Proceedings of the 2nd International Conference on Mathematical Physics and Computational Simulation
DOI: 10.54254/2753-8818/41/20240107

The application of convex function and GA-convex function

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Abstract. A convex function is a function that maps from a convex subset of a vector space to the set of real numbers. Convex functions have some important properties, such as non-negativity, monotonicity, and convexity, which can help us derive and prove inequalities. This paper explores the concepts of convex functions and GA-convex functions, demonstrating their utility in proving a variety of common and complex inequalities. Beginning with an overview of convex functions and their extension to GA-convex functions, the study shows how these mathematical tools can be effectively utilized in the context of inequality proofs. By leveraging the properties of these functions, the paper successfully establishes rigorous proofs for a range of inequalities, highlighting the versatility and applicability of convex and GA-convex functions in mathematical analysis. The properties convex and GA-convex functions allow us to use it to determine the direction of inequalities, prove inequalities, determine the optimal solution of inequalities, and even prove Cauchy inequalities.

Keywords: Convex function, GA-convex function, Application.

1. Introduction

The concavity and convexity of functions have many applications in proving inequalities. Cha conducted research on formulas related to the theorems of convex functions, deriving several important inequalities, which were further applied to prove inequalities and solved conditional extremum problems in 2004 [1]. In 2005, Xia derived the Jensen's inequality from the concavity, convexity, and continuity of functions [2]. Wu provided the definition of square-convex functions and methods for determining square-convex functions. Then the Jensen-type inequality for square-convex functions was established in 2005 [3]. In 2010, Song and Wan obtained a more concise Hadamard-type inequality for GA-convex functions through their study of GA-convex functions [4]. Shi et al. obtained a new refinement of the Hermite-Hadamard-type inequality for GA-convex functions in 2013 [5]. In the same year, Shi et al. derived some new weighted Hadamard-type inequalities for differentiable GA-convex functions [6]. Wu and Mao proved the Hermite-Hadamard inequality on a special region in 2022 [7].

This article mainly introduces convex functions and GA-convex functions. The paper first introduces the definition of convex functions and its equivalent definitions, extends it to n numbers, and then proves several common inequalities using its properties in section 2. This paper transitions from convex functions to GA-convex functions, introduces its definition, proves its properties, creates an inequality, and then proves a more complex inequality relationship in section 3.

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2. Convex Function and its application

2.1. Properties of concave-convex function

The definition of concave-convex function will be introduced first, followed by an explanation of its properties.

Definition 2.1. ([8]) The original definition of convex functions is derived from geometric intuition. Assuming curve $C: y = f(x), x \in [a,b]$, take $x_1, x, x_2 \in [a,b]$ such that $x_1 < x < x_2$. The equation of the chord passing through the points $A(x_1, f(x_1)$ and $B(x_2, f(x_2)$ is

$$F(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) = \frac{x_2 - x}{x_2 - x_1}f(x_1) + \frac{x - x_1}{x_2 - x_1}f(x_2) \tag{1}$$

So $f(x)$ is concave upwards or downwards in interval $[a,b]$,

$$f(x) \geq (or \leq)\frac{x_2 - x}{x_2 - x_1}f(x_1) + \frac{x - x_1}{x_2 - x_1}f(x_2) \tag{2}$$

Property 2.2. Suppose $f(x)$ is concave upwards or downwards in interval $[a,b]$, then it holds that $f(\alpha x_1 + \beta x_2) \geq (or \leq)\alpha f(x_1) + \beta f(x_2)$.

Proof: Let

$$\theta = \frac{x_2 - x}{x_2 - x_1} \Leftrightarrow x = x_1 + (x_2 - x_1)\theta \Leftrightarrow x = \frac{x_2 - x}{x_2 - x_1}x_1 + \frac{x - x_1}{x_2 - x_1}x_2 \quad (0 < \theta < 1) \tag{3}$$

If the x_1 and x_2 in equations (1) and (3) are interchanged, the result remains unchanged. This means that the above results are independent of whether x_1 is greater than or less than x_2, as long as $x \in (x_1, x_2)$. Therefore, set

$$\alpha = \frac{x_2 - x}{x_2 - x1} > 0, \beta = \frac{x - x_1}{x_2 - x1} > 0, \alpha + \beta = 1, x = \alpha x_1 + \beta x_2 \tag{4}$$

So $f(x)$ is concave upwards or downwards in interval $[a,b]$ that can be replaced by another form:

$$f(\alpha x_1 + \beta x_2) \geq (or \leq)\alpha f(x_1)\beta f(x_2) \tag{5}$$

Definition 2.3. Let $f(x)$ be defined on interval $[a,b]$, $x_1, x_2 \in [a,b]$, $\alpha > 0, \beta > 0$, $\alpha + \beta = 1$, if

$$f(\alpha x_1 + \beta x_2) \geq (or \leq)\alpha f(x_1) + \beta f(x_2) \tag{6}$$

Then it indicates that $f(x)$ is concave up or concave down on the interval $[a,b]$.

