Order-six Complex Hadamard Matrices Constructed by Schmidt Rank and Partial Transpose in Operator Algebra

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Abstract. Hadamard matrices play a key role in the study of algebra and quantum information theory, and it is an open problem to characterize 6×6 Hadamard matrices. In this paper, we investigate the problem in terms of the Schmidt rank. The primary achievement of this paper lies in establishing a systematic approach to generate 6×6 Hadamard matrices and H-2 reducible matrices through partial transpose. First, if the Schmidt rank of a Hadamard matrix is at most three, then the partial transpose of the Hadamard matrix is also a Hadamard matrix. Conversely, if the Schmidt rank is four, then the partial transpose is no longer a Hadamard matrix. Second, we discuss the relationship between Schmidt rank and H-2 reducible matrices. We prove Hadamard matrices with Schmidt-rank-one are all H-2 reducible, and prove that some Schmidt-rank-two matrices are H-2 reducible. Finally, we confirm that the partial transpose of an H-2 reducible Schmidt-rank-one or two Hadamard matrix remains H-2 reducible.

Keywords: Hadamard matrix, Quantum Operator, Schmidt rank, partial transpose, H-2 reducible

1. Introduction

A complex Hadamard matrix is a matrix with entries of equal modulus and orthogonal rows and columns. Because of its unitary nature, each element takes the form $\frac{1}{\sqrt{n}}e^{i\theta_{jk}}$.

Hadamard matrices have extensive applications in various fields, including spectroscopy, error correction codes, signal processing, and cryptography [1]. The emergence of quantum computing has further fueled interest in Hadamard matrices, as they facilitate the transformation of ground state bits into superimposed qubits, which enables high-level parallel computation [2].

Higher-order Hadamard matrices specifically offer the advantage of generating “qudits” with an increased number of superimposed bases, thus enabling enhanced parallelization. Hadamard matrices of orders one through five have been completely classified, leaving the order-six matrices as the smallest unresolved case. Specifically, the study of order-six Hadamard matrices has garnered considerable attention due to the associated MUB-6 problem, which asks for the maximum number of mutually unbiased bases in $\mathbb{C}^6$ [3, 4]. It is established that if $n = p^k$ where p is prime and $k > 0$, $n + 1$ MUBs can be constructed in $\mathbb{C}^n$. Order six remains an intriguing case, because six is the first integer great than one that is neither prime nor prime power [5].

To approach the MUB-6 problem, we need to progress towards characterizing and generating order-six Hadamard matrices. This paper explores the potential of Schmidt rank. We investigate the conditions under which a Hadamard matrix remains a Hadamard matrix after partial transpose. The Schmidt rank emerges as a valuable criterion for assessing this property [6]. This study concludes

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that the partial transpose of an order-six Hadamard matrix with Schmidt-rank-four cannot yield another Hadamard matrix. We further explore Schmidt rank's relationship with H-2 reducible matrix, a class of parameterized Hadamard matrices. This study establishes that Hadamard matrices with Schmidt-rank-one are all H-2 reducible, and Schmidt-rank-two matrices are likely to be all H-2 reducible.

2. Preliminaries

In this section, we introduce the fundamental knowledge and facts used in the paper. We define Hadamard matrices in Sec. 2.1. In Sec. 2.2, we discuss the existence and construction of real Hadamard matrices. Then, we introduce H-2 reducible matrix as a class of parameterized matrix in 2.3. Next in Sec. 2.4 and 2.5, we respectively introduce the notion of Schmidt rank and partial transpose of a bipartite matrix, so as to characterize the Hadamard matrices studied in later sections. In Sec. 2.6, we introduce two kinds of decomposition for unitary matrices, namely the CS decomposition and that for four by four unitary matrices.

2.1. Hadamard Matrix

A Hadamard matrix is a unitary matrix whose entries have the same modulus.

Lemma 2.1. *A matrix complex equivalent to a Hadamard matrix is also a Hadamard matrix.*

Set $U = PHQ$ where P and Q are complex permutation matrices and H is a Hadamard matrix. Since complex permutation matrices are unitary and have entries of modulus 1, U is a unitary matrix with entries of equal modulus, so U is a Hadamard matrix.

Lemma 2.2. *Every complex Hadamard matrix is complex equivalent to the dephased Hadamard matrix.*

We write Hadamard matrix U as

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} e^{i\alpha_{1,1}} & \cdots & e^{i\alpha_{1,n}} \\ \vdots & \ddots & \vdots \\ e^{i\alpha_{n,1}} & \cdots & e^{i\alpha_{n,n}} \end{bmatrix}. \tag{1}$$

We can apply two complex permutation matrices to U to create a Hadamard matrix V with entries $\frac{1}{\sqrt{n}}$ in the first row and first column.

$$V = \begin{bmatrix} e^{-i\alpha_{1,1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-i\alpha_{n,1}} \end{bmatrix} U \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{(\alpha_{1,1}-\alpha_{1,2})i} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{(\alpha_{1,1}-\alpha_{1,n})i} \end{bmatrix}. \tag{2}$$

Without loss of generality, we construct Hadamard Matrices by starting with the dephased Hadamard matrix. All other Hadamard Matrices can be derived through complex equivalence.

2.2. Existence of Real Hadamard Matrices

1×1 and 2×2 real Hadamard matrices exist.

$$H_1 = \begin{bmatrix} 1 \end{bmatrix}, \tag{3}$$

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{4}$$

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Sylvester's construction Hadamard matrix of order nm can be constructed through

$$H_{nm} = H_n \otimes H_m. \tag{5}$$

where H_n and H_m are Hadamard matrices of order n and m respectively.

