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Investigating the drag impact on pendulum through numerical models and the stability analysis

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Abstract. As a traditional physical model, the pendulum plays a crucial role in the dynamics. However, most research focuses on the ideal pendulum structure which deviates from reality. In order to study the pendulum in practical situations, this paper not only investigates the ideal model but also the simple pendulum motion with drag. Firstly, this work derived analytical and numerical solutions of the simple pendulum motion equations, which come from physical phenomena, and compared these two solutions. Moreover, this study investigated the error of this numerical model and accomplished further research on demonstrating the conditions to meet the stability of the model.

Keywords: Pendulum, Numerical modelling, Air resistance, Ordinary differential equation, Stability analysis.

1. Introduction

1.1. Background

Christian Huygens is considered the inventor of the pendulum. The definition of the pendulum is that an object hangs at a fixed point so that it can move back and forth when it is affected by gravity, and its whole swing back and forth has a consistent time interval. Newton's second law of motion produced a second-order differential equation, $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\sin\frac{g}{L}\theta$, where g is the gravitational constant, L is the length of the string and θ is the angle formed by the string of pendulum and the y-axis. There are several ways to solve this equation, such as Jacobian elliptic functions and hypergeometric functions. In addition, we include another pendulum equation with air resistance, $\ddot{\theta} + \kappa\dot{\theta} + \frac{g}{L}\sin\theta = 0$, where κ is the damping constant that describes the drag.

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2. Simple pendulum without the influence of the air resistance

2.1. Investigation of the analytical solution

To get the analytical solution of the simple pendulum without the influence of the air resistance, we need to use the torque formula for rotational kinematics

$$\tau = I\alpha, \tag{1}$$

where I is the rotational inertia and α is the angular acceleration.

After expanding this formula, we obtain

$$-\sin\theta \times L = mL^2 \times \alpha, \tag{2}$$

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2},\end{aligned} \tag{3}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{-\sin\theta mg}{mL} = -\frac{g}{L}\sin\theta. \tag{4}$$

When deriving the analytical solution, we cannot get an explicit result if we retain $\sin\theta$. Hence, aiming to transform this non-linear ODE (ordinary differential equation) to a linear form, we transform $\sin\theta$ into θ by the small angle approximation[1] for the following deduction

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L}\theta. \tag{5}$$

After solving this linear ODE, we obtain the general solution

$$\theta(t) = c_1\cos(\sqrt{\frac{g}{L}}t) + c_2\sin(\sqrt{\frac{g}{L}}t), \tag{6}$$

where c_1 and c_2 are both constant.

Applying the initial condition $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$ into the equation, we obtain

$$c_1 = \theta_0, \tag{7}$$

$$c_2 = 0, \tag{8}$$

where θ_0 is the initial angle from the negative vertical axis.

Then, the analytical solution for the simple pendulum without the influence of the air resistance is

$$\theta(t) = \theta_0\cos(\sqrt{\frac{g}{L}}t). \tag{9}$$

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2.2. Investigation of the numerical solution

We have obtained the equation below in the process of deducing the analytical solution

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta = 0. \tag{10}$$

Discretizing this equation by the Forward Euler's Difference gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\theta_{i+1} - \theta_i}{\Delta t}, \tag{11}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{\Delta t^2}, \tag{12}$$

$$\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{\Delta t^2} = -\frac{g}{L}\sin\theta_i, \tag{13}$$

where Δt is the time slice.

According to this formula, we could get the following results by numerical simulation (Figure 1).

