Proceedings of the 3rd International Conference on Social Psychology and Humanity Studies
DOI: 10.54254/2753-7048/72/2025.21414

A Study of the Logic of Common Beliefs and Common Knowledge

Zhaonian Zhang[1,a,*]

[1]*Marxist College, Beijing Forestry University, Beijing, 100083, China*
a. 13910721403@163.com
**corresponding author*

Abstract: The research on the philosophical logic of common belief and common knowledge has always been carried out together. There has been a lot of research on the logic of common knowledge. In addition to the simplest cognitive logic, there are also dynamic studies on the logic of common knowledge, the logic of common knowledge with weight, and the logic of common declaration of common knowledge. But the study of the philosophical logic of common beliefs has always been relatively marginalized. This paper tries to employ literature methods to present the language, models, and semantics of simple common belief logic according to the system of cognitive logic form about common knowledge by elaborating on the history of logical development and the standard research methods of modal logic.

Keywords: common belief, common knowledge, philosophical logic, history of logic

1. Introduction

Modal logic has made significant progress in the study of common knowledge, but the logic concerning common belief has not been clearly established yet.

The formation of common belief and its relevance to common knowledge have an important position in the study of logic. Is common belief a kind of strengthened common knowledge? Or is the content of common beliefs more relaxed and weaker than common knowledge? In addition, the beliefs of the subject are often formed and easy to change, but some are firm. These phenomena play an extremely important role in the cognition of human individuals and groups. Therefore, This study elaborates on the history of logical evolution and the normal research methods of modal logic in an attempt to use literature methods to provide the language, models, and semantics of basic common belief logic in accordance with the system of cognitive logic form about common knowledge. Research can contribute to a deeper understanding of human thinking laws, to enrich the theory of belief, knowledge and cognitive logic itself, and even to the study of related fields such as artificial intelligence and game theory.

2. Overview of cognitive logic and common knowledge

Cognitive logic studies knowledge reasoning as a branch of modal logic from the perspective of an axiom system. It is well known that epistemology has a long philosophical tradition dating back to ancient Greece[1]. However, cognitive logic, as a branch of modal logic, is widely used in philosophy, computer science, artificial intelligence, economics and linguistics in the 1960s.

DOI: 10.54254/2753-7048/72/2025.21414

In a sense, cognitive logic can also be said to be established by Aristotle. In his preanalysis and post-analysis, it can be seen that he studied the basic problems of modern cognitive logic. In 1947, Karnap (R.Carnap) published his book *Meaning and Necessity*, which discussed the sentences with "believe" and "conclude". This is probably the earliest study of the cognitive logic. The first book to discuss cognitive logic in detail, "Knowledge and Belief," written by Hintika in 1962[2]. Hintika uses models to describe the semantics of knowledge from a new perspective to examine the corresponding transition from modal logic to cognitive logic to a new level. At this point, cognitive logic research is maturing and beginning to focus on an increasing number of individuals.

At the same time, in the 1970s, the game theorists represented by Oman also independently developed the cognitive logic and used it to represent the players knowledge of others behavior, trying to provide a logical explanation for the equilibrium concept of game theory according to the preferred behavior of rational subjects. This is actually the first article to give a formal definition of common knowledge in the Agreement of Inconsistency in 1976. The discussion of common knowledge has been discussed since 1969 in Lewiss Covenant and Friedells The Structure of Shared Awareness). In 1988, Barwise further discussed common knowledge in three concepts of common knowledge.

The concept of common knowledge has its application across multiple disciplines and in multiple fields. common knowledge In the analysis of Nobel Prize-winning economist Robert Aumann, if there are posterior probabilities of common knowledge among agents, these probabilities must be the same, leading them to agree, not disagree. In economics, the common belief is crucial to understanding the behavior of financial markets such as the stock market, because the market value is often determined by people's common belief. Common knowledge is also very important in social interaction, which can help people predict the behavior of others and thus make reasonable decisions. For example, in traffic rules, drivers need to know that other drivers also know that they should drive on the left side, and such common knowledge helps to avoid traffic accidents. The concept of common knowledge is related to the development of a child's theory of mind, meaning that children beginning to understand the mental state of others at the age of four may be different from themselves. In the Sally-Anne experiment, children needed to predict where Sally would go to find the marble without knowing it was moved, which requires some common knowledge, and common knowledge is also important in the interaction between adults, especially when coordinated actions or collective decisions are needed. In legal relations, it is necessary to assume that the law is common knowledge, which helps to ensure that the legal relationship between individuals and the state is clear and enforceable. In the economic decisions of governments or companies, common beliefs are essential to understanding market behavior, because market value is often determined by people's common beliefs. Although common knowledge is an idealized concept, it is seen as a necessary assumption in law and policy making to ensure universal compliance and enforcement of the law. Game theory uses the concept of common knowledge to describe perfect rationality, and its absence can lead to different strategy choices. Common knowledge is also important in social software and cognitive science.

