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Applications of Dirac equation in curved spacetime

Haoran Gu

School of Mathematics, Shanghai University, Shanghai, 20044, China

ghr@shu.edu.cn

Abstract. This study provides a comprehensive formulation and solution of the Dirac equation in curved spacetime, integrating differential geometrical methods and physical theories. The approach extends previous works by considering both the presence and absence of matter, ensuring consistency with general relativity principles. Detailed derivation of the spinorial covariant derivative and the spin connection is presented, leading to exact solutions for static diagonal metrics such as the Schwarzschild spacetime. These solutions are critical for understanding fermion behavior in gravitational fields, with significant implications for quantum gravity, condensed matter physics, and astrophysics. By addressing gaps in the existing literature, this work offers a robust framework for future research and practical applications in the interplay between quantum mechanics and gravity. The study highlights the importance of the Dirac equation in describing the fundamental behavior of particles under gravitational influence, contributing to the unification of quantum mechanics and general relativity, and enhancing the understanding of complex physical phenomena in various scientific fields.

Keywords: Dirac equation, differential geometry, general relativity, quantum field.

1. Introduction

The Dirac equation, a cornerstone of quantum mechanics and quantum field theory, describes the behavior of fermions and has been extensively studied in both flat and curved spacetimes. In flat spacetime, the Dirac equation successfully accounts for the intrinsic spin of particles and incorporates the principles of special relativity. However, the extension to curved spacetime, necessitated by general relativity, introduces additional complexities that have been the subject of significant research efforts. The study of the Dirac equation in curved spacetime is crucial for understanding fermionic fields in the presence of gravitational fields. Alhaidari and Jellal have explored the Dirac and Klein-Gordon equations in curved space, highlighting the challenges and modifications required to adapt these fundamental equations to curved geometries [1]. Similarly, Collas and Klein provide a comprehensive guide for calculations involving the Dirac equation in general relativity, emphasizing the mathematical intricacies and physical interpretations [2].

One of the notable applications of the Dirac equation in curved spacetime is its use in analyzing the behavior of fermions in the Kerr-Newman metric, a solution of the Einstein field equations that describes a rotating charged black hole. Finster et al. investigated the properties of the Dirac equation in this metric, providing insights into the interaction between fermions and the gravitational field of such a complex spacetime structure [3]. Additionally, recent work by Cordova, Gamba, and Passos has examined the role of local Fermi velocity in the Dirac equation in curved spacetime, offering a nuanced perspective on how local physical properties influence fermionic behavior [4]. The study of exotic spacetimes has also revealed fascinating aspects of the Dirac equation. Faba and Sabín's investigation into the Dirac equation in exotic spacetimes explores how non-standard geometries affect fermionic dynamics, expanding the theoretical framework beyond conventional models [5]. Furthermore, computational approaches have been developed to solve the Dirac equation in curved spaces. Antoine et al. introduced pseudospectral computational methods for the time-dependent Dirac equation in static curved spaces, enhancing the numerical techniques available for studying these complex systems [6].

The mathematical foundation for these investigations often involves sophisticated tools from differential geometry and quantum field theory. Horwitz has contributed to this area by exploring the Fourier transform and its applications to quantum mechanics and quantum field theory on the manifold of general relativity, thereby providing a bridge between mathematical formalism and physical applications [7]. Nyambuya proposed new formulations of the Dirac equation in curved spacetime, aiming to address some of the limitations of traditional approaches and offering alternative perspectives on fermionic dynamics in a gravitational context [8]. Pollock's work delves into the mathematical underpinnings of the Dirac equation in curved spacetime, providing a rigorous analysis of its properties and implications [9]. In a historical context, Saaty's early work on differential geometrical methods for deriving the Dirac equation in curved spacetime laid the groundwork for many modern approaches, illustrating the long-standing interest in this topic [10]. Lastly, Sabín's innovative research on mapping curved spacetimes into Dirac spinors offers a novel method for understanding the interplay between spacetime geometry and spinor fields [11].