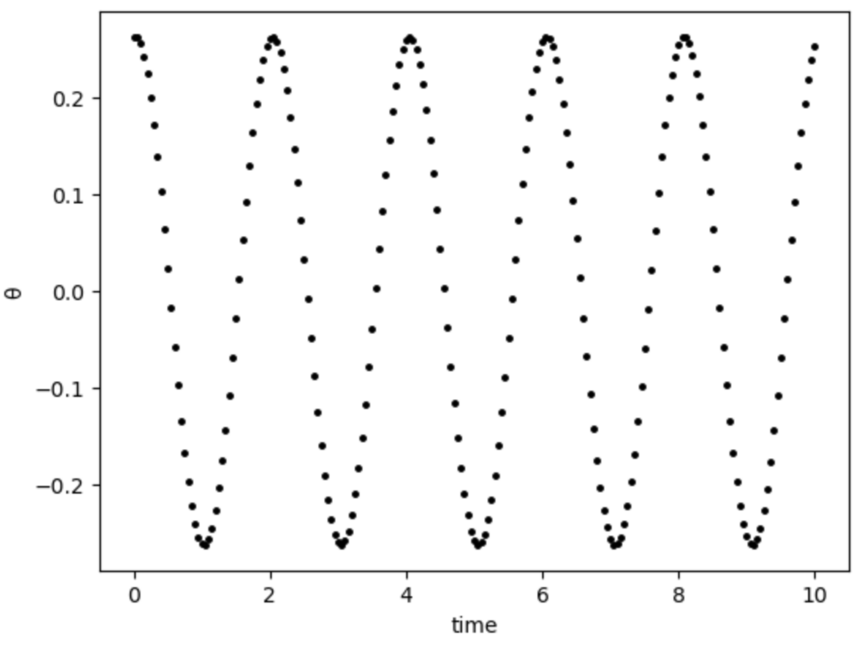

Figure 1. The numerical solution of the simple pendulum without the influence of the air resistance. (Parameters: θ_0=$\frac{\pi}{12}$, Δt=0.05(second), g=9.81, L=1(meter))

2.3. The comparison between the analytical solution and the numerical solution

We use a graph (Figure 2) to compare the numerical and analytical solutions.

The blue curve represents the analytical solution, while the black dots represent the numerical solution. This comparison reveals the differences between analytical and numerical solutions (the truncation error[2]). Obviously, with increases in time, the error between them will become bigger and bigger.

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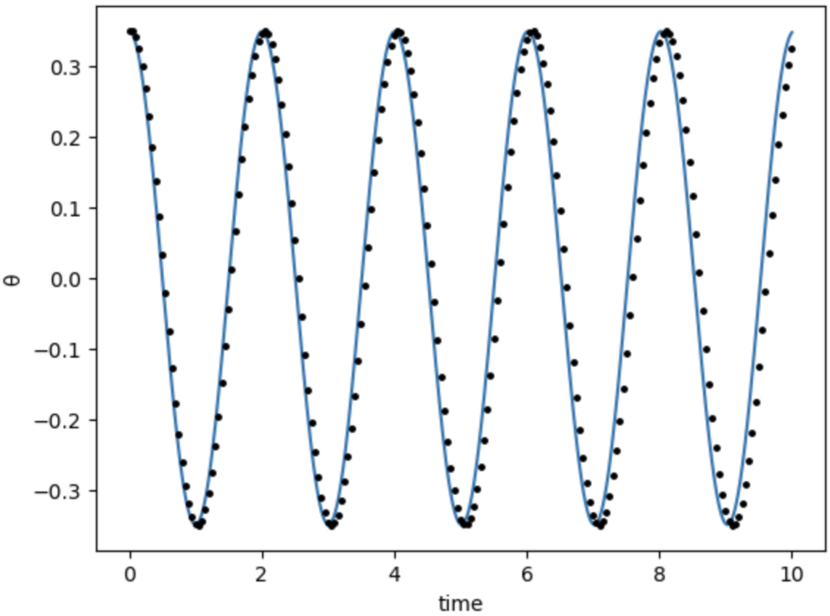

Figure 2. The comparison of the numerical and analytical solutions of the simple pendulum without the influence of the air resistance. (Parameters: θ_0=$\frac{\pi}{12}$, Δt=0.05, g=9.81, L=1)

Figure 3 shows the difference between these two solutions with a large initial angle. We can see that the error is enormous, which shows the inaccuracy of this numerical model. This considerable error is because we use a small angle approximation to change $\sin\theta$ into θ when deducing the analytical solution.

