The International Conference on Computing Innovation and Applied Physics (CONF-CIAP 2022)
DOI: 10.54254/2753-8818/2/20220086

# Elegant Theory of Complex Analysis

**Yisun Chen**

Shanghai Foreign Language School Affiliated to SISU, No.295 Zhongshan Bei Yi Rd, Shanghai, PRC 200083

fz0528@shisu.edu.cn

**Abstract.** A complex number is an element in a number system containing both real numbers and the imaginary unit $i$, satisfying $i^2 = -1$. Since their discovery, complex numbers have been a powerful means of mathematical calculation. Complex analysis is a part of mathematical analysis that investigates complex numbers and their analyticity, holomorphicity, etc. Many renowned mathematical giants once had their own research in complex analysis, such as Cauchy, Gauss, Euler, etc. On the grounds that it deals with functions of complex numbers, complex analysis is a helpful area in the whole mathematics field. There are plenty of applications of complex analysis in both the mathematical field and the physics field. In this paper, the history of complex numbers and complex analysis is presented. Also, some contents of complex variables are shown, including the basic properties of complex numbers, the derivative and integral of functions of complex numbers, and several critical theorems in the area of complex analysis.

**Keywords:** Complex number, Complex analysis, Complex variables

## 1. Introduction

Complex analysis, also known as the theory of functions of a complex variable, is a part of mathematical analysis that investigates complex numbers and their analyticity, holomorphicity, etc. It is based on functions of complex variables and complex numbers.

Complex analysis is definitely an important branch of math. For students who majors in math, complex analysis is such an appropriate topic to be a capstone of their undergraduate study as it integrates many of the topic in the mathematical field. Also, it has connections with other branches of mathematics——the theorems and results in complex analysis also appear in lots of other areas of research in both pure and applied mathematics. With a solid foundation of experience of complex analysis, students are able to form a more clear and deeper mathematical framework. For the whole scientific community, it is imperative as well since complex analysis is enable to generalize the analytical method from real variables to complex variables. Many famous mathematical giants once had their own research in complex analysis——Cauchy, Gauss, Euler, just to name a few.

As the basis of complex analysis, complex number itself is of great importance. Complex numbers are a powerful means of mathematical calculation such as multiplying vectors and rotating the angles. These numbers contain both the real numbers and the imaginary part called $i$. A claim has to be made here that many people consider “real numbers” as the numbers which exist and which people come across every day. This concept is in some sense confounding because imaginary numbers aren’t numbers that “do not exist” in the other way. The concept of numbers is abstract and mathematicians

DOI: 10.54254/2753-8818/2/20220086

have always regarded "real numbers" and "imaginary numbers" as on the same logical level. The utility of complex numbers can simplify the calculation of mathematical or physical problems to a rather great extent. The existence of complex plane——formed by the real axis and the imaginary axis ——introduces the concept of "numbers" from one-dimensional to two-dimensional. For the whole mathematic field, the appearance of complex numbers brings renewal of notion. Also, the area of complex numbers is a synthesis of algebra, trigonometry, geometry, etc. Thus, it has many real-life applications.

Complex analysis is an extremely useful tool because it has a quite high application value. As computers appeared in the end of the last century, there are a wider range of applications of complex analysis. Many scientific problems can be easily solved when using the complex variables besides the real domain. Plenty of areas of research start from complex analysis, including pure and applied mathematics, quantum mechanics, electrical engineering, nuclear engineering, hydrodynamics, etc. For instance, complex analysis has an intrinsic property that it is two dimensional. Many theorems and results in complex analysis can be shown geometrically on the complex plane. Thus, it can be related to some areas, especially in physics, where the problems are presented in a plane, like electric fields and ideal fluid flow. [1] Also, complex analysis can be used in aerodynamics to design various kinds of wings. [2] In the area of electrical and mechanical engineering, to be more specific, it is common to find the time function using integral transform in complex analysis. In applied mathematics, algorithms of complex analysis have been used to find conformal maps [3] and solve Riemann-Hilbert problems. [4] In physics, the most classical problems are about fluid mechanics, like the Hele-Shaw problem. Even in biology, a field that one has never thought of being related to analysis, complex analysis can still contribute to the area of signal processing using Fourier transform method. Therefore, the vast range of applications makes complex analysis even more critical.

