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Fractional fourier transform and its application

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Abstract. The Fourier Transform (FT) is a linear transformation for the primitive function. It takes some set of functions to be an orthogonal basis. Its physical meaning is to transfer the primitive function onto each set of base functions. Because it can convert functions between the time and frequency domains, the FT is widely employed in many fields. The Fractional Fourier Transform (FrFT) is an improvement and progress based on the FT. This paper will define the FT and FrFT. Then the distinction between FrFT and FT is discussed. Finally, specific examples of its application in processing digital image are provided. FrFT is the process of transforming an image function into a series of periodic functions. The FrFT is used as a powerful mathematical tool to understand non- smooth signals, nonlinear systems and complicated phenomena, which is significant and has broad possibilities in the fields of signal processing, communication, image processing, optical imaging and quantum information processing.

Keywords: Fourier Transform; Fractional Fourier Transform; Decentralized image resoration; Face recognition.

1. Introduction

Decomposing a function into a superposition of sine and cosine functions of different frequencies can be achieved using the FT, a mathematical tool. Converting a function from the time domain to the frequency domain with the FT enables visualizing the frequency components contained in the function. As a method of signal analysis, the FT helps us understand signals and data from a new perspective through spectral analysis, signal processing. Besides, it has very important research significance in digital image processing. FrFT is promoting FT, which is a significant advancement in digital image processing technology.

In 2024, Wang discussed the application of signal processing and image filtering [1]. In 2023, Xia studied the application of SAR image ship wake detection by the Radon Transform [2]. In 2016, Li, Yan, Zeng et al. learnt about the application of automatic identification and localization of profiling zone according to the One-dimensional FT to extract frequency-domain features in the profile amplitude region [3]. In 2011, Sun talked about the application of textile images [4]. In 2008, Zhao, Li and Xuan discussed the applying the remote sensing image alignment based on FT [5]. In 2006, Liu, Guo and Feng researched the applying the remote sensing image alignment [6].

This paper will define FT and FrFT in Section 2. Section 3 will examine the use of FrFTs in decentralized image restoration and face recognition, which will be elaborated on.

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2. Definition of FrFT

2.1. Fundamentals of Fourier Series

A Fourier Series is a special form of function expansion that is a mathematical tool for decomposing a periodic signal into a series of sine and cosine functions. Its general representation is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,. \tag{1}$$

The above three coefficients in the equation are:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx\,, \tag{2}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx, \tag{3}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx. \tag{4}$$

2.2. FT

The FT is a mathematical tool for transforming a signal from the time domain (the space domain) to the frequency domain, expressing a function that meets certain conditions as a trigonometric function or a linear combination of their integrals. By introducing imaginary numbers from the Fourier Series, the FT can be obtained:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dx\, e^{i\omega t}\, d\omega\,. \tag{5}$$

From the above equation, the FT equation is given as:

$$F(x) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega t}\, dx\,. \tag{6}$$

2.3. FrFT

By linking the relatively independent time and frequency domains, the FT displays the frequency components of the whole signal. This enables it explores deterministic signals and smooth signals. Due to mostly non-smooth signals existing in the natural world, a variety of latest time-frequency analysis theories and methods have been put forward to solve this problem, including FrFT .

Linear operators are used to describe FrFT. In the time-frequency plane, the FrFT operator can be rotated by an arbitrary angle, if the Fourier Transform is considered as a counterclockwise rotation of 90 degrees from the time axis to the frequency axis. In this sense, the FrFT is considered a generalized FT.

The definition of the FrFT varies under different perspectives. The definition of the one-dimensional and two-dimensional FrFT in terms of the integral form will be given following.

2.3.1. Definition of integral form. The basic definition of the FrFT is given from the point of view of the linear integral transform, which reflects the most basic properties of the FrFT and is the most rigorous form of mathematical definition of the FrFT.

The FrFT's integration definition is given below [7]:

Definition 1: The p-order FrFT of the function $f(x)$ is

$$g(u) = F^p\{f(x)\} = \int_{-\infty}^{+\infty} f(x)\, K_p(u,x) f(x) dx \tag{7}$$

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where K_p is the FrFT kernel function defined as following:

$$K_p(u,x) = \begin{cases} \sqrt{\frac{1-jcota}{2\pi}} \exp\left[j\left(\frac{x^2+u^2}{2tana} - \frac{xu}{sina}\right)\right] & a \neq n\pi \\ \delta(x-u) & a \neq 2n\pi \\ \delta(x+u) & a = (2n \pm 1)\pi \end{cases}, \quad (8)$$

in the formula $a = p\frac{\pi}{2}, p \neq 2n$. The order p is taken with 4 as the least positive period. $p = 0$ is the original; $p = 1$ is the FT, and when p is transformed between $[0,1]$. It is called FrFT.

The fractional order FT is considered to be a generalized FT, as stated by the definition of the FrFT. The FrFT is a form of intermediate state transition between the function and the FT. To express the degree of such an intermediate state, the order parameter can be used through the continuous fractional transform, which also takes into account the signal time and frequency domain [8].