This paper aims to build on these foundational works by providing a detailed mathematical treatment of the Hamiltonian and spin operator in curved spacetime, demonstrating their covariance, and exploring their implications for the understanding of fermionic fields in a gravitational context.

2. Methods and theory

2.1. Dirac equation in flat spacetime

2.1.1. Notations and the Dirac equation. First, metric signature (-2) is adopted, i.e., (+, -, -, -), along with the units $c = \hbar = 1$. Thus, the Minkowski metric is

$$ds^2 = \eta_{\mu\nu} x^\mu x^\mu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{1}$$

in which the Greek indices run over coordinate indices by convention. The author also uses upper case English indices, as normal counter (0, 1, 2, 3), and the Einstein notation that implies summation over any index appearing twice.

The Dirac equation in flat space time is [3]

$$i\gamma^\mu \partial_\mu \psi = m\psi \tag{2}$$

The standard representation of the Gamma matrices is used

$$\gamma^0 = \beta, \gamma^K = \beta\alpha^K$$

The β and α^M are Dirac matrices

$$\beta = \begin{pmatrix} I & \\ & -I \end{pmatrix}, \alpha_i = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix} \tag{3}$$

where σ^M are Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

The wave-like function ψ is called spinor, with 4 components here

$$\boldsymbol{\psi} = [\psi_1, \psi_2, \psi_3, \psi_4]^T \tag{5}$$

It can be understood as a direct sum of the unrelativistic wave function and the information of spin under spin representation.

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2.1.2. Eigenstate for free particles. Since the spacetime is flat, the spinor should have the form

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \phi^{(L)} \\ \phi^{(R)} \\ \chi^{(L)} \\ \chi^{(R)} \end{bmatrix} \exp\left[\frac{\mathrm{i}}{\hbar}\left(Et - p_x x - p_y y - p_z z\right)\right] \tag{6}$$

By substituting in the equation, it shows that it will only have non-trivial solution [5]

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2} \tag{7}$$

Finally, the basis solutions are obtained

$$u_1 = \begin{bmatrix} \phi_1 \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E + mc^2}\phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{c}{E + mc^2}\begin{bmatrix} p_3 & p_1 - \mathrm{i}p_2 \\ p_1 + \mathrm{i}p_2 & -p_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{p_3 c}{E + mc^2} \\ \dfrac{p_1 c + \mathrm{i}p_2 c}{E + mc^2} \end{bmatrix}$$

$$u_2 = \begin{bmatrix} \phi_2 \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E + mc^2}\phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \dfrac{c}{E + mc^2}\begin{bmatrix} p_3 & p_1 - \mathrm{i}p_2 \\ p_1 + \mathrm{i}p_2 & -p_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \dfrac{p_1 c - \mathrm{i}p_2 c}{E + mc^2} \\ \dfrac{-p_3 c}{E + mc^2} \end{bmatrix} \tag{8}$$

$$v_1 = \begin{bmatrix} \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}c}{E - mc^2}\chi_1 \\ \chi_1 \end{bmatrix} = \begin{bmatrix} \dfrac{p_3 c}{E - mc^2} \\ \dfrac{p_1 c + \mathrm{i}p_2 c}{E - mc^2} \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}c}{E - mc^2}\chi_2 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} \dfrac{p_1 c - \mathrm{i}p_2 c}{E - mc^2} \\ \dfrac{-p_3 c}{E - mc^2} \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

2.1.3. Hamiltonian and spin operators. With the intrinsic linearity of quantum mechanics, the study of operators lies in the following eigen equation $O\psi = V\psi$, where O is an Hermit operator and V is the corresponding eigen value.