## 2. Complex analysis

*2.1 History development of complex numbers*

The history of complex numbers can date back to the 16th century. Complex numbers originated from mathematicians' demand for solving cubic equations. During the 1500s, mathematicians were confused by finding the solutions of the cubic equation $x^3 + cx = d$.

One of the mathematicians, Targaglia, eventually found out the formula to find the solutions. He defined two numbers $u$ and $v$ such that $\begin{cases} u - v = d \\ uv = (\frac{c}{3})^3 \end{cases}$. Then, the solution was that $x = \sqrt[3]{u} - \sqrt[3]{v}$.

Targaglia then told the formula to another mathematician, Girlamo Cardano but without the proof. However, Cardano combined the knowledge in geometry and proved the formula. He then rewrote it into a more direct way:

$$x = \sqrt[3]{\sqrt{(\frac{d}{2})^2 + (\frac{c}{3})^3} + \frac{d}{2}} - \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} - \frac{d}{2}} \tag{1}$$

Cardano figured out the solutions to the cubic equation, whereas he failed to explain the confusion that when $c$ and $d$ are negative, there would be square roots of negative numbers in the solution. This paved the way for the introduction of $\sqrt{-1}$.

Rafael Bombelli, who was almost an engineer instead of mathematician, once noticed Cardano's solution to the cubic equation. He regarded Cardano's method as a synthesis of the real domain and something that could be discovered as a brand-new area in mathematics. He identified $\sqrt{-1}$ to be the phrase "plus of minus". Also, he identified the opposite $(-\sqrt{-1})$ as "minus of minus" ($meno\ di\ meno$) and abbreviated it as $mdm$. [5] He then found out the identity that $i^2 = -1$ and made the claim that "plus of minus times minus of minus makes minus", where the last "minus" actually meant $-1$. The claim was written in mathematical equation, which was $(i)(-i) = -1$. [6]

DOI: 10.54254/2753-8818/2/20220086

After he identified complex numbers, he demonstrated the combination of real numbers and complex numbers. He claimed that the two parts of a solution of the equation $x^3 = 15x + 4$, $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$, only differed in the sign. Thus, he wrote that $\begin{cases} \sqrt[3]{2+\sqrt{-121}} = a + b\sqrt{-1} \\ \sqrt[3]{2-\sqrt{-121}} = a - b\sqrt{-1} \end{cases}$. With his correct proof of this expression, he indicated that real numbers are actually complex numbers, too. In the 16th century, Cardano also came across the issue with the "discovery" of imaginary numbers. However, along with this quadratic equation, he almost ignored it and commented that the imaginary numbers were "as subtle as they are useless". [7] As Bombelli made this discover, he was seen as the one who initiated the world of "new algebra". [8] Since then, more and more mathematicians have developed the area of imaginary quantities.

During the late 17th century, René Descartes first mentioned the term "imaginary" to express complex numbers. Besides, he tried to associate the complex numbers with geometrical meanings, which he succeeded when solving the equation $z^2 = az - b^2$. Several decades later, German mathematician Gottfried Wilhelm Leibniz started to find some properties about complex numbers when doing related calculations. For example, he found that $\sqrt{a+\sqrt{-b}} + \sqrt{a-\sqrt{-b}} = \sqrt{2a + 2\sqrt{a^2+b}}$ by squaring the left part of the equal sign. He also summed up the previous experience and proved those mathematicians' formula. Furthermore, his most valuable contribution is that he officially used the term "imaginary". Unlike Descartes, Leibniz explained the use of the word "imaginary"——the amphibian between being and not being——and gave meaning to this new terminology. He set the beginning of modern research on complex quantities.

The next significant development of complex numbers was by Leonhard Euler. He provided the most basic and fundamental concept in the current field of complex numbers——he coined the symbol $i$ to stand for $\sqrt{-1}$. That means, to be specific, $i = \sqrt{-1}$. [9] However, the mathematician didn't believe that complex numbers did exist. In 1747, d'Alembert proved that there were no "categories" in the field of complex number and that every complex number can be written in the form of $a + bi$. This eliminated a bunch of paradoxes stating that complex numbers could be classified into a few categories.

