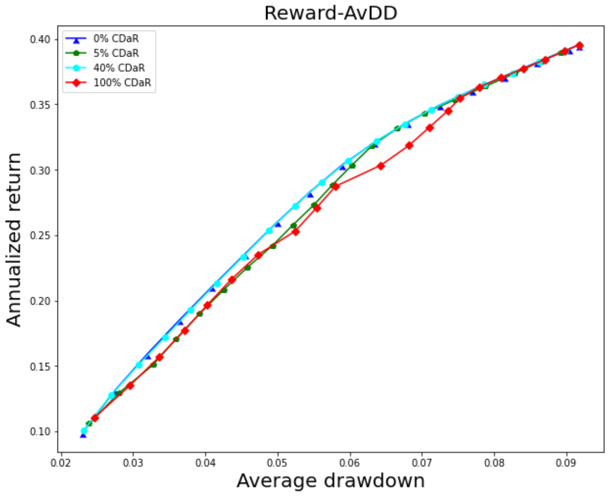

Figure 8: Reward-AvDD graphs.

Photo credit: Original

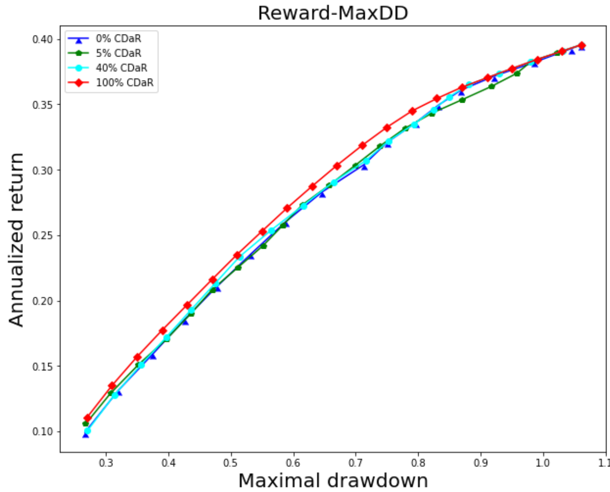

Figure 9: Reward-MaxDD graphs.

Photo credit: Original

Reward-AvDD graphs for the portfolios with (1) = 0, 0.05, 0.4, and 1 CDaR limitations are shown in Figure 9. The scenario where (1) = 0 CDaR corresponds to MaxDD has a concave efficient frontier that dominates other graphs, as is predicted.

The charts of MaxDD Ratio and AvDD Ratio are shown in Figure 10 and Figure 11, respectively.

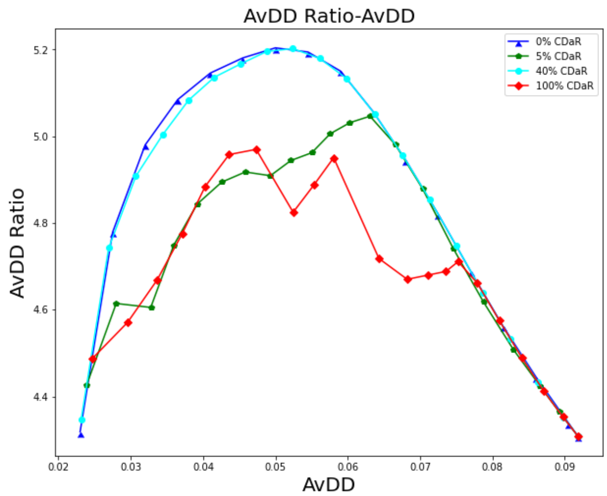

Figure 10: AvDDRatio.

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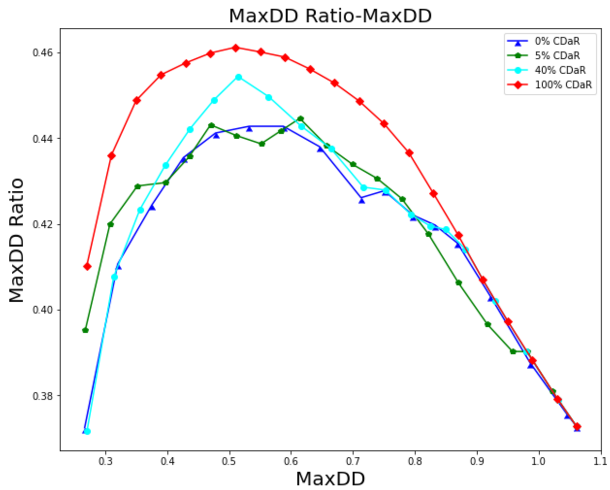

Figure 11: MaxDDRatio.

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4. Conclusion

Markowitz's asset selection model has undergone multiple empirical tests since its inception in the 1950s and has continued to develop to this day.

This thesis uses nearly 20 years of historical daily total return data for 10 stocks, aggregate the daily data into monthly observations, and based on these monthly observations, calculate all appropriate optimized inputs for the full Markowitz model (MM) and the index model (IM). This paper finds areas of permissible portfolios with additional constraints (efficient boundary, minimum risk portfolio, optimal portfolio, and minimum return portfolios boundary). Markowitz model fully utilizes the covariance matrix to generate superior portfolios. However, its results are numerically unstable. Meanwhile, its assumptions of normality, stationarity and homoskedasticity are invalidated. Index model simplify Markowitz model to generate more robust results. However, it introduces additional assumptions of independence, normality, and homoskedasticity of regression residuals, which, are invalidated as well. Drawdown (CDaR) model is assumption-free. Moreover, it reduces the nonlinear optimization problems into linear programming problems, which can generate numerically stable results.

Markowitz's theory has gradually transitioned financial research towards quantification, which is a significant progress in theory. However, from the empirical results of this article, it can be seen that in reality, financial market data is difficult to meet a series of assumptions of the model, which leads to the problem of low simulation efficiency. Therefore, future research can start in this area.

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