To prove this construction we can use the properties of Kronecker product.

$$\begin{aligned} &(H_n \otimes H_m)(H_n \otimes H_m)^\dagger \\ =&(H_n \otimes H_m)(H_n^\dagger \otimes H_m^\dagger) \\ =&(H_n H_n^\dagger \otimes H_m H_m^\dagger) \\ =&(I_n \otimes I_m) \\ =&(I_{nm}). \end{aligned} \tag{6}$$

We can construct all H_{2^k} with H_2 with Sylvester's construction.

Lemma 2.3. *Hadamard matrices of odd dimension cannot be real.*

Proof. The real Hadamard matrix exist if we can find rows whose entries add up to zero. An odd number of elements, every one of which are either -1 or 1, cannot add up to zero, so odd dimensional Hadamard matrices do not exist. □

Lemma 2.4. *Besides H_1 and H_2, all real Hadamard matrices have dimension of multiple of 4.*

Proof. As every Hadamard matrices can be derived through equivalence (a special case of complex equivalence), if the dephased real Hadamard matrix does not exist, then there is no real Hadamard matrix for the dimension.

To make the rows orthonormal, every row (except the first row) in such matrix of dimension n should contain $\frac{n}{2}$ ones and $\frac{n}{2}$ negative ones. Let a and b be distinct row indices such that $a, b \neq 1$ in a given matrix H. Let r be the number of columns k for which $H_{a,k} = H_{b,k} = 1$. Then, it follows that there are also r columns k where $H_{a,k} = H_{b,k} = -1$. The dot product of row a and row b can then be mathematically expressed as:

$$r(1)^2 + r(-1)^2 + (n-2r)(-1) = 4r - n = 0. \tag{7}$$

So we conclude that n has to be the multiple of 4. □

Yet, it is still unsolved whether there is a real Hadamard matrix for every dimension of multiple of 4. This is known as the Hadamard Conjecture, which is still open today.

2.3. H-2 Reducible Matrix

If there exist a order 2 Hadamard submatrix $\begin{bmatrix} h_{ij} & h_{ik} \\ h_{lj} & h_{lk} \end{bmatrix}$ in a order 6 Hadamard matrix H, such that $1 \leq i, j \leq l, k \leq 6$, then H is a H-2 reducible Hadamard matrix [7].

Lemma 2.5. *H-2 reducible 6×6 Hadamard matrices can be fully characterized under a three-parameter family [8].*

2.4. Schmidt Rank

Separate a $m \times m$ matrix in to k^2 $n \times n$ matrices such that $kn = m$. For example, a 6×6 matrix can be separated into four 3×3 matrices. As shown below, U_6 is separated into the 3×3 matrices A, B, C, D.

$$U_6 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{8}$$

The **Schmidt rank** of a matrix is defined as the number of linearly independent matrices between the separated small matrices. For example, if $ABCD$ are all linearly independent, then the Schmidt rank of U_6 is four.

Consider if U_6 is separated into nine 2×2 matrices, the maximum Schmidt rank is 4, as there can be only at maximum four linearly independent 2×2 matrices.

Lemma 2.6. *If A,B,C, and D are invertible matrices, Sr*$((A \otimes B)H(C \otimes D)) =$ *Sr*(H) *for any matrix H.*

Proof. Let $K = (A \otimes B)H(C \otimes D)$.
Matrix H can be written as

$$H = \sum_{i,j=0}^{1} |i\rangle \langle j| \otimes M_{i,j} = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix}, \tag{9}$$

and

$$K = (A \otimes B)H(C \otimes D) = \sum_{i,j=0}^{1} (A(|i\rangle \langle j|)C) \otimes (BM_{i,j}D), \tag{10}$$

which decompose the four blocks of K as a linear combination of $M_{i,j}$. Thus, Sr$(K) \leq$ Sr(H).
As A,B,C, and D are reversible,

$$H = (A^{-1} \otimes B^{-1})K(C^{-1} \otimes D^{-1}). \tag{11}$$

By the same token, Sr$(H) \leq$ Sr(K), so Sr$(H) =$ Sr(K). □

2.5. Partial Transpose

Partial transpose of a matrix takes only the transpose of blocks and leaves the rest unchanged. The partial transpose operator is defined as follow:

$$H^\Gamma = T_a \otimes I_b. \tag{12}$$

Applying partial transpose to 6×6 Hadamard matrix U_6, where $a = 2$ $b = 3$, we get

$$U_6 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{13}$$

$$U_6^\Gamma = \begin{bmatrix} A & C \\ B & D \end{bmatrix}, \tag{14}$$

where A, B, C, and D are three by three blocks.

2.6. Unitary Matrix Decomposition

We begin by introducing the so-called CS-decomposition.

Lemma 2.7. *Let N be an even integer. Every $N \times N$ unitary matrix U can be decomposed as*

$$U = \begin{bmatrix} L_0 & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} D_c & -D_s \\ D_s & D_c \end{bmatrix} \begin{bmatrix} R_0 & 0 \\ 0 & R_1 \end{bmatrix}. \tag{15}$$

L_0, L_1, R_0 and R_1 are $(N/2) \times (N/2)$ unitary matrices. D_c and D_s are diagonal matrices such that

$$\begin{aligned} D_c &= \operatorname{diag}(\cos\phi_1, \cos\phi_2, \cdots, \cos\phi_{N/2}), \\ D_s &= \operatorname{diag}(\sin\phi_1, \sin\phi_2, \cdots, \sin\phi_{N/2}). \end{aligned} \tag{16}$$

where $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_{N/2} \leq \frac{\pi}{2}$. □

Next, we introduce a canonical decomposition for 4×4 unitary matrices [9].