As complex numbers are rapidly developing in their algebraic meanings and properties, mathematicians were also developing the field for complex numbers' geometric possibilities. During the same time of Leibniz, mathematician John Wallis first used a geometric expression of complex numbers. His finding was based on Descartes's method, which used the Pythagorean theorem and the properties of isosceles triangles. Meantime, he presented a concept that complex numbers can be demonstrated as a segment in a plane. Later, William Rowan Hamilton added on his theory that the plane should be a kind of "$abi$" plane, which is parallel to the commonly-used xy-coordinate system.

After few decades, some mathematicians discovered some imperative formulas and properties. For instance, Abraham de Moivre put forward his theorem that $(\cos(\theta) + i\sin(\theta)^n = \cos(n\theta) + i\sin(n\theta)$. Also, Leonhard Euler discovered the formula that $e^{i\theta} = \cos\theta + i\sin\theta$. These both count for the fundamental basis of complex numbers.

As Carl Friedrich Gauss proved the fundamental theorem of algebra using complex numbers, [10] the development of the complex numbers area has been increasing promptly. More and more core theorems have appeared, such as the Cauchy Riemann Theorem, the residue theorem, the Cauchy's integral theorem etc. [11] Moreover, a novel important branch of mathematics appeared——complex analysis. In this area, mathematicians focus on problems not only related to pure and applied math, but also all kinds of applications, from electrical engineering to quantum mechanics. For instance, a new way to solve linear boundary-value problems, which is called "unified method" of Fokas has caught many people's attention. [12] For years, researchers have long been trying for scientific breakthrough and solving real-life problems using complex numbers and complex analysis.

DOI: 10.54254/2753-8818/2/20220086

*2.2 Complex analysis*

First, we have to be clear with the definition of complex numbers. It is an element of a number system that contains real numbers and an element $i$, the imaginary unit. $i$ satisfies that $i = \sqrt{-1}$. Every complex number can be written in the form $a + bi$ where $a$ and $b$ are real numbers and $a$ is called the real part, $b$ is called the imaginary part. The commonly used symbol for complex numbers is $z$.

From the definition, we can find the most basic rule of $i$, which is $i^2 = -1$.

Then the complex plane is introduced, which is a Cartesian plane. The complex plane has one real axis and one imaginary axis. Suppose a complex number $z = x + iy$, the Cartesian representation will be $(x, y)$ on the complex plane. Also, another critical representation is the polar representation of complex numbers. We use $\rho$ to stand for the modulus of $z$, which means the distance from the point $z$ to the origin. The modulus of a complex number can also be written as $|z|$. We use $\theta$ to represent the argument of $z$. Thus, $z = \rho(\cos\theta + i\sin\theta)$ and $\rho = \sqrt{x^2 + y^2}$. Based on Euler's discovery, $z = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$.

The complex conjugate of $z$ is a fundamental but important concept. The symbol we use is $\bar{z}$. Suppose $z = x + iy$, then $\bar{z} = x - iy$ and the Cartesian representation will be $(x, -y)$. As to the polar representation, $\rho$ of $\overline{z}$ equals $\rho$ of $z$ since $\sqrt{x^2 + y^2} = \sqrt{x^2 + (-y)^2}$.$\theta$ of $\bar{z}$ equals the opposite number of $\theta$ of $z$. A crucial formula is that $z\bar{z} = x^2 + y^2 = \rho^2$.

After basic knowledge and definition, some simple properties of complex numbers are mentioned, such as the addition, subtraction, multiplication, and division of complex numbers.

Suppose $z_1 = x_1 + iy_1 = \rho_1 e^{i\theta_1}\ and\ z_2 = x_2 + iy_2 = \rho_2 e^{i\theta_2}$

Then, $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$ $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1)\ and\ also\ z_1z_2 = \rho_1\rho_2 e^{i(\theta_1+\theta_2)} = \rho_1\rho_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$ $\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{i(\theta_1-\theta_2)}\ (z_2 \neq 0)$.