In explaining and preventing social phenomena (e. g., bank runs), or in the intention of understanding others beliefs. Finally, common knowledge also has a clear role in social norms and individual decision-making and can be used through social software to design policies to encourage or deter certain behaviors.

Throughout the history of the development of common knowledge logic, the common knowledge logic system is not developed independently, but is based on the development of modal logic, cognitive logic, dynamic logic and other logic branch systems[3]. To analyze the process of human cognition is the study and development of modal logic system. A proposition may initially be false, but the addition of modal words transforms it into a true proposition, thereby expanding the scope of

DOI: 10.54254/2753-7048/72/2025.21414

human cognition.. With the study of the modal logic deduction, logicians expand the modal language in many directions.

Hintikka has established the cognitive modal logic system. He related the cognitive concepts with the modal words "inevitable" and "possibility", replaced the modal operator with cognitive operators, and studied the single-subject cognitive logic."Subject A knows that proposition φ" can be expressed by the modal operator "□φ" or by the cognitive operator "$K_a\varphi$"."Subject a believes φ" can be represented by the modal operator "◇φ" or the cognitive operator "$B_a\varphi$". The proposition that the subject knows must be true, so "$K_a\varphi\rightarrow\varphi$" is an intellectual axiom. But the proposition that the subject believes is true and false, which is the difference between belief and knowledge.

Prioran has developed a temporal logic system. He adds temporal operators to cognitive language, such as "Fφ" means proposition φ is true at some moment in the future; "Hφ" means proposition φ is true at some moment in the past[4]. Combining the subjects cognition with the tense, with the change of time, the cognitive information changes, and the subject follows the reasoning rules to introduce some new conclusions. Such a cognitive change state will continue indefinitely. Logicians believe that a discrete temporal framework with a starting point such as a finite past and an infinite future is more conducive to the subjects cognition.

Plazaja Focus on studying the logic of common announcement[5]. He combined knowledge operators and behavior operators and developed a dynamic cognitive logic system. The cognition between subjects includes two levels, one level is to reason the new knowledge from the existing knowledge; the other is that the new knowledge causes the change of the subject behavior state and then produces new knowledge. The behavior operator "<π> φ" indicates the state of information φ after the program π is executed in the current state. Of course, in addition to simple behaviors, there are also complex behaviors that change the subjects cognition, such as "<π *> " representing the iteration of the program π.

3. Overview of the research on the beliefs

The common belief is related to the study of "belief" itself, which is an important factor for human beings to know themselves and the world, and is a common phenomenon in human thinking activities. Hot topics in philosophy and logic include the acquisition of knowledge in artificial intelligence, the simulation of human thinking, reasonable reasoning in legal trials, and information security in databases.

The study of "belief" dates back to ancient Greece. The philosopher Plato for the first time explicitly put forward the concept of "belief" from the perspective of epistemology, and made a preliminary study on its essence, object and so on. He believes that belief is the ability and psychological idea that do not contain one-sided opinion and acquire such opinion, it is between imagination, and reason and reason, and its object is something that can be felt. From its definition, we shows that he regards belief as a state of mind. Scholars Hume (D. Hume), Griffith (A. P. Grifith), Armstrong (D. Armstrong), and Moser (P. K. Moser) also studied belief in depth along Platos heart theory. In addition, scholars have studied beliefs from other perspectives. Russell explores the question of belief by combining both mental and physical factors, arguing that " belief is a state of both mental and physical inclusion[6]."Bain (A. Bain) holds a belief behavior theory, arguing that belief has no meaning other than behavior. Braithwaite (R. B. Braith waite), Pierce (S. Pierce), Montague (W. P. Montague), and Quine (Quine) all see belief as a behavioral tendency. Price (H. H. Price) and Russell regard belief as a propositional attitude belief is a positive attitude of the subject to the proposition. Some scholars regard belief as a kind of belief and certain knowledge. These discussions pertain to beliefs at a conceptual level. In recent years, some scholars have sorted out the definition of belief of predecessors, some scholars have made in-depth discussions on the beliefs of some philosophers, and some scholars have studied the concepts of belief with knowledge and truth.

DOI: 10.54254/2753-7048/72/2025.21414

On the basis of belief concept analysis, scholars try to study it formally through logical methods.

