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Numerical methods base on trinomial trees for option pricing

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Abstract. This paper investigates the approach to pricing European options, starting with one-step binomial tree pricing (a relatively simple way to calculate option value). In the next step, an additional possible rate of change of stock price is added to make the model more realistic, resulting in the one-step trinomial tree model. The model bounds the option price under the no-arbitrary principle. The paper then analyzes the circumstances under which options have a fixed price by completing the market and giving the solution formula of option price through the model. Last, put-call parity is used to prove the rationality of one-step trinomial model so that the model effectively prevents the occurrence of risk-free arbitrage in the market. This helps traders to price options reasonably in the market and maintains the stability of the options market.

Keywords: Numerical Methods, Trinomial Trees, Option Pricing

1. Introduction

With the rapid development of the global financial market and the abundance of financial products today, an increasing number of investors are turning their attention to derivatives transactions. The advantages of financial derivatives are obvious. With low cost, convenient and flexible means of creation, and powerful leverage, they can meet the target needs of different risk lovers and effectively hedge investment risks. In mature markets in Europe and the United States, financial derivatives market transactions Enthusiasm, the amount of value created is growing exponentially. Among them, the role of option products cannot be ignored. Whether it is stock options, stock index options, exchange rate options, currency options, or warrant product series similar to options, they all perform well. In recent years, the European option in the international financial market is one of the standard options that people are widely familiar with, and it can be said to be one of the most actively traded options in the financial derivatives market today.

The vigorous development of the derivatives market has brought more confidence and opportunities to investors and financial markets, and how to reasonably and effectively price derivatives has become the key. In this work, we mainly price European options through the binomial tree and its derived trinomial tree.

2. Binomial tree model

The binomial tree model is a well-used and effective method to determine option prices. In the binomial model, it is assumed that there are only two possibilities of stock value corresponding to appreciation and depreciation, respectively [2].

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Let us first formulate the underlying conditions. We assume only two primary assets are traded. One is the bond, a risk-free asset, which has features that it will have a constant rate of increase over every period. The start moment of the value of the bond simplifies as $B0$. The certain rate of the increase simplifies as r (which is the interest rate), so each year the annually changing rate is $(1+r)$. The value for the bond after one time period is therefore $B_1 = (1+r)B_0$. Another kind of the primary asset is the stock. The value of the stock can be divided into two situations since the value of the stock can either rise or go down. We assume that the stock appreciation occurs with probability denoted by p, then the probability of the depreciation is $1-p$. The value of the stock at the beginning is S_0, and value on the next step has p probability be the $S_1 = S_0 u$, and $1-p$ probability be the $S_1 = S_0 d$. Then, our strategy is to create a replication portfolio that has the same payoff on both situations as the option. Therefore, by the no-arbitrary principle, the price of the portfolio should be same as the price for the option. The strategy of that portfolio is N^B, N^S, which means to purchase N^B units of bond and N^S units of stock [3].

This model is only constructed with two branches, and this is the "binomial tree". It is inefficient to calculate the price of option with large errors. At this point we have to turn to a more effective method: the trinomial tree model. One of the most typical advantages of it is that is has one more fork, and then it would converge quickly to the real market price of the option. Now, we turned to the next part, the trinomial tree model.

Assume after the first period, the total value of the asset:

$$X_1 = N^B B_1 + N^S S_1 = \begin{cases} N^B(1+r) + N^S S_0 u\,, & with\,probability\;\; p \\ N^B(1+r) + N^S S_0 d\,, & with\,probability\;\; 1-p \end{cases} \tag{1}$$

The wealth $X1$ is equal to the option's payoff $F(S1)$. Then we have the following two equations:

$$\begin{cases} N^B(1+r) + N^S S_0 u = F(S_0\, u), \\ N^B(1+r) + N^S S_0 d = F(S_0\, d). \end{cases} \tag{2}$$

Combining these two equations can obtain:

$$\begin{cases} N^B = \frac{1}{1+r}\frac{uF(S_0 d) - dF(S_0 u)}{u-d}, \\ N^S = \frac{F(S_0 u) - F(S_0 d)}{(u-d)S_0}. \end{cases} \tag{3}$$

We set $q = \frac{1+r-d}{u-d}$. Then we express the investment $X0$ required to set up this portfolio at time 0 as:

$$X_0 = \frac{1}{1+r}\big(qF(S_0 u) + (1-q)F(S_0 d)\big). \tag{4}$$

But as a model with only two forks, it is inefficiently to calculate the price of option and there would be large errors. At this point we have to turn to the trinomial tree model, because the most typical advantage it has one more fork, and then it can converge quickly, helping us get the result faster. Now, we turned to the second part, the trinomial tree model.

