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Multi-Period Portfolio Optimization under Cash-In Strategies

Shuyi Wang[1,a,*]

[1]*Department of Mathematics, University of North Carolina at Chapel Hill, North Carolina, 27514, USA*

a. shuyiwang2002@outlook.com

**corresponding author*

Abstract: Portfolio optimization has been crucial in its applications to finance. Investors allocate capital in terms of various assets and wisely adjust each proportion in quest of higher returns and lower risks. However, common optimization strategies are usually short- sighted, lacking the ability of gaining higher returns over a long period. This paper addresses Multi-Period Portfolio Optimization problem in more applicable settings to provide efficient strategies. Firstly, a summary of conclusions from former works and scholars is given. Since most previous works assume self-financing conditions, this work extends this constraint and considers more complex cash-in and cash-out situations. It also derives the expression of the optimal policy for trading at each period as well as the objective functions. Finally, numerical experiments are done under both self-financing and cash-in situations to prove the validity of conclusions. This work has profound implications in finance because it adopts conditions more consistent with real-world scenarios and provides more farsighted strategies. Moreover, the solution offers a prototype for similar optimization in other fields like electrical engineering and stochastic processes.

Keywords: Multi-Period Portfolio, Long-Term Optimization, Self-Finance, Cash-In Strategies

1. Introduction

The academic investigation of portfolio optimization was initiated from the last century. In Harry Markowitz's classic work Portfolio Selection, he was the first to formally define portfolio problems as "starting with the relevant beliefs about future performances and ending with the choice of portfolio" and formulate scenarios in terms of rigorous mathematics[1]. In portfolio problems, researchers have a brief grasp of the future trends of capitals, such as various types of stocks and their corresponding accretion means and variances. Having the information beforehand, scholars proceed in maximizing their expected returns while minimizing risks of losing wealth. Most of the time the dimension of such problems is very large because of the amount of possible assets in the real world, thus presenting intractable calculation difficulties. Instead of obtaining the absolute optimal policy, approximate suboptimal policies are usually more common and computationally cheaper in such problems. Miguel Lobo and his team made initial attempts by solving the convex quadratic program and accounting for linear transactions costs and discount breakpoints [2]. Their numerical results produced a suboptimal solution and an upper bound on the optimal solution, which turned out to be promising at that time.

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In addition to normal portfolio optimization, multi-period ones have been more complicated but applicable to realistic scenarios. Instead of allocating portfolios once, most investors hope to consider the long run and maximize their returns based on long-term targets, where the multi-period optimization problem comes in. Stephan Boyd formulated the problem nicely with reasonable mathematical assumptions and solved optimal policy without constraints and suboptimal policy with constraints [3]. Giuseppe Calafiore solved multi-period sequential decision problems by maximizing a cumulative risk measure and satisfying portfolio diversity constraints at each period [4]. In 2017, Stephan Boyd's work Multi-Period Trading via Convex Optimization discussed the entire framework of multi-period optimization, gave his explanation of every commonly accepted setup, and presented numerical results at the end[5].

Though most of the conditions have been covered in Boyd's work, the common assumption of self-finance has yet been relaxed. Self-finance conditions require that the algebraic sum of each trading must be zero, restricting any addition or subtraction of money via tradings. In this work, the condition is relaxed through careful manipulation of Lagrange methods and mathematical induction. This work also derives the suboptimal policy under cash-in constraints and simulates accretion curves of wealth under three specific cash-in strategies to test its validity. The numerical results yield promising shapes of consistency with theories. The results are meaningful for both academics and applications. In academia, it sheds lights on how cash-in strategies affect overall portfolio optimization and implies investigation of more complex strategies for future studies. In the real world, the paper gives investors more flexibility by allowing for adjustment of their aggregate amount of financial assets in markets.

2. Preliminaries

To formulate the problem in multiple periods, the entire time span is broken into T equivalent smaller periods. Each period represents a complete business day and consists of two steps. The first involves adjusting the holdings of each asset through purchases and sales, where the target is to find the best policy. The second involves gaining returns and losses on all assets, with a random return model with some fundamental regularities. Then the investor enters the next period with this modified portfolio.