Lemma 2.8. *Any 4×4 unitary matrix W can be decomposed as*

$$W = (U_A \otimes U_B) U (V_A \otimes V_B), \tag{17}$$

where U_A, U_B, V_A and V_B are 2×2 unitary matrices, and

$$U = \begin{bmatrix} c_0 + c_3 & 0 & 0 & c_1 - c_2 \\ 0 & c_0 - c_3 & c_1 + c_2 & 0 \\ 0 & c_1 + c_2 & c_0 - c_3 & 0 \\ c_1 - c_2 & 0 & 0 & c_0 + c_3 \end{bmatrix}, \tag{18}$$

where the coefficients are

$$c_0 = \cos x \cos y \cos z + \mathrm{i} \sin x \sin y \sin z, \tag{19}$$

$$c_1 = \cos x \sin y \sin z + \mathrm{i} \sin x \cos y \cos z, \tag{20}$$

$$c_2 = \sin x \cos y \sin z + \mathrm{i} \cos x \sin y \cos z, \tag{21}$$

$$c_3 = \sin x \sin y \cos z + \mathrm{i} \cos x \cos y \sin z, \tag{22}$$

and x, y, and z are on the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$. □

3. 6×6 Complex Hadamard matrix

It was proven that for dimension one to five, there exist a family of dephased Hadamard matrices to which all Hadamard matrices of the same dimension are complex equivalent to. [5] However, for dimension six, the first natural number that is neither prime nor prime power, such dephased matrix is not discovered.

If the Schmidt rank of a 6×6 Hadamard Matrix is 1, 2, or 3, the partial transpose is still a Hadamard matrix as follows.

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = (U \otimes V) \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} (W \otimes X), \tag{23}$$

$$H^\Gamma = (W^T \otimes V) \begin{bmatrix} F_1 & F_3 \\ F_2 & F_4 \end{bmatrix} (U^T \otimes X). \tag{24}$$

The above decomposition holds due to the fact that every qubit-qutrit unitary matrix of Schmidt rank at most three is a controlled unitary matrix [10].

In this section, we present the following result. This is also the **main contribution** of this paper.

Theorem 3.1. *The partial transpose of a 6×6 Hadamard matrix of Schmidt rank four is not a Hadamard matrix.*

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To study the conjecture, we consider a 6×6 Hadamard matrix H_6 and its partial transpose K_6 as follows.

$$H_6 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{25}$$

$$H_6^\dagger = \begin{bmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{bmatrix}, \tag{26}$$

$$K_6 = H_6^\Gamma = \begin{bmatrix} A & C \\ B & D \end{bmatrix}, \tag{27}$$

$$K_6^\dagger = \begin{bmatrix} A^\dagger & B^\dagger \\ C^\dagger & D^\dagger \end{bmatrix}. \tag{28}$$

By definition of Hadamard matrix,

$$H_6 H_6^\dagger = I_6 = \begin{bmatrix} AA^\dagger + BB^\dagger & AC^\dagger + BD^\dagger \\ CA^\dagger + DB^\dagger & CC^\dagger + DD^\dagger \end{bmatrix}, \tag{29}$$

$$K_6 K_6^\dagger = I_6 = \begin{bmatrix} AA^\dagger + CC^\dagger & AB^\dagger + CD^\dagger \\ BA^\dagger + DC^\dagger & BB^\dagger + DD^\dagger \end{bmatrix}, \tag{30}$$

The 5 conditions we can conclude are

$$\begin{cases} AA^\dagger + BB^\dagger = I_3, \\ AA^\dagger = DD^\dagger, \\ BB^\dagger = CC^\dagger, \\ AC^\dagger + BD^\dagger = 0, \\ AB^\dagger + CD^\dagger = 0. \end{cases} \tag{31}$$

Another crucial condition is every element of the matrices $A, B, C,$ and D have modulus of $\frac{1}{\sqrt{6}}$. Also note ABCD are linearly independent.

Proof of Theorem 3.1 We can rewrite a Hadamard matrix with CS Decomposition (2.7).

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} L_0 D_c R_0 & -L_0 D_s R_1 \\ L_1 D_s R_0 & L_1 D_c R_1 \end{bmatrix}. \tag{32}$$

By equation (31), we can derive the following equations if H remains a Hadamard matrix after partial transpose.

$$\begin{aligned} &AA^\dagger + BB^\dagger = L_0 D_c D_c L_0^\dagger + L_0 D_s D_s L_0^\dagger, \\ &AA^\dagger = L_0 D_c D_c L_0^\dagger = L_1 D_c D_c L_1^\dagger = DD^\dagger, \\ &BB^\dagger = L_0 D_s D_s L_0^\dagger = L_1 D_s D_s L_1^\dagger = CC^\dagger, \\ &AC^\dagger + BD^\dagger = L_0 D_c D_s L_1^\dagger - L_0 D_c D_s L_1^\dagger = 0, \\ &AB^\dagger + CD^\dagger = -L_0 D_c R_0 R_1^\dagger D_s L_0^\dagger + L_1 D_s R_0 R_1^\dagger D_c L_1^\dagger = 0. \end{aligned} \tag{33}$$

From $AA^\dagger = DD^\dagger$, we can derive

$$\begin{aligned} L_0 D_c D_c L_0^\dagger &= L_1 D_c D_c L_1^\dagger, \\ L_1^\dagger L_0 D_c^2 &= D_c^2 L_1^\dagger L_0. \end{aligned} \tag{34}$$

We write $D_c = \text{diag}(\cos\phi_0, \cos\phi_1, \cos\phi_2)$. We analyze by 4 cases.