2.2. The Application of Convex Functions in Proving Inequalities

In this subsection, common inequalities are proven using the properties of convex functions. First, a lemma is introduced.

Lemma 2.4. Each Let $f(x)$ be convex upwards and downwards on $[a,b]$, $\forall x_1, x_2, \ldots, x_n \in [a,b]$, there exists,

$$f\left(\frac{\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_n}}{\mathbf{n}}\right) \leq (\mathbf{or} \geq)\frac{\mathbf{f(x_1)} + \mathbf{f(x_2)} + \cdots + \mathbf{f(x_n)}}{\mathbf{n}} \tag{7}$$

Proof: By induction, when $n = 1,2$, the proposition can be proven using (6). Assuming it holds for $n = k$, prove that it also holds for $n = k + 1$, $\forall x_1, x_2, \ldots, x_n \in [a,b]$,

$$f(\frac{x_1 + x_2 + \cdots + x_{k+1}}{k + 1}) = f(\frac{k}{k + 1} * \frac{x_1 + x_2 + \cdots + x_k}{k} + \frac{x_{k+1}}{k + 1})$$

Let $\alpha = \frac{k}{k+1}, \beta = \frac{1}{k+1} \Rightarrow \alpha + \beta = 1, \frac{x_1 + x_2 + \cdots + x_k}{k} \in [a,b]$, then

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$$f\left(\frac{x_1+x_2+\cdots+x_{k+1}}{k+1}\right)=f\left(\alpha*\frac{x_1+x_2+\cdots+x_k}{k}+\beta x_{k+1}\right)$$
$$\leq(or\geq)\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \tag{8}$$

Example 2.5. Let $a_1,a_2,\ldots,a_n>0$. Prove:

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\leq\sqrt[n]{a_1a_2\ldots a_3}\leq\frac{a_1+a_2+\cdots+a_n}{n} \tag{9}$$

Proof: First prove the right half of the equation.

$$\sqrt[n]{a_1a_2\ldots a_3}\leq\frac{a_1+a_2+\cdots+a_n}{n}\Leftrightarrow a_1a_2\ldots a_n\leq[\frac{a_1+a_2+\cdots+a_n}{n}]^n$$
$$\Leftrightarrow\frac{\ln a_1+\ln a_2+\cdots+\ln a_n}{n}\leq\ln\frac{a_1+a_2+\cdots+a_n}{n} \tag{10}$$

The inequality can be proven using convex function $f(x)=lnx$ and the **Lemma 2.4**. Replacing a_k with $\frac{1}{a_k}(k=1,2,\ldots,n)$ can prove the left half of the inequality.

3. GA-Convex Functions

3.1. Characteristics of GA-Convex Functions

The definition of GA-Convex Functions will be introduced first, followed by an explanation of its properties.

Definition 3.1.([9]) The Let $f(x)$ be a function defined on $I\in(0,+\infty)$. For any $x_1,x_2\in I$ and $t\subseteq(0,1)$, it exists,

$$f\left(x_1^tx_2^{1-t}\right)\leq tf(x_1)+(1-t)f(x_2) \tag{11}$$

Then $f(x)$ is called a GA-subconvex function on I,if the inequality sign is reversed; otherwise, it is termed a GA-superconvex function on that interval.

Theorem 3.2. If a function $f(x)$ is GA-convex on the interval $(a,b)\in(0,+\infty)$, then for any $x_1,x_2\in(a,b)$ and for $t\in(0,1)$, the function $f(e^x)$ is GA-subconvex function on the interval (lna,lnb).

Proof: Let any $x_1,x_2\in(a,b)$, and $t\in(0,1)$, then

$$f\left(x_1^tx_2^{1-t}\right)=f\left(e^{lnx_1^tx_2^{1-t}}\right)=f\left(e^{tlnx_1+(1-t)lnx_2}\right)$$
$$\leq tf\left(e^{lnx_1}\right)+(1-t)f\left(e^{lnx_2}\right)=tf(x_1)+(1-t)f(x_2) \tag{12}$$

Where $f(x)$ is GA-convex on (a,b). For any $x_1,x_2\in(lna,lnb)$, since $f(x)$ is GA-subconvex function on (a,b), for any $t\in(0,1)$, it holds

$$f\left(e^{tx_1+(1-t)x_2}\right)=f\left((e^{x_1})^t(e^{x_2})^{1-t}\right)\leq tf(e^{x_1})-(1-t)f(e^{x_2}) \tag{13}$$

Therefore, $f(e^x)$ is GA-subconvex function on interval (lna,lnb).