Assume there're n assets in total. We denote $X_t \in \boldsymbol{R}^n$ as the portfolio holdings at time t, where t corresponds to the index of time period and each dimension represents the money in each holding. We define $w_t \in \mathbf{1}^T x_t$ as the total wealth at time t, where 1 is a vector with 1's in all of its entries. $u_t \in \boldsymbol{R}^n$ is another n-dimension vector that represents the trade we make at time t, with its positive entries as buying and negative entries as selling. The main goal is to find the best policy for each ut, written as $\varphi_1,\ldots,\varphi_T$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$,$u_t = \varphi_t(x_t)$, that optimizes the final wealth as well as the risk. In addition, the portfolio holdings and trades at each period are under some constraints, denoted as $(u_t, x_t) \in C_t$, which will be discussed later. After the trade of each time, we denote $x_t^+ = x_t + u_t \in \boldsymbol{R}^n$ as the post-trade portfolios of holdings.

At the end of each period, we also define a return vector $r_t \in \boldsymbol{R}_+^n$. This random variable represents the return coefficients of portfolios. We take the component-wise product of the return vector with the post-trade portfolio to simulate the random changes in each asset's market values. This process will be the main reason why our portfolio holdings can increase under self-finance trades. To normalize the process, we generate a square matrix $A_t = diag\ (r_t)$ to transfer the component-wise product to the regular matrix product. In other words, the portfolio holdings for next period can be expressed as $x_{t+1} = A_t x_t^+$. For the convenience of later notations, we also denote the first and second moment of the random variable r_t as $\bar{r} = \mathbf{E} r_t$ and $P_t = \Sigma_t + \bar{r}\bar{r}_t^T$.

Finally, we arbitrarily set a desired wealth after T intervals, denoted as w^{des}. Then the objective we want to optimize should be $J = E(w_{t+1} - w^{des})^2$.This objective seems less intuitive because if the final wealth is higher the desired value, it is still penalized. The reason behind is that higher returns come with higher risks, and if one gets higher returns from trades, it is still unfavorable because of the higher risk than expected.

3. Optimization for Self-Finance

Firstly, consider the unconstrained case, where the only condition is that trades are self-finance, formulated as $C_t = C^{\text{basic}} = \{(x, u) \mid 1^T u = 0\}$. It simply means that the purchases and sales in each trade should algebraically add up to 0. This requirement corresponds to real-world situations that a fixed amount of money is used to increase its value through investment.

From the condition and objective, [3] has shown in detail that the optimal policy for each trade can be expressed in an affine function of the current portfolio holdings. It's written as

$$u_t = \varphi_t(x_t) = K_t(x_t - g_t) \tag{1}$$

Where

$$K_t = -I + \frac{1}{\mathbf{1}^T P_t^{-1} \mathbf{1}^{-1}} P_t^{-1} \mathbf{1} \mathbf{1}^T$$
$$g_t = w_{t+1}^{\text{tar}} P_t^{-1} \bar{r}_t$$
$$w_t^{\text{tar}} = w_{t+1}^{\text{tar}} (\mathbf{1}^T P_t^{-1} \bar{r}_t)$$

The derivation process for formulas above uses induction and Lagrange multiplier method. A brief summary is given below. Denote Vt(xt) as the function to minimize at time t before the trade. Denote V +(xt + ut) as the function to minimize at time t after the trade. In the paper, it is shown that

$$V_t(x_t) = a_t(1^T x_t - w_t^{\text{tar}})^2 + b_t \tag{2}$$

$$V_t^+(x_t + u_t) = a_{t+1}\big((\bar{r}_t^T(x_t + u_t) - w_{t+1}^{\text{tar}})^2 + (x_t + u_t)^T \Sigma_t (x_t + u_t)\big) + b_{t+1} \tag{3}$$

First, the team show that the expression for V_T^+ satisfied the form above, by directly finding the expectation of the last objective function. Then, they assume the form for V_T^+ is correct and derive the expression for V_t^+ by using a simple Lagrange argument. In the process, the expression of optimal affine trading policy is automatically derived. Lastly, they assume the form for V_T^+ is correct and derive the expression for V_t^+ by calculating the expectation of objective functions. More details in derivation are in [3] and not the focus of this paper.

As shown in the formulas, the main challenge of finding the trading policy is the second moment of the return vector for each period. Since all return vectors are i.i.d. in the experiment design (to be discussed later), there is only one universal second moment. This value is mfound in the experiment with moment generating functions.