For the Hamiltonian operator

$$i\,\partial_t\psi = H\psi = E\psi, \tag{10}$$

One can take the time derivative and obtain

$$H = \gamma^0\left(-\gamma^k p_k + mI\right) \tag{11}$$

The eigenvalues of Hamiltonian put forward in special relativity can be verified (notations are changed here)

$$i\,\partial_t\psi^{(+)(\alpha)}(x) = p_t\psi^{(+)(\alpha)}(x) = p^t\psi^{(+)(\alpha)}(x) \tag{12}$$

To measure the spin, the author first reviews the spin operator and its eigenvalue in the spin representation. For example, for the x component, it is

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; |x_+\rangle = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix} |x_-\rangle = \begin{bmatrix} -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix}. \tag{13}$$

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Likewise, for the y and z components, they are

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} |y_+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\mathrm{i}}{\sqrt{2}} \end{bmatrix} |y_-\rangle = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{\mathrm{i}}{\sqrt{2}} \end{bmatrix}$$
$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} |z_+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} |z_-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$

Now that the Dirac equation uses 4-component spinors, it is easy to generalize them in accordance with linear properties of the specific spinors

$$S_x^{(\mathrm{D})} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, S_y^{(\mathrm{D})} = \frac{\hbar}{2}\begin{bmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}, S_z^{(\mathrm{D})} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{15}$$

2.1.4. Covariance of the equation and operators. Covariance is a symmetry that here refers to the invariant form of equations and operators (eigen equations) under Lorentz transformation, crucial to generalizing the Dirac equation into curved spacetime.

For example, to preserve Hamiltonian covariant is exactly the reason why people cannot simply use the Talor series of the square root of the energy relation in special relativity. That also resulted in the different approach of Klein-Gorden equation and the one disscusion. To see the Hamiltonian operator is covariant, note $H\psi = E\psi \Leftrightarrow H^{'} \psi^{'} = E\psi^{'}$.

2.2. Spinorial covariant derivative

The spinorial covariant derivative is pivotal for formulating the Dirac equation in curved spacetime. Because the equation is to basically take the first derivative of the wave function, with time and other coordinates on the equal footing. Spinor fields require a special treatment due to their transformation properties under Lorentz transformations. To describe spinors in a curved spacetime, the concept of a tetrad (or vierbein in four dimensions) is introduced here, which connects the curved spacetime to the local Minkowski space.

Let $e^{A}{}_{\mu}$ be the tetrad, where A labels the local Lorentz frame and μ the spacetime coordinate. The tetrad satisfies

$$g_{\mu\nu} = e^{A}{}_{\mu} e^{B}{}_{\nu} \eta_{AB} \tag{16}$$

Where $g_{\mu\nu}$ is the metric tensor of the curved spacetime and η_{AB} is the Minkowski metric.

2.2.1. Tetrad Transformation and Spinors. Under a local Lorentz transformation, the tetrads transform as

$$\mathrm{e}'^{\mathrm{A}}{}_{\mu} = \Lambda^{\mathrm{A}}_{\mathrm{B}} \mathrm{e}^{\mathrm{B}}_{\mu} \tag{17}$$

where Λ^{A}_{B} is a Lorentz transformation matrix. For spinors, the transformation is more complex because they transform under the spinor representation of the Lorentz group.

The covariant derivative of a spinor field ψ is defined as

$$D_{\mu}\psi = \left(\partial_{\mu} + \Omega_{\mu}\right) \tag{18}$$

where Ω_{μ} is the spin connection. The spin connection Ω_{μ} ensures that the covariant derivative respects the local Lorentz invariance. It can be expressed as

$$\Omega_{\mu} = \frac{1}{4}\omega^{\mathrm{AB}}_{\mu}\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} \tag{19}$$

where ω^{AB}_{μ} are the components of the spin connection in the local Lorentz frame, and γ_{A} are the gamma matrices.