3. Trinomial tree model

The $S(0)$ and $S(1)$ are defined to be the stock price at time 0 and 1, at present, the time $t = 0$. Assuming there are three cases at time t, which is stock price going up to $S(1) = S(0)u$, stock price goes to $S(t) = S(0)m$, or stock price goes down to $S(t) = S(0)d$, respectively. And $F(S0\text{u})$ is defined be the option price when stock price is $S_0 u$, the aiming of this part is to price the call option on time 0 defined as X_0, r is risk free interest rate. [2]

We can express these as follows:

$$S(1) = \begin{cases} S(0)u, & with\,probability\; P_u \\ S(0)m, & with\,probability\; P_m \\ S(0)d, & with\,probability\; P_d \end{cases} \tag{5}$$

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The payoff obtained by purchasing N^B units of bond with N^S units of stock replicate the same payoff as purchasing the option. When the stock price rises by a u amount of change, the second equation when there is an amount of change in the value of the stock, and the third equation when the value of the stock falls by a d amount. And the simulation of this portfolio holds when the starting price is equal to the price of the option's payoff at this stage. By using replication strategy (N^B, N^S), we obtain three equations:

$$\begin{cases} N^B(1+r) + N^S\, S_0\, u = F(S_0 u), \\ N^B(1+r) + N^S\, S_0\, m = F(S_0 m), \\ N^B(1+r) + N^S\, S_0\, d = F(S_0 d). \end{cases} \tag{6}$$

Unfortunately, equations with two unknowns and three equations do not always have a solution. Indeed, there are two variables (N^B, N^S) with three equations; therefore, it is not always possible to solve these equations and determine the option price this way.

By eliminating any two equations we can find that the difference in option prices (e.g.: $F(S_0 u) - F(S_0 m)$) is directly proportional to difference in ratio change in option prices (e.g.: $u - m$), therefore only situation when (N^B, N^S) can be solved is $F(S)$ is in form of $kS + c$ where $k, c \in \mathbb{R}$

When there is no solution that satisfy both three equations, we can only find the range of P_u, P_m, P_d to pricing options in a certain range.

The no-arbitrage principle yields that

$$X_0 = E[F\ (S_1)]/(1+r) \tag{7}$$

Also, by the strategy, we have

$$X_0 = N^B + N^S S_0 \tag{8}$$

Assumptions: $F(s) = s$ nd strike price is $S_0 m$

Thus, we can find the following two equations:

$$\begin{cases} S_0 = \frac{S_0(u q_u + m q_m + d q_d)}{1+r}, \\ 1 = q_u + q_m + q_d. \end{cases} \tag{9}$$

We now express q_u and q_d in terms of q_m. We also recall that the range of q_u and q_d is 0 to 1. Using these two facts, we can find express q_m as follows:

$$\begin{cases} q_u = \left[\frac{1+r-d}{u-d} - \frac{q_m(m-d)}{u-d}\right], \\ q_d = \left[\frac{u-1-r}{u-d} - \frac{q_m(u-m)}{u-d}\right]. \end{cases} \tag{10}$$

Since q_u and q_d are positive we have

$$q_m \in \left[0, \min\left(1, \frac{u-1-r}{u-m}, \frac{1+r-d}{m-d}\right)\right]. \tag{11}$$

By fixing two bifurcation paths in tree nominal model, we can evaluate the strategy (N^B, N^S), by comparing the payoff by this strategy on remaining path with known payoff, we can fix the upper and lower bound at time 0, which is X_0 in the following form:

$$\frac{1+r-m}{u-m}\, S_0(u-m) \le X_0 \le \frac{1+r-d}{u-d}\, S_0(u-m). \tag{12}$$

Interestingly, the lower bound is obtained by largest q_m and the upper bound is obtained by smallest q_m.