In 1951, the philosopher von Wright (G. H. von Wright) first proposed the use of logical formalization to explore knowledge and beliefs, and described them in axiom. In 1962, the logician Syndica (J. Hintikka) proposed a semantic model that uses the possible world semantics to establish the cognitive proposition.

In cognitive logic, knowledge and belief are distinguished mainly through T axioms. For knowledge, if a certain subject knows a certain proposition p, then that proposition p is true. But for beliefs, if the subject a believes in the proposition p, then the proposition p is not necessarily true. This is the most salient distinction between knowledge and belief. Although the logical methods of exploring knowledge and belief are similar, they are still different. Therefore, scholars often refer to the logic of exploring belief as the logic of belief.

In belief logic, scholars describe belief as a modal operator, giving its language and model, the possible world for semantic interpretation, and establishing the axiom system. On this basis, add preference to it. Performed preference studies; added the declaration operator and conducted a dynamic study, including honest declaration and lying[7].

Researchers make public and private declarations of their research, among other things; additionally, belief is a "degree" concept, signifying the intensity of belief. Scholars have made formal studies of the strength of belief, adding the degree of belief to the belief operator in formalized form as $B^a\varphi$, indicating that the subject **a** believes information in **σ** degree.

4. Change and challenges

4.1. Classical belief logic

4.1.1. Definition (language)

Given a countable atomic proposition set P, An agent subject set A, subject $a \in A$, and a belief logical language (Lb) is defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B^a\varphi$$

(where $p \in P$) $B^a\varphi$ indicating that the subject **a** believes φ belief.

The dual of $B^a\varphi$ is defined as $\neg B^a \neg\varphi$ and is represented by B^a φ represents that subject a believes that information is possibilities.

The combination formula containing other conjunctions can be defined by the conjunctions and.among

(1) $\varphi \rightarrow \psi$ is defined as $\neg(\varphi \wedge \neg\psi)$;

(2) $\varphi \vee \psi$ is defined as $\neg(\neg\varphi \wedge \neg\psi)$;

(3) $\varphi \leftrightarrow \psi$ is defined as $\neg(\varphi \wedge \neg\psi) \wedge \neg(\psi \wedge \neg\varphi)$;

(4) $\bot = \varphi \wedge \neg\varphi = \neg\top$

4.1.2. Definition (model)

Given a countable atomic proposition set P, a agent set A, the belief logic A model is a triple multiple group M=(W, R, V), where W is a non-empty set of possible worlds; R: A→ (W×W), writing each R (a) as R_a. Ra represents the accessibility relationship of subject a, with constancy; transitivity and Euclidean property.

V:P→P (W) is an assignment function that assigns each propositional letter a set of possible worlds.

DOI: 10.54254/2753-7048/72/2025.21414

4.1.3. Definition (semantics)

Given a belief logic model M (W, R, V), a possible world w∈W,

We define a formula φ on w in M as true (denoted M, w⊨φ) as follows:

M, w⊨p if and only if w∈V (p);

M, w⊨¬φ if and only if M, w⊭φ ;

M, w⊨φ∧ψ if and only if M, w⊨φ and M, w⊨ψ ;

M, w⊨Bαφ if and only if for all v∈W, if wR_av, then M, v⊨φ

A compound formula is true on any of the possible worlds wϵW in a belief logic model M,

We call the formula true in M, denoted as M; a composite formula is true in any possible world wϵW in the belief logic model M, we call the formula valid in the belief logic model class, written as.

4.1.4. The Axioms of belief logic

The axiom system K、D、4、5 is defined as follows

Prop: all axiom and rules in a propositional logic;

Ax 1: (Baφ∧Ba (φ→ψ)) →Baψ ; (axiom K)

Ax 2: ¬Ba ⊥; (axiom D)

Ax 3: Baφ→BaBaφ ; (axiom 4)

Ax 4:¬Baφ→Ba¬Baφ ; (axiom 5)

Reasoning Rule: Ru1 (Modus ponens (MP)):

If⊢φ, and⊢φ→ψ, then⊢ψ .

Ru 2 (Generalization (Gen)):

If⊢φ , then the $\vdash B_a\varphi$.

The traditional belief logic can only depict the beliefs of a certain subject, but not the common and common beliefs between multiple subjects.

4.2. Common belief Logic

According to Johan van Benthem's interpretation of the cognitive logic with "common announcement" in his article "logics of communication and change", the common announcement is a cognitive event in which all agents are transparently informed that a formula is now established[8]. This is modeled by a modal operator [!ϕ]. The formula shaped like [!ϕ] is read as "true after the declaration of ϕ". If an operator Cb ϕ is added to express that ϕ is common knowledge between the subject group b, a common declaration logic (PAL-C) with common knowledge is obtained. Language L_{pal} and L_{palc} are interpreted in the standard model of epistemological logic. If we remove the mode operator [!ϕ] from the common announcement logic PAL-C, out of the thought of technical,we get a standard cognitive logic with common beliefs.Consider the callback later.