There are properties between any two complex numbers. The first is that we cannot compare complex numbers directly, which means we cannot write $z_1 > z_2$. Instead, we can only compare the modulus of two complex numbers, such as $|z_1| > |z_2|$. The second property is that the triangle inequality in the real domain can also be applied to complex numbers. We get the consequence that for any two complex numbers, $\big||z_1| - |z_2|\big| \leq |z_2 - z_1|$.

After knowing many basic calculations and operations of complex numbers, we are able to manage some functions of complex numbers. We define $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ as a function of $z$ where $u(x, y)$ is the real part of $z$ and $v(x, y)$ is the imaginary part of $z$. The mapping of the function is also worth understanding. For instance, for the function $w = z^2$, we know that $u = x^2 - y^2$ and $v = 2xy$. Thus, we can think of transforming the function geometrically from the $xy$ plane to the $uv$ plane. After the transformation, it's easy for us to find the images of the certain hyperbolas.

Next, we now define the continuity and differentiability of functions $w = f(z)$, which paves the way for the derivatives and integrals of complex functions. We say that $f(z)$ is continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

Now, it's time to look at the concept of derivatives in the area of complex numbers. Let $f$ be a function whose domain contains a neighborhood $|z - z_0| < \varepsilon$ of a point $z_0$. The derivative of $f$ at $z_0$ is $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, and the function $f$ is said to be differentiable at $z_0$ when $f'(z_0)$ exists.

After learning the definition of the derivatives of complex numbers, we can skip to one of the most fundamental yet significant equations in the field of complex numbers——Cauchy-Riemann Equations. To proof the equations are correct, we have to make some preparations. We first assume that $f'(z_0)$ exists, and we have $z_0 = x_0 + iy_0$, $\Delta z = \Delta x + i\Delta y$, $\Delta w = f(z_0 + \Delta z) - f(z_0) =$

$[u(x_0+\Delta x, y_0+\Delta y)+iv(x_0+\Delta x, y_0+\Delta y)]-[u(x_0,y_0)+iv(x_0,y_0)]$. Then, by dividing the last two equations, we get:

$\frac{\Delta w}{\Delta z}=\frac{u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)}{\Delta x+i\Delta y}+i\frac{v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)}{\Delta x+i\Delta y}$. Next, we do two caseworks to solve the equation.

1. Horizontal approach

We can reach that $\Delta y=0$ and let $\Delta x$ tend to be $0$. The equation above will turn into:

$$f'(z_0)=\lim_{\Delta x\to 0}\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x}+i\lim_{\Delta x\to 0}\frac{v(x_0+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}$$

which is the same as $f'(z_0)=u_x(x_0,y_0)+iv_x(x_0,y_0)$ (2)

2. Vertical approach

We can write $\Delta x=0$ and let $\Delta y$ tend to be $0$. The equation will turn into:

$$f'(z_0)=\lim_{\Delta y\to 0}\frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{\Delta y}-i\lim_{\Delta y\to 0}\frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}$$

which is the same as $f'(z_0)=-i\left[u_y(x_0,y_0)+iv_y(x_0,y_0)\right]$ (3)

As we assumed that $f'(z_0)$ exists, there must be $(2)=(3)$. Thus, we can get the $Cauchy-Riemann\ equations$: $u_x(x_0,y_0)=v_y(x_0,y_0)\ and\ u_y(x_0,y_0)=-v_x(x_0,y_0)$.

Notice that in general, these conditions are not sufficient to get $f'(z_0)$ to exist but under very mild extra conditions on $u$ and $v$ they are sufficient.

Now we move on to the integrals of complex functions. For the definite integrals of functions $w(t)$, when $w(t)=u(t)+iv(t)$ where $u$ and $v$ are real-valued numbers, the definite integral of $w(t)$ over an interval $a\le t\le b$ is defined as $\int_a^b w(t)dt=\int_a^b u(t)dt+i\int_a^b v(t)dt$. Given this definition, we can also have $Re\int_a^b w(t)dt=\int_a^b Re[w(t)]dt$ and $Im\int_a^b w(t)dt=\int_a^b Im[w(t)]dt$. Then, we mainly focus on the contour integrals of complex functions. The definition of this kind of integral is in terms of the values of $f(z)$ along a given contour $C$, which extends from a point $z=z_1$ to another point $z=z_2$. It is written as $\int_C f(z)dz$ or $\int_{z_1}^{z_2} f(z)dz$. After knowing this, we assume that $z=z(t)\quad(a\le t\le b)$ represents a contour $C$, which extend from a point $z_1=z(a)$ to another point $z_2=z(b)$. Also, we suppose that the function is continuous on the interval $a\le t\le b$. Now, we can eventually define the contour integral of $f$ along $C$ in terms of the parameter $t$:

$$\int_C f(z)dz=\int_a^b f[z(t)]z'(t)dt \tag{4}$$

In an easy way, we can also call this integral the integral of a function along a path, where path is a continuous function.

Now, let's move on to a specific, also special, topic in the area of integrals of complex numbers: integrals along contours.

Before we look at integrals along contours, we should first focus on the definition of "contour".

We define a set of points $z=(x,y)$ to be an arc if there exists

$$x=x(t)\text{ and }y=y(t)\quad(a\le t\le b) \tag{5}$$

given that $x(t)$ and $y(t)$ are both continuous functions of the real number $t$ in the complex plane.

The definition above naturally produces a mapping of $t$ into the $xy$ plane. For the sake of convenience, we can just describe the points of the arc $C$ through $z=z(t)\quad(a\le t\le b)$ where $z(t)=x(t)+iy(t)$. Then, the arc $C$ is called a "simple arc" as long as the arc is simple and $z(t_1)\ne z(t_2)$ given that $t_1\ne t_2$. This also means that the arc does not cross itself at any point. Moreover, when arc $C$ is "simple" except that $z(b)=z(a)$, we can say that arc $C$ is a "simple closed curve". A curve like this has a positive orientation when it is counterclockwise.

After the basic definition, we move forward a single step. Now, we suppose that the parameter $t$ is also a function of some real numbers $\tau$ and a real-valued function $\Phi$, which means $t=\Phi(\tau)\quad(\alpha\le\tau\le\beta)$. Then, we can rewrite $z$ by $z=Z(\tau)\quad(\alpha\le\tau\le\beta)$ where $Z(\tau)=z[\Phi(\tau)]$.

Now we take the derivative of $x(t)$ and $y(t)$ and suppose that $z'(t) = x'(t) + iy'(t)$ for $x'(t)$ and $y'(t)$ are continuous on the interval $a \le t \le b$, then the arc is called a "differentiable arc" with the function $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ integrable over the interval $a \le t \le b$. Notice that according to the definition made in calculus, the length of arc $C$ can be represented as $L = \int_a^b |z'(t)|dt$. With our assumption above that $t = \Phi(\tau)$, we can get $L = \int_\alpha^\beta |z'[\Phi(\tau)]|\Phi'(\tau)d\tau$. As $Z'(\tau) = z'[\Phi(\tau)]\Phi'(\tau)$, we can also have the expression for the length that $L = \int_\alpha^\beta |Z'(\tau)|d\tau$.

If $z = z(t)$ is a differentiable arc and $z'(t) \neq 0$ on the entire interval $a < t < b$, then the unit tangent vector has the expression $\mathrm{T} = \frac{z\prime(t)}{|z'(t)|}$. Since $t$ can be any value on the interval $a < t < b$, T is continuous when it turns. An arc that has these properties can be referred to as "smooth". Thus, if an arc $z = z(t) \quad (a \le t \le b)$ is smooth, we can have that $z'(t)$ is continuous on the interval $a \le t \le b$ and that it is nonzero on the interval $a < t < b$.

Now, finally, we can reach the definition of "contour". A contour, as known as piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Only if the contour $C: z = z(t) \quad (a \le t \le b)$ has the property that $z(a) = z(b)$ can $C$ be called a "simple closed contour".

After we know the definition of contours and the integral of contours, some very important theorems have to be mentioned in order to solve some specific problems.