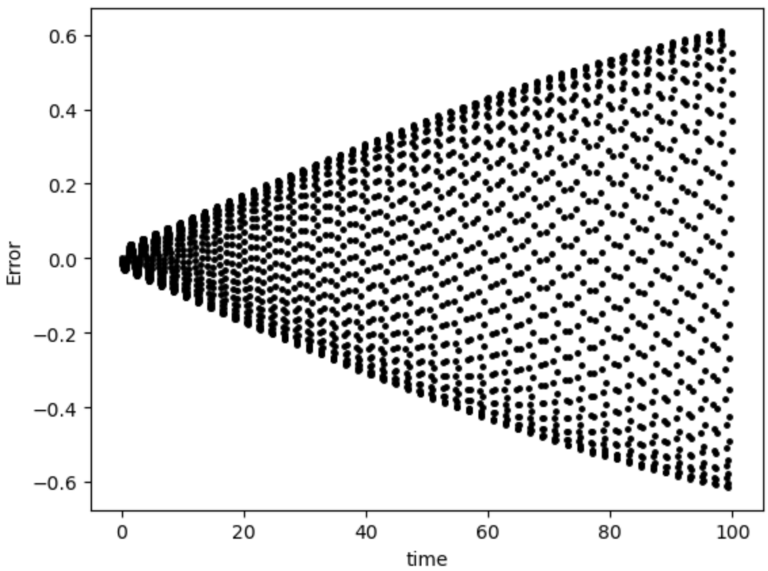

Figure 3. The error of the simple pendulum without the influence of the air resistance when the initial angle $\theta_0 = \frac{\pi}{9}$. (Parameters:θ_0=$\frac{\pi}{9}$,Δt=0.05,g=9.81,L=1)

We can reduce the initial angle θ_0 to reduce the error. The error is less than 1 percent when the initial

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angle θ_0 is less than $\frac{\pi}{12}$. As shown in Figure 4, when we select $\frac{\pi}{15}$ as the initial angle θ_0, the error is reduced.

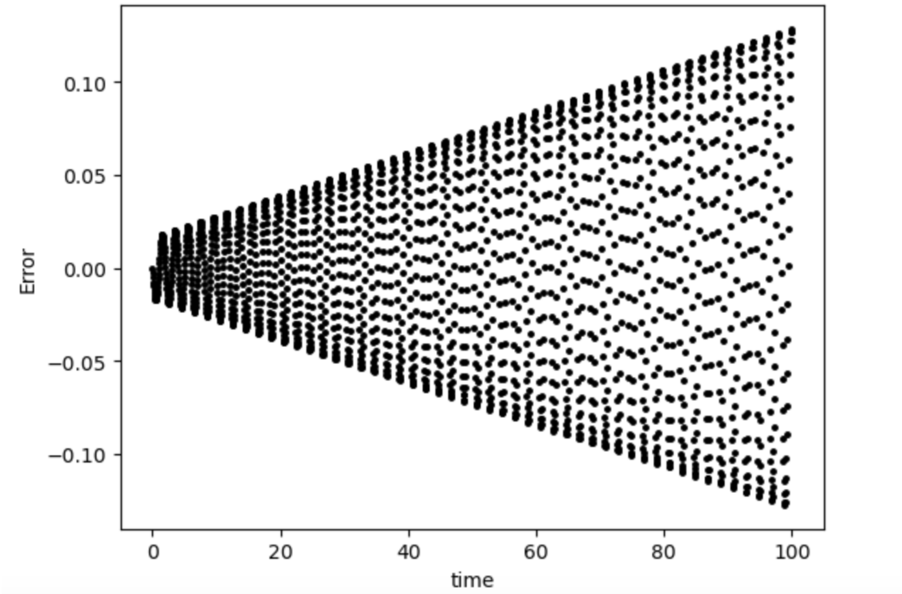
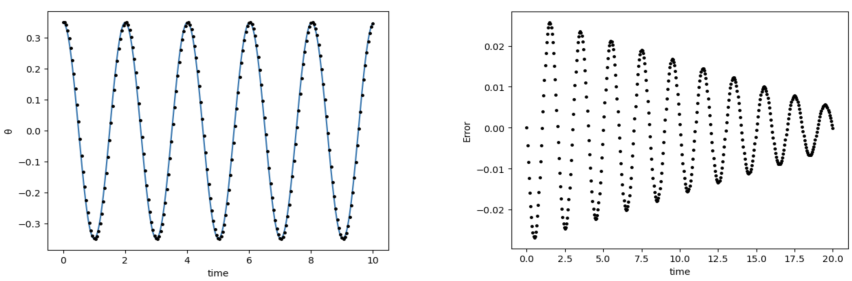


Figure 4. The error of the simple pendulum without the influence of the air resistance when the initial angle $\theta_0 = \frac{\pi}{15}$. (Parameters:$\theta_0$=$\frac{\pi}{15}$,$\Delta t$=0.05,$g$=9.81,$L$=1)

According to Figures 5(a) and 5(b), if θ is used instead of $\sin\theta$ when discovering the numerical solution, the numerical solution fits the analytical one better.



(a) (b)

Figure 5. The error of the simple pendulum without the air resistance when using a small angle approximation to calculate the numerical solution. (a) The comparison of the numerical and analytical solutions of the simple pendulum without the influence of the air resistance when using a small angle approximation to calculate the numerical solution.;(b)The corresponding truncation error. (Parameters:θ_0=$\frac{\pi}{12}$, Δt=0.05, g=9.81, L=1)

3. Simple pendulum under the influence of the air resistance

3.1. Investigation of the analytical solution

When investigating the analytical solution of a simple pendulum under the influence of air resistance, we need to divide the model into two parts. The first part is the pendulum bob, and the second is the string connecting the pendulum bob and the supporting point.