Theorem 3.3. Let a function $f(x)$ be twice differentiable on the interval $I\in(0,+\infty)$. Then $f(x)$ is GA-convex on the interval I if and only if the following conditions hold:

(1) Let $f(x)$ be GA-convex on I, the inequality $xf''(x)+f'(x)\geq 0,\forall x\in I$ must hold all x in I.

(2) Let $f(x)$ be GA-concave on I, the inequality $xf''(x)+f'(x)\leq 0,\forall x\in I$ must hold all x in I.

Proof: It is easy to establish the connection between the second derivative of $f(e^x)$ on the interval (lna,lnb) and the concavity/convexity of the function.

Theorem 3.4. Suppose $f(x)$ is GA-concave on the interval I , $x_i \in I, \lambda_i \in R(i=1,2,\ldots,n), \lambda_1+\lambda_2+\cdots+\lambda_n=1$ It holds $f\left(x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_k^{\lambda_k}\right) \leq \lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_k f(x_k)\left(\sum_{i=1}^{k}\lambda_i=1,\lambda_i>0\right)$.

Proof: This theorem can be proved by induction. Then, it is easy to get if $f(x)$ is GA-Concave on interval I:

$$f\left(\sqrt[n]{x_1x_2\ldots x_n}\right) \leq \frac{1}{n}\sum_{n}^{i=1} f(x_i) \tag{14}$$

3.2. Applications of GA-convex functions.

Theorem 3.5. ([10]) Suppose function $f:[a,b]\to(0,+\infty)$ is GA-Concave, it holds

$$\left(\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \left(\frac{1}{lnb-lna}-\frac{a}{b-a}\right)f(a)+\left(\frac{b}{b-a}-\frac{1}{lnb-lna}\right)f(b) \tag{15}$$

If function f is GA-Convex, inverting the inequality sign is sufficient.

Proof: First prove the inequality on the right-hand side. It can be proved easily by taking the logarithm on both sides. Let $x=a^{\frac{lnb-lnx}{lnb-lna}}b^{\frac{lnx-lna}{lnb-lna}}$ and $\frac{lnb-lnx}{lnb-lna}+\frac{lnx-lna}{lnb-lna}=1$.

Let $t=\frac{lnx-lna}{lnb-lna}$, it is easy to infer $t\in(0,1)$. By the properties of GA-Concave, the following formula can be derived.

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_0^1 f(a^{1-t})f(b^t)\,d(a^{1-t}b^t) \leq \int_0^1 [(1-t)f(a)+tf(b)]\,d(a^{1-t}b^t).\\
&= a\int_0^1 [(1-t)f(a)+tf(b)]\,d\left(\frac{a}{b}\right)^t\\
&= a[(1-t)f(a)+tf(b)]\left(\frac{b}{a}\right)^t \Big|_0^1 - a\int_0^1 \left(\frac{b}{a}\right)^t d[(1-t)f(a)+tf(b)]\\
&= bf(b)-af(a)+a(f(a)-f(b))\int_0^1 \left(\frac{b}{a}\right)^t dt\\
&= \left(\frac{b-a}{lnb-lna}-a\right)f(a)+\left(b-\frac{b-a}{lnb-lna}\right)f(b)
\end{aligned} \tag{16}$$

Dividing both sides by $b-a$ will get the inequality on the right-hand side. By the same way, the inequality on the left-hand side can be proved. Let $\triangle=b-a, a+\frac{i}{n}\triangle\in[a,b], i=1,2,\ldots,n$. By the definition of a definite integral and **Theorem 3.4**, the following formula can be derived.