The first theorem is Cauchy-Goursat Theorem. Let $z = z(t)$ $(a \le t \le b)$ represents a simple closed contour $C$ which is described counterclockwise. The condition for applying the theorem is that the function $f$ is analytic at each point interior to and on the contour $C$. We start with this equation: $\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$ and $f(z) = u(x,y) + iv(x,y), z(t) = x(t) + iy(t)$ . Since $f[z(t)]z'(t)dt = \{u[x(t), y(t)] + iv[x(t), y(t)]\} \cdot [x'(t) + iy'(t)]$ , we can get $\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy$.

Then, we have to apply the Green's theorem in calculus, which says that $\int_C Pdx + Qdy = \iint_R (Q_x - P_y)dA$. Since $f$ is continuous on $R$ and is analytic, $u$ and $v$ are also continuous on $R$. For the same reason, since $f'$ is continuous on $R$, the first-order partial derivatives of $u$ and $v$ are also continuous on $R$. Thus, we can use the theorem and write $\int_C f(z)dz = \iint_R (-v_x - u_y)dA + i\iint_R (u_x - v_y)dA$.

However, problem arises that according to the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$, so the two double integrals must have the value of 0. That gives us the Cauchy-Goursat Theorem: If a function $f$ is analytic at all points interior to and on a simple closed contour $C$, then

$$\int_C f(z)dz = 0. \tag{6}$$

The next theorem to introduce is the Cauchy integral formula. The theorem says that given that function $f$ is analytic anywhere inside and on a simple closed contour $C$, which is positively oriented, if $z_0$ is any point interior to $C$, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0}. \tag{7}$$

Also, the formula can be written as $\int_C \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0)$ for easy calculation.

The theorem and formula indicate that if $f$ is analytic within and on a simple closed contour $C$, the values of $f$ inside the contour $C$ will be determined only by the values of $f$ on $C$. Thus, when it is hard to find the values of a function inside a simple closed contour, we may think of this formula and transfer the key to the problem solving to finding the values on the contour, which is usually easier for us to find the solution.

At last, an imperative theorem has to be introduced, which is the Cauchy's residue theorem. Before the theorem is shown, we have to recall the definition of two things.

The first is the Laurent's theorem. Suppose that $f$ is analytic throughout a domain $r_1 < |z - z_0| < r_2$, which has a center at $z_0$, and $C$ is any positively oriented simple closed contour which is around $z_0$ and lies in the domain. Then, at each point in the domain, there is always a special representation for $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \tag{8}$$

where $a_n$ and $b_n$ have the property that $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$ $(n = 0,1,2,\cdots)$, $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$ $(n = 1,2,\cdots)$.

This series representation enables us to expand the function under some specific conditions.

The second definition we are going to recall is the residue. As is introduced before, the Laurent series representation tells us that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \ (0 < |z - z_0| < r_2) \text{ and } b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}} \tag{9}$$

for any positively oriented simple closed contour $C$. When $n = 1$, we have $b_1 = \frac{1}{2\pi i}\int_C f(z)dz$. $b_1$ here is the coefficient of $\frac{1}{z-z_0}$, which is named as the "residue" of $f$ at the singular point $z_0$.
We usually express the residue like this:

$$b_1 = \underset{z=z_0}{Res} f(z). \tag{10}$$

We can then get the equation $\int_C f(z)dz = 2\pi i \underset{z=z_0}{Res} f(z)$. The use of residues can simplify the evaluation of many integrals of complex functions.

Eventually, it's time to introduce the Cauchy's residue theorem. Suppose $C$ is a positively oriented simple closed contour. If $f$ is analytic interior to and on $C$ except for a finite number of singular points $z_k (k = 1,2,\cdots,n)$ inside $C$, then we have:

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z). \tag{11}$$

This is a very strong and powerful theorem because it can make the evaluation of contour integrals of complex functions much easier. If we are going to solve a problem in which a specific contour is provided and the function is given, we can first write the function $f$ in the Laurent series representation and then use the Cauchy's residue theorem.