3.1. Case 1: $\phi_0 < \phi_1 < \phi_2$.
As D_c is diagonal, we can see $L_1^\dagger L_0$ is also diagonal such that

$$\begin{aligned} L_1^\dagger L_0 &= \begin{bmatrix} e^{ib_0} & 0 & 0 \\ 0 & e^{ib_1} & 0 \\ 0 & 0 & e^{ib_2} \end{bmatrix} = F_0, \\ L_0 &= L_1 F_0. \end{aligned} \tag{35}$$

Then similarly from $A^\dagger A = D^\dagger D$, we can derive $R_0 = F_1 R_1$ where F_1 is also a diagonal matrix. We can rewrite the Hadamard matrix as

$$H = \begin{bmatrix} L_1 F_0 D_c F_1 R_1 & -L_1 F_0 D_s R_1 \\ L_1 D_s F_1 R_1 & L_1 F_0 D_c R_1 \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} F_0 D_c F_1 & F_0 D_s \\ D_s F_1 & D_c \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix}. \tag{36}$$

By Lemma (2.6) the Schmidt rank of H is same as the matrix in the middle. As the four blocks in the middle matrix are all diagonal matrices, the Schmidt rank of this case is at most 3, which satisfy statement (3.1).

3.2. Case 2: $\phi_0 = \phi_1 < \phi_2$.
In this case, we can derive from $AA^\dagger = DD^\dagger$ that

$$L_0 = L_1 G_0 = L_1 \begin{bmatrix} G_{00} & 0 \\ 0 & x_{00} \end{bmatrix}, \tag{37}$$

where G_{00} represents a two by two block. Similarly, we can derive from $A^\dagger A = D^\dagger D$ that

$$R_0 = G_1 R_1 = \begin{bmatrix} G_{11} & 0 \\ 0 & x_{11} \end{bmatrix} R1, \tag{38}$$

where G_{11} represents a two by two block.
Then we rewrite the Hadamard matrix as

$$H = \begin{bmatrix} L_1 G_0 D_c G_1 R_1 & -L_1 G_0 D_s R_1 \\ L_1 D_s G_1 R_1 & L_1 D_c R_1 \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} G_0 D_c G_1 & -G_0 D_s \\ D_s G_1 & D_c \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix}. \tag{39}$$

By Lemma (2.6), the rank of H is equal to the matrix in the middle.
Name the middle matrix K. This matrix is unitary and can be written in the form

$$K = \begin{bmatrix} K_0 & 0 & K_1 & 0 \\ 0 & m_0 & 0 & m_1 \\ K_2 & 0 & K_3 & 0 \\ 0 & m_2 & 0 & m_3 \end{bmatrix}, \tag{40}$$

where K_0, K_1, K_2,and K_3 are all two by two blocks.
As K is unitary, the following 4×4 matrix W is also unitary.

$$W = \begin{bmatrix} K_0 & K_1 \\ K_2 & K_3 \end{bmatrix}. \tag{41}$$

By Lemma (2.8), W can be decomposed as

$$W = (U_A \otimes U_B) U (V_A \otimes V_B), \tag{42}$$

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where U_A, U_B, V_A and V_B are 2×2 unitary matrices. By Lemma (2.6), $\mathrm{Sr}(W) = \mathrm{Sr}(U)$, so we then investigate the rank of U.

$$U = \begin{bmatrix} c_0 + c_3 & 0 & 0 & c_1 - c_2 \\ 0 & c_0 - c_3 & c_1 + c_2 & 0 \\ 0 & c_1 + c_2 & c_0 - c_3 & 0 \\ c_1 - c_2 & 0 & 0 & c_0 + c_3 \end{bmatrix}. \tag{43}$$

The partial transpose of U is

$$U^\Gamma = \begin{bmatrix} c_0 + c_3 & 0 & 0 & c_1 + c_2 \\ 0 & c_0 - c_3 & c_1 - c_2 & 0 \\ 0 & c_1 - c_2 & c_0 - c_3 & 0 \\ c_1 + c_2 & 0 & 0 & c_0 + c_3 \end{bmatrix}. \tag{44}$$

Both U and its U^Γ should be unitary, which means they need to satisfy: First, the square of the modulus of each entry in a row or column sums to one; Second, Every pair of rows and columns are orthogonal.

By property 1, we can see from the first column of U and U^Γ that

$$\begin{gathered} |c_0 + c_3|^2 + |c_1 + c_2|^2 = |c_0 + c_3|^2 + |c_1 - c_2|^2 = 1, \\ (c_1 + c_2)(c_1^* + c_2^*) = (c_1 - c_2)(c_1^* - c_2^*), \\ c_1 c_2^* + c_2 c_1^* = 0. \end{gathered} \tag{45}$$

Similarly, we can derive from first column of U and second column of U^Γ

$$\begin{gathered} (c_0 + c_3)(c_0^* + c_3^*) = (c_0 - c_3)(c_0^* - c_3^*), \\ c_0 c_3^* + c_3 c_0^* = 0. \end{gathered} \tag{46}$$