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,dx &= \lim_{x\to+\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(a+\frac{i}{n}\Delta\right) \geq \lim_{x\to+\infty}\sqrt[n]{\prod_{i=1}^{n}\left(f\left(a+\frac{i}{n}\Delta\right)\right)}\\
&= f\left(\lim_{x\to+\infty} exp\left[ln\sqrt[n]{\prod_{n}^{i=1}\left(a+\frac{i}{n}\Delta\right)}\right]\right) = f\left(\lim_{x\to+\infty} exp\left[\frac{\sum_{i=1}^{n} ln\left(a+\frac{i}{n}\Delta\right)}{n}\right]\right)\\
&= f\left(exp\left\{\frac{1}{\Delta}\lim_{x\to+\infty}\frac{\Delta\sum_{i=1}^{n}\ln\left(a+\frac{i}{n}\Delta\right)}{n}\right\}\right) = f\left(exp\left\{\frac{1}{b-a}\int_a^b lnx\,dx\right\}\right) = f\left(\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\right)
\end{aligned} \tag{17}$$

When $f(x)=lnx^{b-a}$, the inequality in (15) holds.

Example 3.6. ([10]) Suppose $b>a>0$,

DOI: 10.54254/2753-8818/41/20240107

$$\sqrt{ab} \leq \frac{b-a}{lnb-lna} \leq \frac{(\sqrt{a}+\sqrt{b})^2}{4} \leq \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}} \leq \frac{4}{9}(\frac{a+b+\sqrt{ab}}{\sqrt{a}+\sqrt{b}})^2 \leq \frac{a+b}{2} \quad (18)$$

Proof: This example can be proven by GA-concave functions and **Theorem 3.5** By substituting $f(x) = x, \frac{1}{x}, \sqrt{x}, \frac{1}{\sqrt{x}}$ into the inequality on the left side of (15), it follows

$$\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}} \leq \frac{a+b}{2}, \frac{b-a}{lnb-lna} \leq \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}$$

$$\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}} \leq \frac{4}{9}(\frac{a+b+\sqrt{ab}}{\sqrt{a}+\sqrt{b}})^2, \frac{(\sqrt{a}+\sqrt{b})^2}{4} \leq \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}} \quad (19)$$

Substituting $f(x) = \frac{1}{x}$ into the inequality on the right side of (15) results in

$$\sqrt{ab} \leq \frac{b-a}{lnb-lna} \quad (20)$$

Next, the proof of **Example 3.6** reduces to prove**:**

$$\frac{b-a}{lnb-lna} \leq \frac{a+b}{2} \quad (21)$$

Suppose $x = \frac{b}{a} > 1$, the original formula can be simplified as $(x+1)lnx > 2(x-1)$.

Construct a function $f(x) = (x+1)lnx$ and utilize the Lagrange Mean Value Theorem $\frac{f(x)-f(1)}{x-1} = f'(\xi)(1 < \xi < x) \Leftrightarrow (x+1)lnx = (\frac{1}{\xi} + ln\xi + 1)(x-1)$.

Due to this common inequality:

$$lnx > 1 - \frac{1}{x} > \left(\frac{1}{\xi} + 1 - \frac{1}{\xi} + 1\right)(x-1) = 2(x+1) \quad (22)$$

Therefore, the inequality (21) is proved.

Replacing a and b with $\sqrt{a}$ and $\sqrt{b}$ in (21) results in $\frac{2(\sqrt{b}-\sqrt{a})}{lnb-lna} \leq \frac{\sqrt{a}+\sqrt{b}}{2}$, multiplying both sides by $\sqrt{a}+\sqrt{b}$, $\frac{b-a}{lnb-lna} \leq \frac{(\sqrt{a}+\sqrt{b})^2}{4}$ can be obtained.

Only the last inequality needs to be proven now.

$$\frac{4}{9}(\frac{a+b+\sqrt{ab}}{\sqrt{a}+\sqrt{b}})^2 \leq \frac{a+b}{2}$$

$$\Leftrightarrow \frac{(a+b)(\sqrt{a}+\sqrt{b})^2}{2} - \frac{4}{9}(a+b+\sqrt{ab})^2 \geq 0$$

$$\Leftrightarrow \frac{1}{18}[(a+b+4\sqrt{ab})(\sqrt{a}-\sqrt{b})^2] > 0 \quad (23)$$

Therefore, the inequality **Example 3.6** is proved.

4. Conclusion

This article first introduces the definition of convex functions from a geometrically intuitive perspective, then extends from two points on an interval to n points, skillfully demonstrating that the harmonic mean is less than or equal to the geometric mean, which is less than or equal to the arithmetic mean. In the subsequent section, it extends the ordinary convex functions to GA-convex functions, studies their sufficient and necessary conditions and properties, and ultimately constructs an inequality to prove the complex inequality chain in the example. It is evident that convex functions can easily be used to prove

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seemingly complex inequalities, but they also require assistance from other tools in mathematical analysis. It is hoped that in the future, building upon the foundation laid by this research, researchers can continue to advance the understanding and application of convex functions in the realm of inequalities.

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