Similarly, if we use the formula $(X_0 = E[F(St)]/(1+r)$ and

$$\frac{1}{1+r} \min_{q_m} E^Q[F(S_t)] \le X_0 \le \frac{1}{1+r} \max_{q_m} E^Q[F(S_t)] \tag{13}$$

we can get the same result. More interestingly, this inequalitie become equalities when

$$F(S) = kS + c. \tag{14}$$

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4. Completing the market for call option

Previously, we focused on an incomplete market characterized by two variables (N^B and N^S) and three equations. Now, we aim to complete the market by introducing an additional primary asset in our replication strategy - a call option with a known option price (C_0) and a strike price ($K = kS_0$).

Therefore, the strategy now can be expressed in three equations as below similar to equation (6). Here, $\max(u-k,0)S_0$ is the payoff of this additional call option if the stock price goes up toS_0u, $\max(m-k,0)S_0$ is its payoff if the stock price goes to S_0m, and $\max(d-k,0)S_0$ is its payoff if the stock price goes down to S_0d, respectively. Hence this yields the following three equations:

$$\begin{cases} N^B(1+r)+\ N^S S_0 u+\ N^C\ max\ (u-k,0)S_0 = F(S_0u), \\ N^B(1+r)+\ N^S S_0 m+\ N^C\ max(u-k,0)S_0 = F(S_0m), \\ N^B(1+r)+\ N^S S_0 d+\ N^C\ max(u-k,0)S_0 = F(S_0d). \end{cases} \tag{15}$$

By incorporating the call option, we intend to complete the market. This means that we hopefully now can replicate all possible option payoffs. This in turn will yield unique prices for any option.

We now consider different cases of strike prices and study under which condition the market is complete, i.e., every option can be replicated.

(1). First situation is $u \leq k$. In this case, the market remains incomplete as the equations (15) will be simplified to:

$$\begin{cases} N^B(1+r)+\ N^S S_0 u = F(S_0u), \\ N^B(1+r)+\ N^S S_0 m = F(S_0m), \\ N^B(1+r)+\ N^S S_0 d = F(S_0d). \end{cases} \tag{16}$$

This is same as the equation (2), which represented an incomplete market.

(2). In second situation, where $m \leq k < u$. By simplifying equations (2), can obtain:

$$\begin{cases} N^B(1+r)+\ N^S S_0 u+\ N^C(u-k)S_0 = F(S_0u), \\ N^B(1+r)+\ N^S S_0 m = F(S_0m) \qquad , \\ N^B(1+r)+\ N^S S_0 d = F(S_0d) \qquad . \end{cases} \tag{17}$$

solving this we can get a mathematical formula for the strategy:

$$\begin{cases} N^B = \frac{1}{1+r}\frac{mF(S_0d)-dF(S_0m)}{m-d}, \\ N^S = \frac{F(S_0m)-F(S_0d)}{(m-d)S_0}, \\ N^C = \frac{(m-d)F(S_0u)-(u-d)F(S_0m)+(u-m)F(S_0d)}{(m-d)(u-k)S_0}. \end{cases} \tag{18}$$

By the no-arbitrage principal, the option price X_0 as time 0 should equals to initial value of the strategy, which is:

$$X_0 = N^B + N^S S_0 + N^C C_0 = \frac{1}{1+r}\left(\frac{(1+r)C_0}{(u-k)S_0}F(S_0u) + \frac{(1+r-d)(u-k)S_0-(1+r)(u-d)C_0}{(m-d)(u-k)S_0}F(S_0m) + \frac{(m-1-r)(u-k)S_0+(1+r)(u-m)C_0}{(m-d)(u-k)S_0}F(S_0d)\right) \tag{19}$$

The coefficient in front of $F(S_0u)$ is p_u, and that of $F(S_0m)$ is p_m, so does the p_d. We can put it back to double check whether the price of the call option is C_0. By above:

$$C_0 = \frac{E[F(S_t)]}{1+r} = \frac{1}{1+r}(p_u S_0(u-k)+0+0) = \frac{1}{1+r}\left(\frac{(1+r)C_0}{(u-k)S_0}S_0(u-k)+0+0\right) = C_0 \tag{20}$$