Then by Property 2, we can derive

$$(c_0 + c_3)(c_1^* - c_2*) + (c_1 - c_2)(c_0^* + c_3^*) = 0, \tag{47}$$

$$(c_0 + c_3)(c_1^* + c_2*) + (c_1 + c_2)(c_0^* + c_3^*) = 0, \tag{48}$$

$$(c_0 - c_3)(c_1^* - c_2*) + (c_1 - c_2)(c_0^* - c_3^*) = 0, \tag{49}$$

$$(c_0 - c_3)(c_1^* + c_2*) + (c_1 + c_2)(c_0^* - c_3^*) = 0. \tag{50}$$

From the above four equations, we can derive the following.

$$(c_0 + c_3)c_1^* + c_1(c_0^* + c_3^*) = 0, \tag{51}$$

$$(c_0 + c_3)c_2^* + c_2(c_0^* + c_3^*) = 0, \tag{52}$$

$$(c_0 - c_3)c_1^* + c_1(c_0^* - c_3^*) = 0, \tag{53}$$

$$(c_0 - c_3)c_2^* + c_2(c_0^* - c_3^*) = 0. \tag{54}$$

We consider two subcases.

Subcase 1 : $\det(\begin{bmatrix} c_1^* & c_1 \\ c_2^* & c_2 \end{bmatrix}) \neq 0$.
Neither c_1 nor c_2 is zero. From (51) and (53), we conclude $c_0 = c_3 = 0$, so $\mathrm{Sr}(U) = \mathrm{Sr}(W) \leq 2$.

Subcase 2 : $\det(\begin{bmatrix} c_1^* & c_1 \\ c_2^* & c_2 \end{bmatrix}) = 0$.
We first assume c_1 is zero. From (20), we conclude x must be zero and one of y and z must be zero. Thus, either c_2 or c_3 is zero, so $\text{Sr}(U) = \text{Sr}(W) \leq 2$. We then assume c_2 is zero. From (21), we conclude y must be zero and one of x and z must be zero. Thus, either c_1 or c_3 is zero, so $\text{Sr}(U) = \text{Sr}(W) \leq 2$. Since we have showed that the maximum rank of W is 2 such that $K_i \in \text{span}\{L, M\}$, $\text{Sr}(K) = \text{Sr}(H) \leq 3$ as $\begin{bmatrix} K_j & 0 \\ 0 & m_j \end{bmatrix} \in \text{span}\{\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$.
In conclusion, statement (3.1) is true for Case 2 ($\phi_0 = \phi_1 \leq \phi_2$) because the maximum Schmidt rank is three.

3.3. Case 3: $\phi_0 < \phi_1 = \phi_2$.
Case 3 is analogous to Case 2. $\text{Sr}(K) = \text{Sr}(H) \leq 3$ as $\begin{bmatrix} m_j & 0 \\ 0 & K_j \end{bmatrix} \in \text{span}\{\begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\}$.
Statement (3.1) is true for Case 3 ($\phi_0 < \phi_1 = \phi_2$) because the maximum Schmidt rank is three.

3.4. Case 4: $\phi_0 = \phi_1 = \phi_2$.
From the last equation of (33), we have

$$\begin{aligned} L_0 D_c R_0 R_1^\dagger D_s L_0^\dagger &= L_1 D_s R_0 R_1^\dagger D_c L_1^\dagger, \\ L_0 R_0 R_1^\dagger L_0 &= L_1 R_0 R_1^\dagger L_1, \\ (L_0^\dagger L_1)(R_0 R_1^\dagger)(L_0 L_1^\dagger) &= R_0 R_1^\dagger. \end{aligned} \tag{55}$$

Setting $X = L_0^\dagger L_1$ and $Y = R_0 R_1^\dagger$, we have

$$\begin{aligned} XYX^\dagger &= Y, \\ XY &= YX. \end{aligned} \tag{56}$$

Two commutative unitary matrices are simultaneously diagonalizable, so

$$\begin{aligned} L_0^\dagger L_1 &= P D_0 P^\dagger, \\ L_1 &= L_0 P D_0 P^\dagger, \end{aligned} \tag{57}$$

$$\begin{aligned} R_0 R_1^\dagger &= P D_1 P^\dagger, \\ R_0 &= P D_1 P^\dagger R_1. \end{aligned} \tag{58}$$

Substitute equation (57) and (58) into (32),

$$\begin{aligned} H &= \begin{bmatrix} L_0 D_c P D_1 P^\dagger R_1 & -L_0 D_s R_1 \\ L_0 P D_0 P^\dagger D_s P D_1 P^\dagger R_1 & L_0 P D_0 P^\dagger D_c R_1 \end{bmatrix} \\ &= \begin{bmatrix} L_0 & 0 \\ 0 & L_0 \end{bmatrix} \begin{bmatrix} \cos(\phi) P D_1 P^\dagger & -\sin(\phi) P P^\dagger \\ \sin(\phi) P D_0 D_1 P^\dagger & \cos(\phi) P D_0 P^\dagger \end{bmatrix} \begin{bmatrix} R_0 & 0 \\ 0 & R_0 \end{bmatrix} \\ &= \begin{bmatrix} L_0 & 0 \\ 0 & L_0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \cos(\phi) D_1 & -\sin(\phi) I \\ \sin(\phi) D_0 D_1 & \cos(\phi) D_0 \end{bmatrix} \begin{bmatrix} P^\dagger & 0 \\ 0 & P^\dagger \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix}. \end{aligned} \tag{59}$$

By Lemma (2.6), the Schmidt rank of H is equal to the Schmidt rank of the matrix in the middle. As the middle matrix is diagonal, its maximum Schmidt rank is 3. Thus, statement (3.1) is true for case 4. □

The above proof does not rely on the modulus of entries in the Hadamard matrices, so we can extend Theorem (3.1) to all 6 × 6 unitary matrices and propose the following theorem.