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Firstly, the formula of air resistance on pendulum bob is studied. For an object moving in a fluid, the following formula expresses the magnitude of the drag force[3] experienced

$$F_d = \frac{1}{2} C \rho A v^2, \tag{14}$$

where C represents the drag coefficient, ρ is the density of fluid through which the object passes, A is the frontal cross-sectional area of the object, and v is the object's velocity.

The drag coefficient is usually related to two variables. The first is the object's shape, and the second is the Reynolds number. The following formula generally calculates the Reynolds number

$$Re = \frac{\rho L |v|}{\mu}, \tag{15}$$

where L is the characteristic diameter of the object, and μ is the fluid's dynamic viscosity. When Reynolds number is in a particular range, the drag coefficient is asymptotically proportional to the Re^{-1}[4]. Therefore, drag force F_d is proportional to speed v. As a result, the formula below expresses the air resistance of the pendulum bob on a simple pendulum model moving in the air

$$F_{db} = -cv, \tag{16}$$

where c is a constant.

The second step is to derive the air resistance formula on the simple pendulum's string. To get this formula, we intercept a small part of the string and get the drag torque on the string. Then we have

$$\tau_s = kD\dot{\theta} \int_0^L r^2 \, \mathrm{d}r = \frac{kL^3 D}{3} \dot{\theta}, \tag{17}$$

where k is a constant, L is the length of the string, D is the string's diameter, and θ is the angle between the string and the vertical line passing through the supporting point.

Finally, through the equation of motion, we can obtain the formula shown below

$$\ddot{\theta} + \kappa \dot{\theta} + \frac{g}{L} \sin\theta = 0, \tag{18}$$

where κ is the damping constant, which is defined by

$$\kappa = \frac{kLD}{3m} + \frac{c}{m}. \tag{19}$$

We cannot get an explicit analytical solution through this formula, so we also use a small angle approximation to transfer $\sin\theta$ into θ. Besides, due to the small air resistance, we have

$$\kappa^2 < \frac{4g}{L}. \tag{20}$$

Therefore, the general solution of the motion of the pendulum is

$$\theta = \mathrm{e}^{-\kappa t/2} [A \cos(\omega t) + B \sin(\omega t)], \tag{21}$$

where A and B are constants, the angular frequency ω is given by

$$\omega = \sqrt{\frac{g}{L} - \frac{\kappa^2}{4}}. \tag{22}$$

After particularizing this general solution using initial conditions ($\theta(t=0) = \theta_0$, $\dot{\theta}(t=0) = 0$), we could finally obtain the analytical solution[5]

$$\theta = \theta_0 \mathrm{e}^{-\kappa t/2} \left[cos(\omega t) + \frac{\kappa}{2\omega} sin(\omega t) \right]. \tag{23}$$

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3.2. Investigation of the numerical solution

When deducing the general equation of the analytical solution, we have obtained the following equation

$$\ddot{\theta} + \kappa\dot{\theta} + \frac{g}{L}\sin\theta = 0. \tag{24}$$

After discretizing this equation by Forward Euler's Difference, we get the following formula

$$\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{\Delta t^2} + \frac{\theta_{i+1} - \theta_i}{\Delta t} = -\frac{g}{L}\sin\theta_i. \tag{25}$$

Based on this formula, we could get the following results by numerical simulation (Figure 6).