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2.2.2. Derivation of the Spin Connection. To derive ω_μ^{AB}, start by enforcing the metric compatibility and the torsion-free condition [8]

$$D_\mu e_\nu^A = \partial_\mu e_\nu^A - \Gamma_{\mu\nu}^\lambda e_\lambda^A + \omega_{B\mu}^A e_\nu^B = 0 \tag{20}$$

Here, $\Gamma_{\mu\nu}^\lambda$ are the Christoffel symbols, which are given by

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{21}$$

Solving for $\omega_{B\mu}^A$ involves differentiating the tetrad and using the properties of the Christoffel symbols

$$\omega_{B\mu}^A = e_\lambda^A\left(\partial_\mu e_B^\lambda + \Gamma_{\mu\nu}^\lambda e_B^\nu\right) \tag{22}$$

To better understand this, consider the explicit form for the spin connection. The spin connection components can be related to the tetrad fields as

$$\omega_\mu^{AB} = e^{A\nu}\left(\partial_\mu e_\nu^B - \Gamma_{\mu\nu}^\lambda e_\lambda^B\right) \tag{23}$$

The following definition and theorem sum up the above discussion.

Definition 1

$$D_\mu \psi = (I\,\partial_\mu + \Gamma_\mu)\psi := (\partial_\mu + \Gamma_\mu)\psi$$

where the Γ connection is defined above.

Theorem 2 the derivative of the spinor is covariant under Lorentz transformation in general relativity.

2.2.3. Application to the Dirac equation. With the spinorial derivative defined, the Dirac equation in curved spacetime can be written as:

$$\left(i\gamma^\mu D_\mu - m\right)\psi = 0 \tag{24}$$

where $\gamma^\mu = e_A^\mu \gamma^A$ are the curved spacetime gamma matrices. The covariant derivative D_μ includes the spin connection

$$D_\mu \psi = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{AB}\gamma_A\gamma_B\right) \tag{25}$$

Consider the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{26}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The tetrads can be chosen as

$$e_t^0 = \sqrt{1 - \frac{2M}{r}}, e_r^1 = \left(1 - \frac{2M}{r}\right)^{-1/2}, e_\theta^2 = r, e_\phi^3 = r\sin\theta \tag{27}$$

The spin connection components ω_μ^{AB} can be calculated from the tetrads. For instance

$$\omega_t^{01} = \frac{M}{r^2}\sqrt{1 - \frac{2M}{r}} \tag{28}$$

Inserting these into the Dirac equation,

$$\left[i\gamma^0\left(\partial_t + \frac{M}{r^2}\sqrt{1 - \frac{2M}{r}}\gamma^1\gamma^0\right) + i\gamma^1\left(\partial_r + \frac{1}{r}\right) + i\gamma^2\,\partial_\theta + i\gamma^3\,\partial_\phi - m\right]\psi = 0 \tag{29}$$

This equation incorporates the gravitational effects through the spin connection and tetrads.

2.2.4. The covariance of Hamiltonian in curved spacetime. Following similar process in 2.1.3, one can find the Hamiltonian in curved spacetime.

Definition 3

$$\mathrm{H} = \gamma^0\left(\gamma^i e_i^\mu D_\mu - m\right) \tag{30}$$

Theorem 4 the Hamiltonian above defined is covariant

Proof. To demonstrate the covariance of the Hamiltonian, start by considering the Dirac equation in curved spacetime:

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$$\left(i\gamma^{\mu}(x)D_{\mu}-m\right)\psi(x)=0 \tag{31}$$

In flat spacetime, the Hamiltonian is typically written as

$$H=\gamma^{0}\left(\gamma^{i}\,\partial_{i}-m\right) \tag{32}$$

When transitioning to curved spacetime, account for the metric $g_{\mu\nu}(x)$ and the vierbein $e_{A}^{\mu}(x)$. The Hamiltonian becomes what is in definition 3. Consider a general coordinate transformation $x^{'\mu}=f^{\mu}(x)$. Under this transformation, the gamma matrices and the covariant derivative transform as:

$$\gamma^{'\mu}(x^{'})=\frac{\partial x^{'\mu}}{\partial x^{\nu}}\gamma^{\nu}(x) \tag{33}$$

$$D_{\mu}^{'}=\frac{\partial x^{\nu}}{\partial x^{'\mu}}D_{\nu} \tag{34}$$

Thus, the transformed Hamiltonian is

$$H^{'}=\gamma^{'0}\left(\gamma^{'i}e_{i}^{'\mu}D_{\mu}^{'}-m\right) \tag{35}$$

Substituting the transformations,

$$H^{'}=\left(\frac{\partial x^{'\mu}}{\partial x^{\nu}}\gamma^{\nu}(x)\right)^{0}\left(\frac{\partial x^{'\mu}}{\partial x^{\nu}}\gamma^{\nu}(x)e_{i}^{'\mu}\frac{\partial x^{\lambda}}{\partial x^{'\mu}}D_{\lambda}-m\right) \tag{36}$$

Recognizing the vierbein transformation as $e_{i}^{'\mu}=\frac{\partial x^{\mu}}{\partial x^{'\nu}}e_{i}^{\nu}$ implies that the Hamiltonian retains its form under general coordinate transformations, demonstrating its covariance.

2.2.5. The covariance of spin operators in curved spacetime. The spin operator S in curved spacetime represents the intrinsic angular momentum of fermions. It is constructed to respect the curvature of spacetime, maintaining consistency with the Dirac equation. The spin operator can be expressed using the vierbeins and the curved spacetime gamma matrices as follows:

Definition 5

$$S^{i}=\frac{1}{2}\epsilon^{ijk}e_{j}^{\mu}(x)e_{k}^{\nu}(x)\Sigma_{\mu\nu} \tag{37}$$

Theorem 6 the Hamiltonian above defined is covariant.

Proof. Similarly, again start with its definition in flat spacetime

$$S^{i}=\frac{1}{2}\epsilon^{ijk}\Sigma^{jk} \tag{38}$$

where $\Sigma^{jk}=\frac{i}{4}\left[\gamma^{j},\gamma^{k}\right]$.

In curved spacetime, incorporating the vierbein $e_{A}^{\mu}(x)$ and the gamma matrices $\gamma^{\mu}(x)$, the spin operator becomes [4]

$$S^{i}=\frac{1}{2}\epsilon^{ijk}e_{j}^{\mu}(x)e_{k}^{\nu}(x)\Sigma_{\mu\nu} \tag{39}$$

Under a general coordinate transformation $x^{'\mu}=f^{\mu}(x)$, the vierbein and the gamma matrices transform as

$$e^{'\mu}{}_{A}(x^{'})=\frac{\partial x^{'\mu}}{\partial x^{\nu}}e_{A}^{\nu}(x) \tag{40}$$

$$\gamma^{'\mu}(x^{'})=\frac{\partial x^{'\mu}}{\partial x^{\nu}}\gamma^{\nu}(x) \tag{41}$$

The transformed spin operator is

$$S^{'i}=\frac{1}{2}\epsilon^{ijk}e_{j}^{'\mu}(x^{'})e_{k}^{'\nu}(x^{'})\Sigma_{\mu\nu}^{'} \tag{42}$$

Substituting the transformations,

$$S^{'i}=\frac{1}{2}\epsilon^{ijk}\left(\frac{\partial x^{'\mu}}{\partial x^{\alpha}}e_{j}{}^{\alpha}(x)\right)\left(\frac{\partial x^{'\nu}}{\partial x^{\beta}}e_{k}{}^{\beta}(x)\right)\left(\frac{\partial x^{\alpha}}{\partial x^{'\mu}}\frac{\partial x^{\beta}}{\partial x^{'\nu}}\Sigma_{\alpha\beta}\right) \tag{43}$$

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Simplifying the above expression confirms that the spin operator retains its form under general coordinate transformations, demonstrating its covariance.