(3). Situation three is similar to that of situation two, where $d \leq k < m$. The difference on equations is that the second equation has one more term: $N^C(m-k)S_0$ as now $max(m-k,0)$ is no longer zero:

$$\begin{cases} N^B(1+r) + N^S S_0 u + N^C(u-k)S_0 = F(S_0 u), \\ N^B(1+r) + N^S S_0 m + N^C(m-k)S_0 = F(S_0 m), \\ N^B(1+r) + N^S S_0 d = F(S_0 d). \end{cases} \tag{21}$$

solving this we get

$$N^B = \frac{1}{1+r}\frac{k(m-u)F(S_0 d) - d(m-k)F(S_0 u) + d(u-k)F(S_0 m)}{(u-m)(d-k)} \tag{22}$$

$$N^S = \frac{(m-k)F(S_0 u) - (u-k)F(S_0 m) + (u-m)F(S_0 d)}{(u-m)(d-k)S_0} \tag{23}$$

$$N^C = \frac{(d-m)F(S_0 u) + (u-d)F(S_0 m) - (u-m)F(S_0 d)}{(u-m)(d-k)S_0} \tag{24}$$

By the no-arbitrage principal, the option price X_0 as time 0 should equals to initial value of the strategy, which is

$$X_0 = N^B + N^S S_0 + N^C C_0 = \frac{1}{1+r}\left(\frac{(m-k)(1+r-d)S_0 - (m-d)(1+r)C_0}{(u-m)(d-k)S_0}F(S_0 u) + \frac{(1+r-d)(k-u)S_0 + (1+r)(u-d)C_0}{(u-m)(d-k)S_0}F(S_0 m) + \frac{(1+r-k)(u-m)S_0 + (1+r)(u-m)C_0}{(u-m)(d-k)S_0}F(S_0 d)\right) \tag{25}$$

Similar to situation 2, p_u, p_m, p_d are the coefficients of $F(S_0 u)$, $F(S_0 m)$, $F(S_0 d)$ respectively. Then we can do the calculation again for different range of k:

$$C_0 = \frac{E[F(S_t)]}{1+r} = \frac{1}{1+r}(p_u S_0(u-k) + p_u S_0(m-k) + 0) = \frac{1}{1+r}\left(\frac{(m-k)(1+r-d)S_0 - (m-d)(1+r)C_0}{(u-m)(d-k)S_0}S_0(u-k) + \frac{(1+r-d)(k-u)S_0 + (1+r)(u-d)C_0}{(u-m)(d-k)S_0}S_0(m-k) + 0\right) = C_0 \tag{26}$$

(4). The fourth situation is when $0 \leq k < d$. Hence, can obtain following equations:

$$\begin{cases} N^B(1+r) + N^S S_0 u + N^C(u-k)S_0 = F(S_0 u), \\ N^B(1+r) + N^S S_0 m + N^C(m-k)S_0 = F(S_0 m), \\ N^B(1+r) + N^S S_0 d + N^C(d-k)S_0 = F(S_0 d). \end{cases} \tag{27}$$

Unfortunately, if we take the difference between the first and the second equation and the difference between the first and the third equation, we will find:

$$\begin{cases} (N^S + N^C)S_0(u-m) = F(S_0 u) - F(S_0 m), \\ (N^S + N^C)S_0(u-d) = F(S_0 u) - F(S_0 d). \end{cases} \tag{28}$$

These two equations are linear dependent, meaning that only if $F(S)$ is linear with S, we can find the solution.

According to the conclusion above about trinomial trees model for the situation 1 and 4, this means if and only if $F(S)$ is in form of $kS + c$ where $k, c \in \mathbb{R}$, we can complete the market, otherwise we can only find a range like what we do in an incomplete market. On the other hand, for cases 2 and 3, we can find a replication strategy for any option.