2.3.2. Definition of a 2D FrFT

Definition 2: Suppose that for a two-dimensional signal of size $M \times N$,which called $f(x,y)$, and $1 \leq x \leq M$,$1 \leq y \leq N$,$M \geq N$. Then the 2D FrFT is defined as

$$F(u,v) = \frac{1}{MN}\sum_{x=1}^{M}\sum_{y=1}^{N} H_{p_1p_2}(x,y,u,v)f(x,y) \quad (9)$$

$$H_{p_1p_2} = H_{p_1}(x,u)H_{p_2}(y,v) \quad (10)$$

In the equation,

$$H_{\varphi_x} = \begin{cases} \frac{1}{2\pi}\sqrt{1-cot\alpha}\sqrt{1-jcot\beta}\exp\left[i\pi\left(x^2cot\varphi_x - 2xucsc\varphi_x + u^2cot\varphi_x\right)\right], & \varphi_x \neq n\pi \\ \delta(x-u), & \varphi_x = 2n\pi \\ \delta(x+u), & \varphi_x = (2n+1)\pi \end{cases}, \quad (11)$$

where p_1, p_2 are the transform order of the FrFT. $\varphi_x = p\frac{\pi}{2}$ is the FrFT of the transform angle. The kernel functions $H_{p_1}(x,u)$ and $H_{p_2}(y,v)$ have the same form. When $p_1 = p_2 = 0$, it is the original signal, and when $p_1 = p_2 = 1$, the 2D FrFT is 2D FT.

As stated below, it is possible to convert the 2D Discrete FrFT process into two 1D Discrete FrFTs.

(1) First, A one-dimensional discrete FrFT is applied to the column vectors of a two-dimensional discrete signal to obtain the transform result F_1.
(2) Then, the one-dimensional discrete FrFT is applied to the row vector of $F1$ to obtain the transform result F_2.
(3) Transposing F_2 is the result of the 2D discrete FrFT.

3. Application of FrFT

Image compression, enhancement, analysis, recovery, and stitching are just some of the many applications of the FrFT in processing the digital image. It helps to improve the quality of images, reduce the size of image files, and obtain more image features by converting images from the null domain to the frequency domain and using the frequency domain information for processing and analysis. The following content will mainly introduce its application in recovering defocused blurry images and face recognition technology.

3.1. Decentralized Image Restoration

The phenomenon of defocusing is very common in real life. In photography, the inaccuracy of focusing will result in blurred photos. To capture images of celestial objects in the starry sky, astronomical telescopes will lead to fuzzy pictures that are difficult to recognize due to defocusing reasons. Medical

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images are also defocused fuzzy phenomenon and the fuzzy images directly affect the diagnostic results. These fuzzy images affect our visual experience and even event processing. So, in some practical applications, appropriate image processing techniques must be used to eliminate the interference of fuzzy factors [9].

The nature of FrFT is the main reason why it is used in defocused fuzzy image recovery. FrFT converts fuzzy images into frequency domains, allowing for the fuzzy operation to be simplified to a simple multiplication operation in the frequency domain. The frequency spectrum is filtered in the frequency domain by creating a suitable filter to eliminate interference caused by the fuzzy process and restore the original image. After filtering the frequency domain data, the data can be inverted. Afterwards, the filtered frequency domain datas are inverted and converted back to the null domain to obtain a clear image after recovery. Compared with other methods, the FrFT can better preserve the local features and details of the image, which maintains the clarity and authenticity of the image in the recovery process. Moreover, by selectively filtering out specific frequency components in the frequency domain, it not only avoids the complex convolution calculation in the null-domain operation and improves the efficiency and speed of the processing, but also more accurately removes the interference caused by blurring and restores the clarity of the image. Most importantly, the FrFT has a flexible parameter adjustment capability, which can adjust the filter parameters according to the specific type and degree of blurring, and adapt to the recovery needs in different scenes. Therefore, restoring defocused blurry images can be accomplished using the FrFT.

3.2. Face Recognition Technology

Face recognition has very important research and application value. It can be widely used in biometric security monitoring human-computer interaction and electronic entertainment. The difficulties of the current research mainly lie in three aspects: the gesture of the face image is difficult to detect, the grasping of face features is not accurate enough, and the interference factor of imaging conditions is much [10].

By altering distances, angles, or lighting conditions, FrFT can transform face images into the frequency domain. To obtain more detailed feature representations, it can extract frequency information at different scales and directions and analyze it. In addition, FrFT can better distinguish the structures and patterns in face images. This helps the system to recognize face features more accurately and reduce the effect of noise on face recognition performance. Moreover, the FrFT can adjust the transform parameters to select the most suitable frequency domain representation according to the specific task requirements. This flexibility allows the system to better resist different types of interference.

In conclusion, because the FrFT is more resistant to interference and more flexible, its application in face recognition can extract richer feature information and deal with multi-scale features. So, this method is also very important for the development of face recognition technology.

4. Conclusion

This paper covers the FrFT in depth. By developing the FrFT to the fractional order, it is better able to describe non-smooth signals and non-linear systems. Hence, it is widely used in digital image processing. Additionally, because it can provide more accurate time-frequency information, it is of great benefit to the analysis of non-smooth signals. The FrFT can enhance the system’s ability to block interference and use spectrum efficiently through multicarrier modulation technology. Moreover, in channel equalization, signal recovery and beamforming fields, FrFT can also be used. Apart from these, image compression, denoising, and encryption use FrFTs extensively. By applying FrFT to an image, image transformation and feature extraction can be realized by extracting specific frequency components of the image. In the field of optics, it can be used for phase reconstruction, wavefront propagation and other related problems. Through the FrFT, the propagation direction and focal length of the light beam can be turned to carry out the optimization and adjustment of the optical system. In the field of quantum communication and quantum computation, FrFT can be used for encoding and decoding of quantum states and the realization of quantum gate operations other than Quantum information processing. As the technology advances, it

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is believed that FrFT can definitely contribute to development of other disciplines and promoting social progress.

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