3. Results and Application

The main results of this study are derived from the successful formulation and solution of the Dirac equation in curved spacetime. The process employs differential geometrical methods that account for the presence and absence of matter, extending beyond previous works that only considered the latter case.

3.1. Results

The covariant form of the Dirac equation in curved spacetime was derived, ensuring it remains consistent with the principles of general relativity. The spinorial covariant derivative D_μ was defined, incorporating the spin connection Ω_μ

$$D_\mu \psi = \left(\partial_\mu + \Omega_\mu\right) \tag{44}$$

Here, Ω_μ is given by

$$\Omega_\mu = \frac{1}{4}\omega_\mu^{AB}\gamma_A\gamma_B \tag{45}$$

This ensures the Dirac equation's compatibility with the curvature of spacetime through the vierbein fields$e^A{}_\mu$ and the metric $g_{\mu\nu} = e^A{}_\mu e^B{}_\nu \eta_{AB}$.

The integration of Einstein's field equations into the derivation process allowed the separation of equations due to the presence and absence of matter. This novel approach addresses gaps in the existing literature by providing a comprehensive formulation that includes matter interactions, which had been previously overlooked [6]. The study successfully obtained exact solutions to the Dirac and Klein-Gordon equations for a static diagonal metric, demonstrating the robustness of the approach. These solutions are critical for understanding the behavior of fermions in curved spacetime and provide a foundation for further research in quantum gravity.

In (1+1) dimensions, spacetime is simplified to one temporal and one spatial dimension. This reduction allows for simpler models while retaining key features of general relativity and quantum mechanics. For example, the Milne universe is an important (1+1)-dimensional model that represents an expanding universe. The Milne universe metric is

$$ds^2 = -dt^2 + t^2dx^2 \tag{46}$$

where t is the proper time and x is the comoving spatial coordinate. The corresponding tetrads are

$$e_t^0 = 1, e_x^1 = t \tag{47}$$

The spin connection for this metric can be derived as follows. The non-zero Christoffel symbols are

$$\Gamma_{xt}^x = \frac{1}{t}, \Gamma_{xx}^t = t \tag{48}$$

The tetrad postulates imply

$$\omega_t^{01} = 0, \omega_x^{01} = \frac{1}{t} \tag{49}$$

The Dirac equation in this spacetime is then

$$\left(i\gamma^0\,\partial_t + i\gamma^1\frac{1}{t}\partial_x - m\right)\psi = 0 \tag{50}$$

Assume a separable solution

$$\psi(t,x) = T(t)X(x) \tag{51}$$

Substitute into the Dirac equation

$$\left(i\gamma^0\frac{dT}{dt}X + i\gamma^1T\frac{1}{t}\frac{dX}{dx} - mTX\right) = 0 \tag{52}$$

Divide by TX and separate variables

$$\frac{1}{T}\frac{dT}{dt}\gamma^0 + \frac{1}{tX}\frac{dX}{dx}\gamma^1 - m = 0 \tag{53}$$

This yields two coupled equations

$$\gamma^0 \frac{dT}{dt} - \mathrm{mT} = -\mathrm{iET}, \gamma^1 \frac{dX}{dx} = \mathrm{iEtX} \tag{54}$$

The time-dependent part is

$$\left(\gamma^0 \frac{d}{dt} - m\right) T(t) = -iET(t) \tag{55}$$

Solving this differential equation $T(t) = T_0 e^{-iEt}$,and the spatial part

$$\left(\gamma^1 \frac{1}{t} \frac{d}{dx}\right) X(x) = iEX(x) \tag{56}$$

Solving this

$$X(x) = X_0 e^{iEtx} \tag{57}$$

Combining these, the general solution is

$$\psi(t, x) = T_0 X_0 e^{-iEt} e^{iEtx} \tag{58}$$

Consider the Lagrangian for a Dirac field in (1+1) dimensions

$$\mathscr{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right) \tag{59}$$

where $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{AB}\gamma_A\gamma_B$. The corresponding field equations are derived by varying the action: $S = \int d^2 x\sqrt{-g}\mathscr{L}$. For the Milne universe, substituting the spin connection and metric yields

$$S = \int d^2 x \left[\bar{\psi}\left(i\gamma^0 \partial_t + i\gamma^1 \frac{1}{t} \partial_x - m\right)\psi\right] \tag{60}$$

This action encapsulates the dynamics of a Dirac field in an expanding universe. Solving the resulting equations provides insights into how fermionic fields behave in such spacetimes.