4. Optimization for Cash in and Cash out

This section will derive the trading strategies where cash in and cash out are allowed. In other words, during the trading process, the investor need not to keep self-finance all the time. They can add or subtract some values to the portfolio in each trade. The constraint of $1^T u_t = 0$ is replaced by $\mathbf{1}^T u_t =$

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d_t , where dt represents changes in the total value of the portfolio. Its positive value means adding money to holdings and negative means subtracting.

Firstly, it is obvious that the objective functions before and after trade are the same as those in the self-finance situation. Since they are both derived from the calculation of expectations, they still have the form (2) and (3). The only difference is in the expressions for a_t, b_t, and w_t^{tar}. Here I will use Lagrange multiplier to derive the optimal policy and expressions of a_t and b_t from (3), which is correct in its form. In other words, I will solve the convex quadratic problem,

$$\begin{aligned} &min.\, a_{t+1}\big((\bar{r}_t^T(x_t+u_t)-w_{t+1}^{\text{tar}})^2+(x_t+u_t)^T\Sigma_t(x_t+u_t)\big)+b_{t+1} \\ &\text{s.t. } \mathbf{1}^T u_t = d_t \end{aligned} \tag{4}$$

To minimize (4) is the equivalent to

$$\begin{aligned} &min.\, (\bar{r}_t^T(x_t+u_t)-w_{t+1}^{tar})^2+(x_t+u_t)^T\Sigma_t(x_t+u_t) \\ &\text{s.t. } 1^T u_t = d_t \end{aligned} \tag{5}$$

First, rewrite the objective function,

$$\begin{aligned} &(\bar{r}_t^T(x_t+u_t)-w_{t+1}^{\text{tar}})^2+(x_t+u_t)^T\Sigma_t(x_t+u_t) \\ &=(x_t+u_t)^T\bar{r}_t^T\bar{r}_t(x_t+u_t)-2w_{t+1}^{\text{tar}}\bar{r}_t^T(x_t+u_t)+w_{t+1}^{\text{tar}\,2}+(x_t+u_t)^T\Sigma_t(x_t+u_t) \\ &=(x_t+u_t)^T P_t(x_t+u_t)-2w_{t+1}^{\text{tar}}\bar{r}_t^T(x_t+u_t)+w_{t+1}^{\text{tar}\,2} \end{aligned}$$

Then, establish a Lagrange function from the objective function and affine constraint, (Since parametrization does not affect Lagrange functions, I am using -2ν instead of ν here)

$$L(u_t,\nu)=(x_t+u_t)^T P_t(x_t+u_t)-2w_{t+1}^{tar}\bar{r}_t^T(x_t+u_t)-2\nu(\mathbf{1}^T u_t-d_t) \tag{6}$$

Find its derivative with respect to ut,

$$\frac{\partial L}{\partial u_t}=2P_t(x_t+u_t)-2w_{t+1}^{\text{tar}}\bar{r}_t-2\nu\mathbf{1} \tag{7}$$

Set (7) to 0,

$$x_t+u_t=P_t^{-1}(w_{t+1}^{\text{tar}}\bar{r}_t+\nu 1) \tag{8}$$

Substitute (8) in $\mathbf{1}^T u_t=d_t$,

$$\begin{aligned} &\mathbf{1}^T(x_t+u_t)=\mathbf{1}^T x_t+d_t \\ &=\mathbf{1}P_t^{-1}(\nu\mathbf{1}+w_{t+1}^{\text{tar}}\bar{r}_t) \\ &\nu=\frac{\mathbf{1}^T x_t+d_t-w_{t+1}^{\text{tar}}\mathbf{1}^T P_t^{-1}\bar{r}_t}{\mathbf{1}^T P_t^{-1}\mathbf{1}} \end{aligned} \tag{9}$$

Substitute (9) into (8) and get

$$u_t=K_t(x_t-g_t)+P_t^{-1}\mathbf{1}\frac{d_t}{1P_t^{-1}\mathbf{1}} \tag{10}$$

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where

$$K_t = -I + \frac{1}{\mathbf{1}^T P_t^{-1} \mathbf{1}} P_t^{-1} \mathbf{1}\mathbf{1}^T$$
$$g_t = w_{t+1}^{\text{tar}} P_t^{-1} \bar{r}_t$$