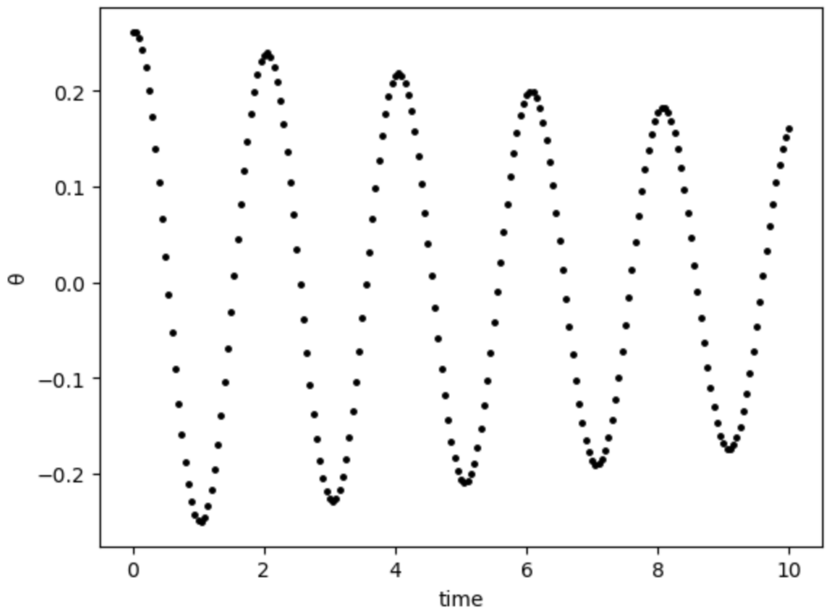

Figure 6. The numerical solution of the simple pendulum with the influence of the air resistance. (Parameters:θ_0=$\frac{\pi}{12}$, Δt=0.05, g=9.81, L=1, κ=9.02×10^{-2})

3.3. The comparison between the analytical solution and the numerical solution

Figure 7 shows the comparison of the numerical and analytical solutions.

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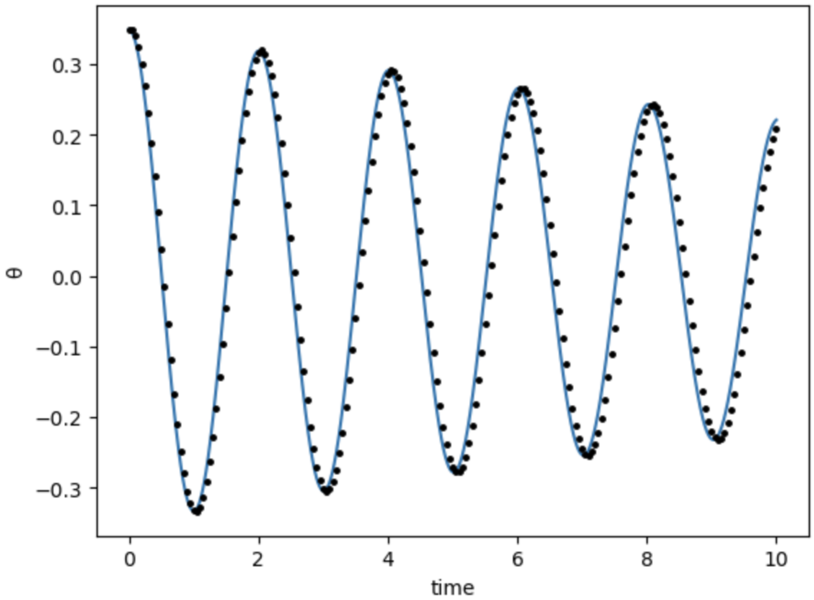


Figure 7. The comparison of the numerical and analytical solutions of the simple pendulum with the influence of the air resistance. (Parameters:θ_0=$\frac{\pi}{9}$, Δt=0.05, g=9.81, L=1, κ=9.02×10^{-2})

The blue curve and the black dots numerical and analytical solutions visualize analytical and numerical solutions. This figure shows the poor fit between these two solutions.

The error of these two solutions is shown in Figure 8. This figure shows us more about the changing trend of the difference between the two solutions. The error increases gradually in the first 10 seconds, but after 10 seconds, the error decreases slowly and stabilizes.