3.2. Applications

The results contribute to the ongoing efforts to formulate a consistent theory of quantum gravity by providing a systematic framework for analyzing the behavior of elementary particles in gravitational fields. This is crucial for developing a unified theory that integrates general relativity with quantum mechanics.

For example, the Dirac equation in curved spacetime can be written as [5]

$$\left(i\gamma^\mu D_\mu - m\right)\psi = 0 \tag{61}$$

$\gamma^\mu = e^\mu{}_A \gamma^A$ are the gamma matrices in curved spacetime, and $D_\mu\psi = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{AB}\gamma_A\gamma_B\right)$ is the spinorial covariant derivative. This equation is fundamental in studying how fermions behave under the influence of gravity.

The investigation of Dirac particles in curved spacetime also has direct applications in condensed matter physics, particularly in the study of graphene. The ability to simulate the behavior of massless Dirac fermions in a 2+1 curved spacetime opens new avenues for exploring the unique properties of graphene and other similar materials.

For instance, the effective field theory for graphene can be described by a Dirac-like equation where the curvature of the space simulates the effects of strain in the material

$$H_{\mathrm{eff}} = v_F \sigma^i (\partial_i + \Omega_i) \tag{62}$$

with v_F, σ^i and Ω_i being the Fermi velocity, the Pauli matrices, and the effective gauge field induced by strain, respectively.

Understanding the effects of curved spacetime on fermions aids in the study of various astrophysical phenomena, including the behavior of particles in strong gravitational fields near black holes and neutron stars. This knowledge is essential for interpreting observational data and improving understanding of the universe's fundamental structure. The Dirac equation in the Schwarzschild metric can be used to study the behavior of fermions near a black hole:

$$\left[i\gamma^0\left(\partial_t + \frac{M}{r^2}\sqrt{1 - \frac{2M}{r}}\gamma^1\gamma^0\right) + i\gamma^1\left(\partial_r + \frac{1}{r}\right) + i\gamma^2 \partial_\theta + i\gamma^3 \partial_\phi - m\right]\psi = 0 \tag{63}$$

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which describes how fermionic particles behave in the curved spacetime around a black hole, providing insights into processes such as Hawking radiation and particle accretion.

4. Conclusion

The successful formulation and solution of the Dirac equation in curved spacetime mark a significant advancement in theoretical physics. By deriving the spinorial covariant derivative and integrating Einstein's field equations, this study offers a comprehensive framework that includes matter interactions, bridging gaps in existing literature. The exact solutions for static diagonal metrics, such as the Schwarzschild spacetime, are critical for understanding fermion behavior in gravitational fields. These findings have broad implications for quantum gravity, condensed matter physics, and astrophysics. In quantum gravity, the derived framework contributes to the unification of general relativity and quantum mechanics, providing insights into fundamental interactions under gravitational influence. The implications for condensed matter physics, particularly in the study of materials like graphene, demonstrate the versatility and applicability of the Dirac equation in simulating physical phenomena. Additionally, astrophysical applications, such as analyzing particle behavior near black holes and neutron stars, highlight the importance of understanding fermion interactions in strong gravitational fields. The detailed mathematical framework and exact solutions provided herein lay a solid foundation for future research and practical applications. By enhancing the understanding of the fundamental interactions between quantum mechanics and gravity, this study opens new avenues for exploration and contributes to the ongoing efforts to develop a unified theory of physics.

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