Here is an example where some of the theorems mentioned above can be applied.
Problem: Let $C_r(z_0)$ be a circle with positive orientation.
(a) For $a \in D_r(z_0)$, show that $\int_{C_r(z_0)} \frac{1}{z-a} dz = 2\pi i$
(b) For $a \notin \overline{D}_r(z_0)$, show that $\int_{C_r(z_0)} \frac{1}{z-a} dz = 0$

(a) According to the conditions we have, since $C: C_r(z_0)$ is a circle with positive orientation, $C$ must be a simple closed contour. We then suppose $f(z) = \frac{1}{z-a}$. Given that $a \in D_r(z_0)$, we can draw the conclusion that the function $f$ is analytic inside and on $C$ except for a singular point $a$ inside $C$. It can be easily discovered that the condition corresponds to the condition of the Cauchy's residue theorem. Hence, we can think of using the theorem to solve the problem. First, let's look back to the theorem again. The theorem says that if a function $f$ is analytic inside and on a simple closed contour $C$, which is described in the positive sense, except for a finite number of singular points $z_k (k = 1,2,\cdots,n)$ inside $C$, then $\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z)$. Thus, $\int_{C_r(z_0)} \frac{1}{z-a} dz = 2\pi i \underset{z=a}{Res} f(z) = 2\pi i \cdot 1 = 2\pi i$.
(b) Since $a \notin \overline{D}_r(z_0)$, $f(z) = \frac{1}{z-a}$ is well defined in the contour $C_r(z_0)$. That means that the function $f$ is analytic at all points inside and on the positively oriented simple closed contour $C_r(z_0)$. This

DOI: 10.54254/2753-8818/2/20220086

looks much like to the prerequisite of the Cauchy-Goursat theorem, so let's recall the theorem.The Cauchy-Goursat theorem tells us that if a function $f$ is analytic at all points interior to and on a simple closed contour $C$, then $\int_C f(z)dz = 0$. Therefore, in our case, $\int_{C_r(z_0)} f(z)dz = \int_{C_r(z_0)} \frac{1}{z-a} dz = 0$.

## 3. Conclusion

Complex analysis plays an important role in the mathematics field. Since complex numbers contains not only real numbers but also the imaginary part $i$, the basic properties are more complicated than those of real numbers. It's vital to always keep in mind those properties and methods, especially introduced in this paper, when solving problems related to complex analysis. However, with complicated properties, complex analysis has the advantage of simplifying calculation. This is the very reason why it is imperative and can be applied in so many scientific fields. As there are numerous applications of complex analysis in mathematics and physics, it is obvious that complex analysis is also a mathematical area with great scientific potential. Once we dig further in this area, more elegant results may appear in the future.

## References

[1] Linda Cummings, Stefan Llewellyn Smith, Paul Martin, and Bartosz Protas (2015) Modern Applications of Complex Variables: Modeling, Theory and Computation
[2] Edgardo V. Gerck, Ed Gerck (2019) Overview of Complex Analysis and Applications https://www.researchgate.net/publication/331025041_Overview_of_Complex_Analysis_and_Applications
[3] R. Wegmann, Methods for numerical conformal mapping. In Handbook of complex analysis, geometric function theory, (R. Kuehnau, ed.), vol. 2, 351–477, Elsevier, 2005.
[4] S. Olver, A general framework for solving Riemann–Hilbert problems numerically, Numer. Math. 122 (2012), 305–340.
[5] Bagni, G. T. (2009) Bombelli's Algebra (1572) and a new mathematical object. For the Learning of Mathematics.
[6] Berlinghoff, W. P. and Gouvea, F. Q. (2002) Math through the ages. Farmington, Maine: Oxton House Publishers, LLC.
[7] Cardano, H. (1545). Artis magnae, sive de regulis algebraicis, liber unus. (n.p.): Joh. Petreius, Ch. 37, Rule II.
[8] Hodgkin, L. (2005) A history of mathematics: From Mesopotamia to modernity. New York, New York: Oxford University Press.
[9] Burton, D. M. (2011) The history of mathematics: An introduction. New York, New York: The McGraw-Hill Companies, Inc.
[10] Christen Peters (2018) The Reality of the Complex: The Discovery and Development of Imaginary Numbers.
[11] Kline, M. (1972). Mathematical Thought from Ancient to Modern Times. Vols. 1-3. New York: Oxford University Press.
[12] A. S. Fokas, A unified approach to boundary value problems, SIAM, Philadelphia, 2008.