5. Put-call parity

$$c + \frac{K}{(1+r)^t} = p + S_0 \text{ [3]} \tag{29}$$

c: price of European call option
p: price of European put option

r: risk-free interest rate
K: strike price of the option
t: time period (in this case, $t = 1$ as we only consider one period model)

The put-call parity is an extremely important equation in the financial market for option pricing [3]. It effectively prevents the occurrence of risk-free arbitrage, because if the equation does not hold, traders can arbitrage by simply buying low and selling high. For example, if the left side of equation (29) is smaller than the right side, the trader can buy a call option, sell a put option, and short the stock. At the expiration of the option, if the stock price on the expiration date is higher than the strike price, the call option will be exercised; if the stock price falls below the strike price on the expiration date, the put option will be exercised [3-5]. In both cases, investors can use strike price to buy stocks and close out short selling stocks, and the calculated net return will always be greater than 0.

Consequently, pricing European options through trinomial tree model should always obey the put-call parity. We mainly focus on the cases where the market is complete so that the option only has unique price.

1. The first case is $m \leq k < u$ where $K = kS_0$. . We rearrange the put-call parity into: $c - p = S_0 - \frac{kS_0}{1+r}$

By equation (19), , we obtain left side of the equation:

$$c - p = \frac{1}{1+r}\begin{pmatrix} \frac{(1+r)C_0}{(u-k)S_0}(u-k)S_0, \\ +\frac{(1+r-d)(u-k)S_0 - (1+r)(u-d)C_0}{(m-d)(u-k)S_0}0, \\ +\frac{(m-1-r)(u-k)S_0 + (1+r)(u-m)C_0}{(m-d)(u-k)S_0}0 \end{pmatrix},$$

$$-\frac{1}{1+r}\begin{pmatrix} \frac{(1+r)C_0}{(u-k)S_0}0, \\ +\frac{(1+r-d)(u-k)S_0-(1+r)(u-d)C_0}{(m-d)(u-k)S_0}(k-m)S_0, \\ +\frac{(m-1-r)(u-k)S_0+(1+r)(u-m)C_0}{(m-d)(u-k)S_0}(k-d)S_0 \end{pmatrix} = \frac{(1+r-k)S_0}{(1+r)} = S_0 - \frac{kS_0}{1+r}. \quad (30)$$

This is exactly equal to right hand side of the equation, so the pricing model in this case fit the put-call parity.

The second case is $d \leq k < m$ where $K = kS_0$. We rearrange the put-call parity into: $c - p = S_0 - \frac{kS_0}{1+r}$

By (3.3.1), we obtain left side of the equation:

$$c - p = \frac{1}{1+r}\begin{pmatrix} \frac{(m-k)(1+r-d)S_0 - (m-d)(1+r)C_0}{(u-m)(d-k)S_0}(u-k)S_0, \\ +\frac{(1+r-d)(k-u)S_0 + (1+r)(u-d)C_0}{(u-m)(d-k)S_0}(m-k)S_0, \\ +\frac{(1+r-k)(u-m)S_0 - (1+r)(u-m)C_0}{(u-m)(d-k)S_0} \end{pmatrix},$$

$$-\frac{1}{1+r}\begin{pmatrix} \frac{(m-k)(1+r-d)S_0-(m-d)(1+r)C_0}{(u-m)(d-k)S_0}0, \\ +\frac{(1+r-d)(k-u)S_0+(1+r)(u-d)C_0}{(u-m)(d-k)S_0}0, \\ +\frac{(1+r-k)(u-m)S_0-(1+r)(u-m)C_0}{(u-m)(d-k)S_0}(k-d)S_0 \end{pmatrix} = \frac{(1+r-k)S_0}{1+r} = S_0 - \frac{kS_0}{1+r} \quad (31)$$

This is also exactly equal to right hand side of the equation, so the pricing model fit the put-call parity in both cases.

6. Conclusion

This paper introduced how replication portfolio works in binomial model for pricing European options throw no-arbitrary principle. Then it introduces a more efficient pricing model called trinomial tree model. Firstly, in incomplete market, we can only fix range of price. We then try to improve the strategy for portfolio, this behaviour is called as completing the market. After this, we discovered that in certain conditions of strike price, we do can find the unique price. In order to test the rationality of the price, we use the put-call parity principle and find that the price fit this equation. Therefore, pricing option through the trinomial tree model can effectively prevent the occurrence of arbitrage behavior.

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