Finally update the expression for $V_t(x_t)$,

$$\begin{aligned} V_t(x_t) &= minV_t^+(x_t + u_t) \\ &= a_{t+1}((x_t + u_t)^T P_t (x_t + u_t) - 2w_{t+1}^{\text{tar}} \bar{r}_t^T (x_t + u_t) + w_{t+1}^{\text{tar}\,2}) + b_{t+1} \\ &= a_{t+1}\left(v^2 1^T P_t^{-1} 1 + w_{t+1}^{\text{tar}\,2}(1 - \bar{r}_t^T P_t^{-1} \bar{r}_t)\right) + b_{t+1} \\ &= a_{t+1} \frac{1}{\mathbf{1}^T P_t^{-1} \mathbf{1}} (1x_t + d_t - \mathbf{1}^T P_t^{-1} w_{t+1}^{\text{tar}} \bar{r}_t)^2 + a_{t+1} w_{t+1}^{\text{tar}\,2} (1 - \bar{r}_t^T P_t^{-1} \bar{r}_t)\Big) + b_{t+1} \end{aligned}$$

which has the form

$$V_t(x_t) = a_t (1^T x_t - w_{t+1}^{\text{tar}})^2 + b_t$$

I conclude with the complete expression of objective functions (2) (3) and optimal policy (10) with their updated parameters

$$a_t = a_{t+1} \frac{1}{\mathbf{1}^T P_t^{-1} \mathbf{1}}$$
$$b_t = a_{t+1} w_{t+1}^{\text{tar}\,2} (1 - \bar{r}_t^T P_t^{-1} \bar{r}_t)\big) + b_{t+1}$$
$$w_t^{tar} = w_{t+1}^{\text{tar}} \mathbf{1}^T P_t^{-1} \bar{r}_t - d_t$$

5. Experiments and Results

5.1. Design

To set up the experiments, this work will adhere to the experiment design in [3]. Apart from some arbitrary parameters, the only complex part is how to generate the random variable $r_t \in \mathbf{R}^n$. We assume that all rt are from an independently identical lognormal distribution, formulated as $\log(r_t) \sim N(\mu, S)$.

The mean μ and variance matrix S are arbitrary but still have some regularities. Here the assumption is that the first asset is risk-free, with $\mu_1 = \mu_{rf}, S_{1j}, S_{i1} = 0$. Then, each asset with larger index will have higher mean return and higher variance (risk). We define $S_{ii} = \left(\frac{\sigma_{max}(i-1)}{n}\right)^2$ and $\mu_i = \mu_{rf} + \rho S_{ii}$ where σ_{max} is the largest risk and ρ is the reward-risk ratio. This setup provides the convenience that we can always index and sort all assets in the order of riskiness without loss of generality. With larger returns, investors must bear larger risks.

Then covariances in the matrix S are set up as below. We first generate a matrix $Z \in \mathbf{R}^{n*n}$ from the standard Gaussian distribution. Then, we calculate

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$$Y = ZZ^T + \lambda \mathbf{1}\mathbf{1}^T$$
$$C = \mathrm{diag}\left(Y_{11}^{-\frac{1}{2}}, \dots, Y_{nn}^{-\frac{1}{2}}\right) Y \mathrm{diag}\left(Y_{11}^{-\frac{1}{2}}, \dots, Y_{nn}^{-\frac{1}{2}}\right)$$

We adjust the value of λ so that the smallest entry in C is approximately -0.1. Then, we copy all off-diagonal entries of the matrix C to S, formulated as $S_{ij} = \left(S_{ii}S_{jj}\right)^{1/2} C_{ij}, i, j = 1, \dots, n$.
Here's a list of parameters chosen:

$$n = 10, T = 40, w^{\mathrm{des}} = 3$$
$$\mu_{rf} = 1\%, \rho = 0.4, \sigma_{\max} = 0.4$$

In the experiments, simulations for both self-finance and cash-in situations are conducted. In the latter case, the algorithm will set an assigned amount of money in advance and use different strategies to add those values to trades in the process. They are respectively increasing linear, decreasing linear, and completely random. To make comparison among strategies, the assigned value will be identical. The strategies are below,

$$\text{increasing linear:} d_t = \frac{p}{T}\frac{2t}{T}$$
$$\text{decreasing linear :} d_t = \frac{p}{T} 2\left(1 - \frac{t}{T}\right)$$
$$\text{completely random:} d_t = \frac{p}{T} \mathrm{rand}_t(0,1)$$

where$p = 0.04$ and $\sum_{t=1}^{T} \mathbf{rand}_t(0,1) = 1$

5.2. Self-Finance Case

The graph shows the simulation of portfolio optimization under self-finance situation. It consists of 50 representative iterations with parameters above. Each iteration uses the optimal policy (1) and attempts to reach the desired wealth after T periods. From the graph, one can interpret that most processes adopt ambitious trades at first by putting more values into riskier assets. It is shown in the graph by larger fluctuations in the first half of each trajectory. After achieving a wealth that can reach the desired wealth, most processes then adopt conservative strategies by putting more values into risk-free assets. Figure 1 shows that most trajectories overlap as an affine line near the end of optimization, which implies validity of the optimal policy.