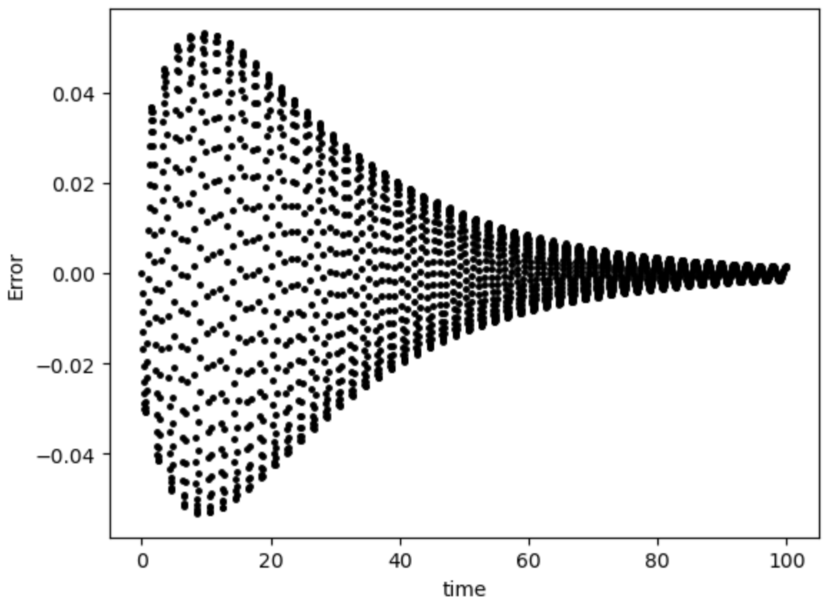


Figure 8. The error of the simple pendulum with the influence of the air resistance when the initial angle $\theta_0 = \frac{\pi}{9}$. (Parameters:θ_0=$\frac{\pi}{9}$, Δt=0.05, g=9.81, L=1, κ=9.02×10^{-2})

The main reason for the considerable short-term increase in the error is that we use a small angle approximation when investigating the analytical solution. When the initial angle is relatively big, the error will be correspondingly large, and this error will get bigger and bigger over time. However, because the numerical solution is stable under this particular condition we choose, the error will eventually become smaller and will tend to be 0 in the end. According to Figure 9, when we reduce the initial angle from $\frac{\pi}{9}$ to $\frac{\pi}{15}$, the error range becomes smaller, and the short-time increase becomes inconsiderable.

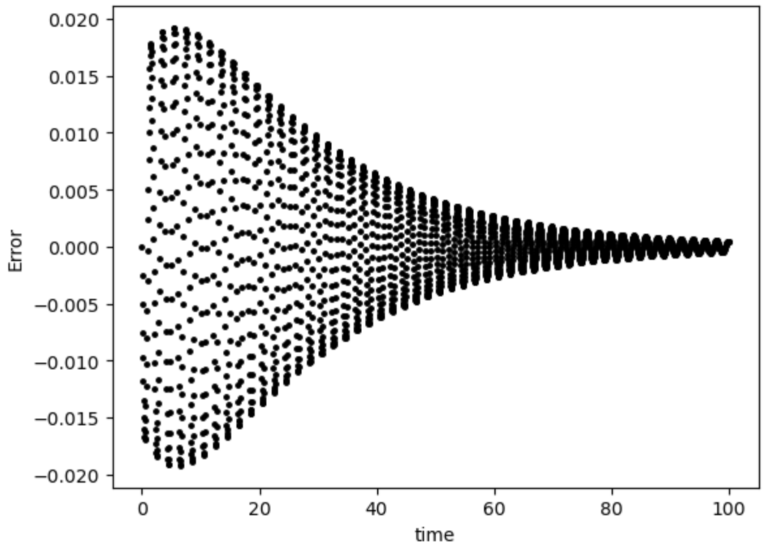
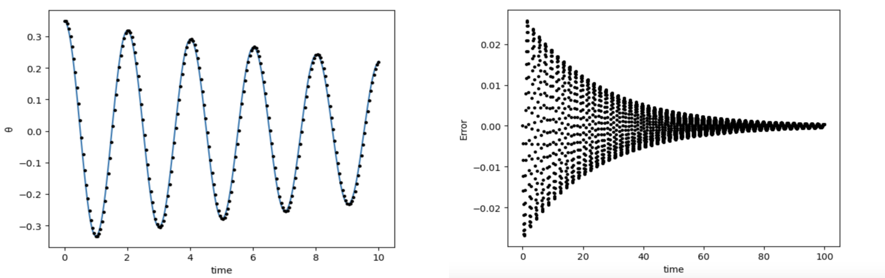


Figure 9. The error of the simple pendulum with the influence of the air resistance when the initial angle $\theta_0 = \frac{\pi}{15}$. (Parameters:$\theta_0$=$\frac{\pi}{15}$, Δt=0.05, g=9.81, L=1, κ=9.02$\times10^{-2}$)

If we use θ to approximate $\sin\theta$ when calculating the numerical solution, the short-term increase vanishes. According to Figures 10(a) and 10(b), the numerical solution is closer to the analytical one, and the error of these two solutions will decrease directly to 0.