Theorem 3.2. *The partial transpose of a 6 × 6 unitary matrix of Schmidt-rank four is not a unitary matrix.*

4. H-2 Reducible 6 $\times$ 6 Hadamard Matrix

4.1. Schmidt-rank-one

Lemma 4.1. *Every 6 $\times$ 6 Hadamard with Schmidt-rank-one is H-2 reducible.*

Proof. The 6 $\times$ 6 Hadamard matrix with Schmidt-rank-one can be decomposed as

$$H_6 = \frac{1}{\sqrt{6}}(P_1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} Q_1) \otimes (P_2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} Q_2), \tag{60}$$

where P_1, P_2, Q_1, and Q_2 are complex permutation matrices. The submatrix $\begin{bmatrix} h_{i,j} & h_{i+3,j} \\ h_{i,j+3} & h_{i+3,j+3} \end{bmatrix}$ with $1 \leq i, j \leq 3$ is always an order 2 Hadamard matrix, making H_6 H-2 reducible. □

4.2. Schmidt-rank-two

Any Schmidt-rank-two order six Hadamard matrix can be written as

$$\mathbb{H}_2(\alpha, \beta, \gamma, V, W) := \left[\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} \otimes I_3\right] \cdot \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix} \cdot \left[\begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix} \otimes I_3\right], \tag{61}$$

$$\mathbb{H}_2(\alpha, \beta, \gamma, V, W) = \begin{bmatrix} (\cos\alpha\cos\beta)V + (\sin\alpha\sin\beta)W & (\cos\alpha\sin\beta)V - (\sin\alpha\cos\beta)W \\ (\sin\alpha\cos\beta)V - (\cos\alpha\sin\beta)W & (\sin\alpha\sin\beta)V + (\cos\alpha\cos\beta)W \end{bmatrix}, \tag{62}$$

where $V = [v_{jk}]$ and $W = [w_{jk}]$ are linearly independent order-three unitary matrices and

$$\alpha, \beta \in [0, \frac{\pi}{4}], \alpha + \beta \geq \frac{\pi}{4}, \gamma \in [0, 2\pi), \tag{63}$$

$$\cos 2\alpha \cos 2\beta + \frac{3(v_{ij}w_{jk}^* + v_{jk} * w_{jk})}{2} \sin 2\alpha \sin 2\beta = 0, \tag{64}$$

$$|v_{jk}|^2 + |w_{jk}|^2 = 2/3, \tag{65}$$

$$(|v_{jk}|^2 - \frac{1}{3}) \cos 2\alpha = 0, \tag{66}$$

$$(|v_{jk}|^2 - \frac{1}{3}) \cos 2\beta = 0. \tag{67}$$

Hence we have two cases. (4.2.1) If $(\alpha, \beta) \neq (\frac{\pi}{4}, \frac{\pi}{4})$, then V and W are both Hadamard matrices, and $|v_{jk}|^2 = |w_{jk}|^2 = 1/3$. (4.2.2) If $(\alpha, \beta) = (\frac{\pi}{4}, \frac{\pi}{4})$, then $v_{jk}^* w_{jk} + v_{jk} w_{jk}^* = 0$, $|v_{jk}|^2 + |w_{jk}|^2 = 2/3$, and V and W are order three unitary matrices [11].

4.2.1. Case 1: $(\alpha, \beta) \neq (\frac{\pi}{4}, \frac{\pi}{4})$

Taking the upper left entry of every block, we obtain the two by two submatrix

$$\begin{bmatrix} \cos\alpha\cos\beta V_{11} + \sin\alpha\sin\beta W_{11} & \cos\alpha\sin\beta V_{11} - \sin\alpha\cos\beta W_{11} \\ \sin\alpha\cos\beta V_{11} - \cos\alpha\sin\beta W_{11} & \sin\alpha\sin\beta V_{11} + \cos\alpha\cos\beta W_{11} \end{bmatrix}, \tag{68}$$

where V_{11} and W_{11} are the upper left entry of V and W respectively.
We can see that this matrix has orthogonal rows and columns as

$$\begin{aligned} &(\cos\alpha\cos\beta V_{11} + \sin\alpha\sin\beta W_{11})(\sin\alpha\cos\beta V_{11}^* - \cos\alpha\sin\beta W_{11}^*) \\ &+ (\cos\alpha\sin\beta V_{11} - \sin\alpha\cos\beta W_{11})(\sin\alpha\sin\beta V_{11}^* + \cos\alpha\cos\beta W_{11}^*) \\ &= \cos\alpha\sin\alpha(|V_{11}|^2 - |W_{11}|^2) \\ &= 0. \end{aligned} \tag{69}$$

Thus, this case of Schmidt-rank-two six by six Hadamard matrix are all H-2 reducible.