Figure 1: Simulation of Self-Finance Portfolio Optimization with Optimal Policy.

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5.3. Cash-in Case

The graphs show the simulation of portfolio optimization under three cash-in strategies. Each of them consists of 50 representative iterations with parameters above. From all three graphs, it's obvious that most of the trajectories reach the desired wealth, with only several outliers that fluctuate near the end of optimization. Particularly, Figure 2 shows that the curves for increasing linear are more uniform and manifest convexity. On the other hand, Figure 3 and 4 show that the curves for decreasing linear and completely random manifest messier patterns before reaching the affine line. The results indicate that when adding amounts of money to portfolio selection, increasing linear strategy might be a reasonable choice in real-world situations.

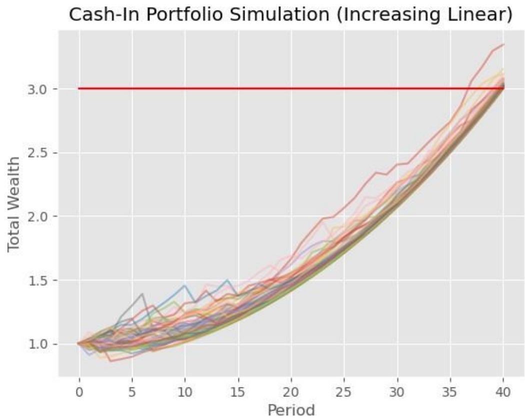

Figure 2: Simulation of Cash-In Portfolio Optimization with Increasing Linear Strategy

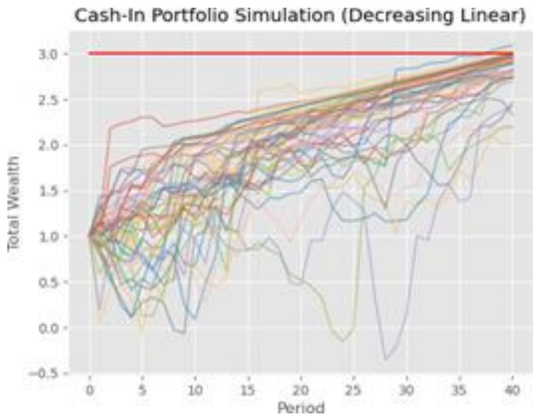

Figure 3: Simulation of Cash-In Portfolio Optimization with Decreasing Linear Strategy.

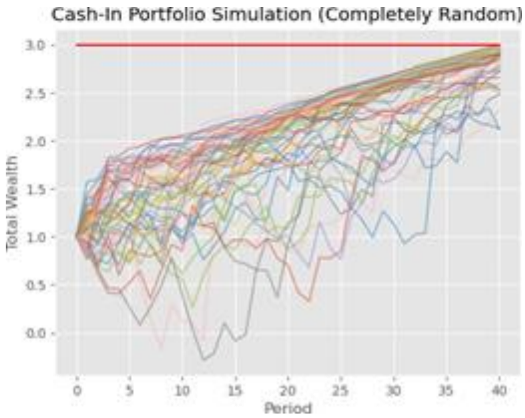

Figure 4: Simulation of Cash-In Portfolio Optimization with Completely Random Strategy.

6. Conclusions

This work investigates the multi-period portfolio optimization problem, where the investors allocate capital to financial assets, wisely arrange their relative proportions, and gain desired profits. In search for more realistic simulation, this research extends the condition of self-finance and derives the optimal policy for both cash-in and cash-out situations as an affine function of holdings. Moreover, numerical experiments on optimization under self-finance and cash-in situations are done to prove that the optimal policy works in consistency with its theoretical performance to gain the desired wealth. Comparisons among specific cash-in strategies are also conducted to suggest increasing linear as the best way to increase capital in investments. The conclusion of this work has profound real-world implications and can be directly applied into similar settings in businesses or other types of long-term optimization problems.

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