(a) (b)

Figure 10. The error of the simple pendulum under the influence of the air resistance when using a small angle approximation to calculate the numerical solution. (a) The comparison of the numerical and analytical solutions of the simple pendulum under the influence of the air resistance when using a small angle approximation to calculate the numerical solution.; (b)The corresponding truncation error. (Parameters:θ_0=$\frac{\pi}{9}$, Δt=0.05, g=9.81, L=1, κ=9.02$\times10^{-2}$)

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4. Stability analysis

4.1. The simple pendulum model considering the influence of air resistance

Starting from the pendulum equation with drag

$$\ddot{\theta} + \kappa\dot{\theta} + \frac{g}{L}\sin\theta = 0. \tag{26}$$

we use a system of ordinary differential equations to express this equation below

$$\begin{cases} u = \frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \frac{\mathrm{d}u}{\mathrm{d}t} + \kappa u + \frac{g}{L}\sin\theta = 0. \end{cases} \tag{27}$$

For a small angle, we use θ to approximate $\sin\theta$. After discretizing this system using Forward Euler's difference, we could get

$$\begin{cases} \theta_{i+1} = \Delta t u_i + \theta_i \\ u_{i+1} = -\Delta t\kappa u_i - \frac{g\Delta t}{L}\theta_i + u_i. \end{cases} \tag{28}$$

Then we express this system in the form of the matrix

$$\begin{pmatrix} \theta_{i+1} \\ u_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & \Delta t \\ \frac{-g\Delta t}{L} & 1 - \Delta t\kappa \end{pmatrix} \begin{pmatrix} \theta_i \\ u_i \end{pmatrix}. \tag{29}$$

We could evaluate the eigenvalues of this matrix, which is

$$\lambda = \left(1 - \frac{\Delta t\kappa}{2}\right) \pm \sqrt{\frac{g\Delta t^2}{L} - \frac{\Delta t\kappa}{2}}i. \tag{30}$$

If magnitudes of eigenvalues are smaller than 1, this model is stable. Therefore, we get the result

$$\frac{g\Delta t}{L} < \kappa. \tag{31}$$

If we do not consider the drag, the value of κ will be 0. Then, there are no parameters that satisfy equation (31). It shows that the numerical model of pendulum motion without drag is usually unstable (Figure 3 and Figure 4 clarify this result). Therefore, this numerical model can only simulate the motion of a pendulum without drag for a short time.

Figure 11 shows an example of an error graph with parameters satisfying equation (31). We can see that the model is stable.

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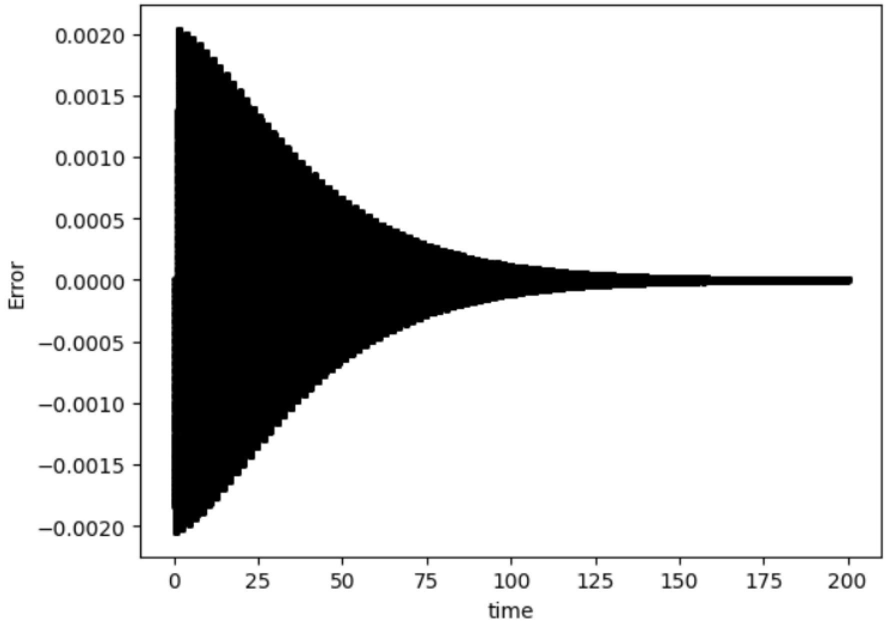

Figure 11. The stable case. (Parameters: θ_0=$\frac{\pi}{12}$, Δt=0.005, g=9.81, L=1, κ=9.02×10^{-2})

Figure 12 shows an unstable case. Parameters contradict ().