4.2.1. Definition (language)

Given a countable atomic proposition set P, An agent set A, subject a∈A, and the common belief logical language Lcb are defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi\wedge\varphi \mid Cb\varphi$$

DOI: 10.54254/2753-7048/72/2025.21414

(where p∈P) Cbφ is express that φ is common belief among agents B.

4.2.2. Define the model (Epistemic models)

Let a finite set of propositional variables P and a finite set of agents N be given.

An epistemic model is a triple M = (W, R, V) such that

- W = ∅ is a set of possible worlds.
- R: N → $\wp$(W × W) assigns an accessibility relation R(a) to each agent a.
- V : P → $\wp$(W) assigns a set of worlds to each propositional variable.

In epistemic logic the relations R(a) are usually equivalence relations. In this paper, we treat the general modal case without such constraints—making 'knowledge' more like belief, as observed earlier.

The semantics are defined with respect to models with a distinguished 'actual world': M,w.

4.2.3. Definition (Semantics)

Let a model M, w with M = (W, R, V) be given. Let a ∈ N, B ⊆ N, and ϕ∈ Lcb. For atomic propositions, negations, and conjunctions we take the usual definition. The truth conditions for the other cases run as follows:

M,w |= □aϕ iff M, v |= ϕ for all v such that (w, v) ∈ R(a)

M,w |= CBϕ iff M, v |= ϕ for all v such that (w, v) ∈ R(B)+,

where R(B) = $\cup_{a\epsilon B}$ R(a), and R(B)+ is its transitive closure.

Here, we chose to define common belief as a transitive closure, common belief is defined as the reflexive transitive closure.

5. Conclusion

Up to this point, we have preliminarily established a modal logic system for common belief. It provides a systematic method for studying the exchange between individual beliefs and common beliefs. We have presented the language, models, and semantics of simple common belief logic. In the multi-agent setting of modal logic Lcb, the common belief is clarified as a fundamental concept.

Currently, modal logic Lcb lacks further axiomatization and has not yet completed the reliability proof, resulting in a lack of proven reliability for natural language inference. Future work will not only focus on proving the reliability of theorems, but also aim to demonstrate the incompleteness of modal logic Lcb, ultimately leading to a fully developed logical outcome. There is a complex connection between common belief and common knowledge. Theoretically, the semantics of common belief are weaker than those of common knowledge, as in our daily lives we do not usually require the user to perform the reflexive act of "believing that one believes" when using the term "belief"; and common beliefs in some cultural and special circumstances may not require the dynamic common knowledge of the prior "public announcement" behavior. These deeper topics are subjects for further formalized research. Studying them is of significant importance, as it can not only develop logical theories but also provide methods and theories for research in cognitive science and artificial intelligence. It is worth further exploration in the future.

References

[1] *Plato. The Republic [M]. Beijing: The Commercial Press, 1986.*

[2] *Hintikka J. Knowledge and belief: an introduction to the logic of two notions[M]. New York: Cornell University Press,1962.*

DOI: 10.54254/2753-7048/72/2025.21414

[3] van Benthem, J.,van Eijck, J.,Gattinger, M.,& Su, K.(2018).Symbolic Model Checking for Dynamic Epistemic Logic–S5 and Beyond. Journal of Logic and Computation, 28(2),367-402.
[4] Prioran. Past,present and future[M].Oxford: Oxford University Press,1967.
[5] Plazaja.Logics of public communications[C]//EMRICH M LPFEIFER M S,HADZIKADIC M,et al.Proceddings of the 4th Inter- national Symposium on Methodologies for Intelligent Systems,1989 .
[6] Russell, B. Human Knowledge [M]. Beijing: The Commercial Press, 1983, page 137.
[7] van Benthem, J., van Eijck, J., Gattinger, M., Su, K.: Symbolic model checking for dynamic epistemic logic. In: van der Hoek, W., Holliday, H.W., Wang, W.f.(eds.) Proceedings of The Fifth International Conference on Logic, Rationality and Interaction (LORI-V) in Taipei, Taiwan, October 28-30, 2015. pp. 366–378 (2015).
[8] van Benthem, J., van Eijck, J., Kooi, B.: Logics of communication and change.Information and computation 204(11), 1620–1662 (2006), http://dx.doi.org/10.1016/j.ic.2006.04.006.