4.2.2. Case 2: $(\alpha, \beta) = (\frac{\pi}{4}, \frac{\pi}{4})$

Lemma 4.2. *If at least one pair of corresponding entries* v_{jk} *and* w_{jk} *have the same modulus, then the Schmidt-rank-two six by six Hadamard matrix is H-2 reducible.*

Proof. We consider the submatrix $\begin{bmatrix} v_{jk} + w_{jk} & v_{jk} - w_{jk} \\ v_{jk} - w_{jk} & v_{jk} + w_{jk} \end{bmatrix}$. The dot product of the two rows is

$$\begin{aligned} &(v_{jk} + w_{jk})(v_{jk}^* - w_{jk^*}) + (v_{jk} - w_{jk})(v_{jk}^* + w_{jk^*}) \\ =&2v_{jk}v_{jk}^* - 2w_{jk}w_{jk}^* \\ =&2(|v_{jk}|^2 - |w_{jk}|^2). \end{aligned} \tag{70}$$

Thus, if the at least one pair of corresponding entries of V and W have the same modulus, the Schmidt-rank-two six by six Hadamard matrix is H-2 reducible . □

Since the sum of square of modulus of corresponding elements in V and W is $\frac{2}{3}$, and the phase angle differ by $\frac{\pi}{2}$, we can parameterize V and W as

$$V = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\alpha_{00} e^{i\beta_{00}} & \cos\alpha_{01} e^{i\beta_{01}} & \cos\alpha_{02} e^{i\beta_{02}} \\ \cos\alpha_{10} e^{i\beta_{10}} & \cos\alpha_{11} e^{i\beta_{11}} & \cos\alpha_{12} e^{i\beta_{12}} \\ \cos\alpha_{20} e^{i\beta_{20}} & \cos\alpha_{21} e^{i\beta_{21}} & \cos\alpha_{22} e^{i\beta_{22}} \end{bmatrix}, \tag{71}$$

and

$$W = \sqrt{\frac{2}{3}} \begin{bmatrix} c_{00}\sin\alpha_{00} e^{i\beta_{00}} & c_{01}\sin\alpha_{01} e^{i\beta_{01}} & c_{02}\sin\alpha_{02} e^{i\beta_{02}} \\ c_{10}\sin\alpha_{10} e^{i\beta_{10}} & c_{11}\sin\alpha_{11} e^{i\beta_{11}} & c_{12}\sin\alpha_{12} e^{i\beta_{12}} \\ c_{20}\sin\alpha_{20} e^{i\beta_{20}} & c_{21}\sin\alpha_{21} e^{i\beta_{21}} & c_{22}\sin\alpha_{22} e^{i\beta_{22}} \end{bmatrix}, \tag{72}$$

where $c_{ij} = \pm i$.

By the orthogonality of columns in V and W, we know

$$\begin{aligned} \cos\alpha_{00}\cos\alpha_{01} e^{i(\beta_{01}-\beta_{00})} &+ \cos\alpha_{10}\cos\alpha_{11} e^{i(\beta_{11}-\beta_{10})} \\ &+ \cos\alpha_{20}\cos\alpha_{21} e^{i(\beta_{21}-\beta_{20})} = 0 \end{aligned} \tag{73}$$

and

$$\begin{aligned} c_{00}c_{01}\sin\alpha_{00}\sin\alpha_{01} e^{i(\beta_{01}-\beta_{00})} &+ c_{10}c_{11}\sin\alpha_{10}\sin\alpha_{11} e^{i(\beta_{11}-\beta_{10})} \\ &+ c_{20}c_{21}\sin\alpha_{20}\sin\alpha_{21} e^{i(\beta_{21}-\beta_{20})} = 0. \end{aligned} \tag{74}$$

From the above two equations, we can deduce the following vectors are orthogonal to each other:

$$\begin{bmatrix} \sin\alpha_{00}\sin\alpha_{01} \\ \sin\alpha_{10}\sin\alpha_{11} \\ \sin\alpha_{20}\sin\alpha_{21} \end{bmatrix}, \begin{bmatrix} \cos\alpha_{00}\cos\alpha_{01} \\ \cos\alpha_{10}\cos\alpha_{11} \\ \cos\alpha_{20}\cos\alpha_{21} \end{bmatrix} \perp \begin{bmatrix} \cos(\beta_{00}-\beta_{01}) \\ \cos(\beta_{10}-\beta_{11}) \\ \cos(\beta_{20}-\beta_{21}) \end{bmatrix}, \begin{bmatrix} c_{00}c_{01}\sin(\beta_{00}-\beta_{01}) \\ c_{10}c_{11}\sin(\beta_{10}-\beta_{11}) \\ c_{20}c_{21}\sin(\beta_{20}-\beta_{21}) \end{bmatrix}, \tag{75}$$

$$\begin{bmatrix} \sin\alpha_{00}\sin\alpha_{02} \\ \sin\alpha_{10}\sin\alpha_{12} \\ \sin\alpha_{20}\sin\alpha_{22} \end{bmatrix}, \begin{bmatrix} \cos\alpha_{00}\cos\alpha_{02} \\ \cos\alpha_{10}\cos\alpha_{12} \\ \cos\alpha_{20}\cos\alpha_{22} \end{bmatrix} \perp \begin{bmatrix} \cos(\beta_{00}-\beta_{02}) \\ \cos(\beta_{10}-\beta_{12}) \\ \cos(\beta_{20}-\beta_{22}) \end{bmatrix}, \begin{bmatrix} c_{00}c_{02}\sin(\beta_{00}-\beta_{02}) \\ c_{10}c_{12}\sin(\beta_{10}-\beta_{12}) \\ c_{20}c_{22}\sin(\beta_{20}-\beta_{22}) \end{bmatrix}, \tag{76}$$

$$\begin{bmatrix} \sin\alpha_{01}\sin\alpha_{02} \\ \sin\alpha_{11}\sin\alpha_{12} \\ \sin\alpha_{21}\sin\alpha_{22} \end{bmatrix}, \begin{bmatrix} \cos\alpha_{01}\cos\alpha_{02} \\ \cos\alpha_{11}\cos\alpha_{12} \\ \cos\alpha_{21}\cos\alpha_{22} \end{bmatrix} \perp \begin{bmatrix} \cos(\beta_{01}-\beta_{02}) \\ \cos(\beta_{11}-\beta_{12}) \\ \cos(\beta_{21}-\beta_{22}) \end{bmatrix}, \begin{bmatrix} c_{01}c_{02}\sin(\beta_{01}-\beta_{02}) \\ c_{11}c_{12}\sin(\beta_{11}-\beta_{12}) \\ c_{21}c_{22}\sin(\beta_{21}-\beta_{22}) \end{bmatrix}. \tag{77}$$