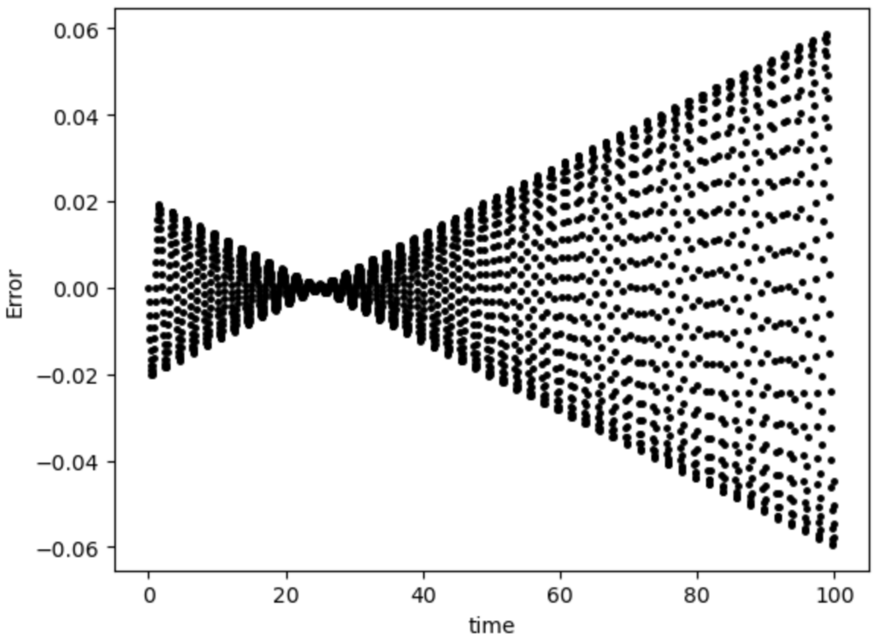

Figure 12. The unstable case. (Parameters:θ_0=$\frac{\pi}{12}$, Δt=0.05, g=9.81, L=1, κ=9.02×10^{-4})

Unfortunately, equation (31) is not a necessary and sufficient condition of stability. Figure 13 shows a counter-example. The parameters contradict (31), but this model is still stable.

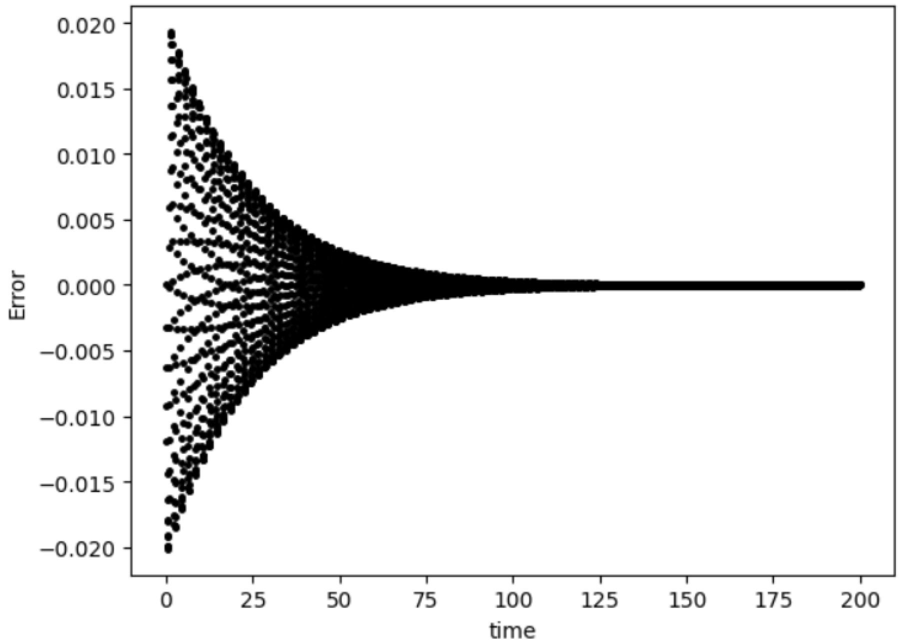

Figure 13. The counter-example of (*). (Parameters:θ_0=$\frac{\pi}{12}$, Δt=0.05, g=9.81, L=1, κ=9.02×10^{-2})

Therefore, equation (31) is a sufficient condition.

5. Conclusion

This paper studies the numerical solutions (using Forward Euler's difference) and stability conditions for the equations of the simple pendulum with and without drag. Like most experiments, we estimate the analytic solution using a small angle approximation. However, unlike other experiments, we have studied the effect of this estimation on the truncation error of the numerical model and have proposed corresponding optimization methods. In this paper, we also do the stability analysis. We have derived stability conditions of the simple pendulum with drag and without drag and a limitation of the numerical model.

There is still much room for improvement in our research. Firstly, we can attempt to derive the analytical solution without a small angle approximation, which leads to a more accurate result. Secondly, we can attempt to deduce the necessary and sufficient stability condition. In follow-up research, we will continue to finish the corresponding work.

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