Linearly Independent If the two vectors on the right of (75), (76), and (77) are linearly independent, then the vectors on the left are linearly dependent such that

$$\begin{aligned} \tan\alpha_{00}\tan\alpha_{01} &= \tan\alpha_{10}\tan\alpha_{11} = \tan\alpha_{20}\tan\alpha_{21}, \\ \tan\alpha_{00}\tan\alpha_{02} &= \tan\alpha_{10}\tan\alpha_{12} = \tan\alpha_{20}\tan\alpha_{22}, \\ \tan\alpha_{01}\tan\alpha_{02} &= \tan\alpha_{11}\tan\alpha_{12} = \tan\alpha_{21}\tan\alpha_{22}. \end{aligned} \tag{78}$$

By multiplying the three equations together and dividing the product by each equation, we get that

$$\begin{aligned} \tan\alpha_{00} &= \tan\alpha_{10} = \tan\alpha_{20}, \\ \tan\alpha_{01} &= \tan\alpha_{11} = \tan\alpha_{21}, \\ \tan\alpha_{02} &= \tan\alpha_{12} = \tan\alpha_{22}. \end{aligned} \tag{79}$$

Since $\alpha \in (0, \frac{\pi}{2})$, the angles α_{jk} and thus the modulus of entries in the same column are also the same. We can further claim that matrices V and W are Hadamard matrices in this case, as every entry has the same modulus. By lemma (4.2), the Schmidt-rank-two six by six Hadamard matrix is H-2 reducible if the vectors on the right hand side of (75), (76), and (77) are linearly independent.

Linearly Dependent We attempted to solve this case through numerical computation on computer. First, we use Scipy library's `unitary_group.rvs(3)` function to generate 3 by 3 unitary matrix V. We then obtain the modulus of matrix W with the condition $|v_{jk}|^2 + |w_{jk}|^2 = 2/3$ (regenerate V if some of its entries' modulus already exceed $\sqrt{\frac{2}{3}}$). Next, we then assign entries in W with the phase angle of corresponding entry in V, either add or subtract $\frac{\pi}{2}$, which results in 2^9 possible cases of W for each V. Finally, we check if the obtained W matrices are unitary and if the six by six Hadamard matrix is H-2 reducible.

However, after iteration over 100,000 generated V matrices, we were not be able to find any W that is unitary. Based on this result, we guess non-Hadamard unitary V and W does not exist.

4.3. H-2 Reducible Matrix and Partial Transpose

Schmidt rank again emerged as a useful indicator for whether the partial transpose of six by six H-2 reducible matrices remain H-2 reducible.

Theorem 4.3. *An H-2 reducible Schmidt-rank-one/two six by six Hadamard matrix remains H-2 reducible after partial transpose.*

Proof. By lemma (4.1), all Schmidt-rank-one are H-2 reducible because the submatrix $\begin{bmatrix} h_{i,j} & h_{i+3,j} \\ h_{i,j+3} & h_{i+3,j+3} \end{bmatrix}$ with $1 \leq i, j \leq 3$ is always an order 2 Hadamard matrix. The partial transpose switches the position of the lower left and upper right entry of the submatrix, which would remain Hadamard.

The same two by two submatrix of case one $(\alpha, \beta) \neq (\frac{\pi}{4}, \frac{\pi}{4})$ Schmidt-rank-two is also Hadamard, so this case also remains H-2 reducible after partial transpose.

The six by six Hadamard matrix in Schmidt-rank-two case two $(\alpha, \beta) = (\frac{\pi}{4}, \frac{\pi}{4})$ has the same lower left and upper right block, so the entire matrix remain unchanged after partial transpose. Thus, this case will also remain H-2 reducible after partial transpose. □

5. Conclusions

In the first part of the study, we thoroughly investigated the impact of partial transposition on the properties of Hadamard matrices. We discovered the significance of the Schmidt rank as a key determinant of the effect of partial transposition on a Hadamard matrix. Our analysis revealed that for Hadamard matrices whose Schmidt rank is less than four, the resultant matrix following a partial transpose remains a Hadamard matrix. However, as proven in Theorem 3.1, a Hadamard matrix with Schmidt-rank-four no longer remains a Hadamard matrix after partial transpose.

This particular transformation is primarily due to the fact that any matrix with a Schmidt rank less than four can be factored out of a Hadamard matrix with a Schmidt rank of four. This pivotal discovery indicates a direct correlation between the Schmidt rank of a Hadamard matrix and partial transposition, a relationship that has potential for further exploration and may yield additional insights into the properties of Hadamard matrices and MUBs.

In the second part of the study, we explored the relationship between Schmidt rank and H-2 Reducibleness. Schmidt-rank-one matrices are all H-2 reducible. We failed to show one case of Schmidt-rank-two matrices are H-2 reducible, but the subsequent computer numerical analysis implied such case is rare (none found in 100,000 iterations). At the end, we further proved that the partial transpose of a H-2 reducible six by six Hadamard matrix with Schmidt rank less than three remains H-2 reducible.

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