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Harmonic analysis approach to the proof of Heisenberg inequality

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Abstract. the Heisenberg uncertainty Principle is a fundamental principle in quantum mechanics, which was developed by the German physicist Werner Heisenberg and was proposed by him in 1927. This principle states that for a pair of physical quantities that share phase space, such as position and momentum, it is impossible to accurately measure their values at the same time. There are several variants of it in harmonic analysis studies, and the article will introduce some of them in R^1 space and L^2 space. In the process of providing the Heisenberg inequality, the article proved the Plancherel identity and Schwartz inequality by using Fourier transform and inverse Fourier transform. Finally, author solved the equation of the wave function $\varphi(x)$. The famous physicists Heisenberg proposed one of the more novel ideas in quantum mechanics – the existence of unobservable orbits cannot be assumed, which did bring great influence in quantum mechanics. The article will introduce the conception of Heisenberg inequality and try to finish the proof.

Keywords: Fourier transform, inverse Fourier transform, Cauchy-Schwartz inequality, Plancherel identity.

1. Introduction

Harmonic analysis is a branch of mathematics that deals with the expansion of functions into Fourier series or Fourier integrals and related problems. It originates from the superposition problems of decomposing a periodic oscillation into simple harmonic oscillation in physics, and has now developed into a discipline with wide application [1]. Harmonic analysis not only involves mathematics, but also plays an important role in many disciplines such as information processing and quantum mechanics. Harmonic analysis is also used in tidal analysis, through which the tidal changes in a certain period can be calculated and the tidal properties of the area can be analyzed. Thus, harmonic analysis of tides is an important method used in Marine engineering for the analysis prediction of tidal changes [2].

Quantum mechanics, as a physical theory, is a branch of physics that studies the motion laws of microscopic particles in the material world. It mainly studies the basic theories of the structure and properties of atoms, molecules, condensed matter, as well as atomic nuclear and elementary particles. Together with relativity, it forms the theoretical basis of modern physics. Quantum mechanics is not only one of the basics theories of modern physics, but also widely used in chemistry and many other modern technologies [3].

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Heisenberg inequality, which is also called by Heisenberg principle of uncertainty, is the bridge between the two theories. And the article will focus on how to prove the Heisenberg inequality using harmonic analysis and apply the results to the quantum mechanics.

2. Methods and Theory

2.1. background knowledge and method

The author can use the method of taking the sum of a series of orthogonal basis to approximate a periodic function, essentially turning it into a sum of functions representing different frequencies [4]

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ik\omega t} \tag{1}$$

To calculate c, the author will use the properties of orthogonal basis to simplify the result. Multiplying $e^{-in\omega t}$ on both sides of the equations, the author will get

$$f(t)e^{-in\omega t} = \sum_{k=-\infty}^{+\infty} c_k e^{i(k-n)\omega t} \tag{2}$$

Taking the definite integral from 0 to T at both ends of the above equation

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt \tag{3}$$

Now having accessed with the definition of Fourier series, the article will introduce Fourier transform to you. The author will begin by expanding the function *f(t)* as Fourier series on the interval $[-T/2, T/2]$

$$f(t) = \sum_{-\infty}^{+\infty} c_n e^{in\omega t}\ , c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega t} \tag{4}$$

On can take the limit of T tends to infinite, then the author will get [5]

$$f(t) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right)e^{i\omega t}d\omega \tag{5}$$

Then, the article has shown the definition of Fourier transform, the new function is only related to the given frequency w, which describes the distribution density of the component in *f(t)*

$$\hat{f} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{6}$$

2.2. structure and content of the article

In the first part of the Sec. 3.1, the author will choose a certain dense subspace with good properties $C_{\downarrow}^{\infty}(R^1)$, in which space, the equality can be proved easily only through properties of complex numbers, integration by parts, Cauchy-Schwartz inequality and the properties of rapidly decreasing function. All the properties will be proved by the author later. In the second Sec. 3.2, the author will generalize the results proved in $C_{\downarrow}^{\infty}(R^1)$ to a more general function space $L^2(R^1)$. The author uses a function series f_n to approximate function *f*, which converges uniformly to 0 in the integral as n approaches infinity, which is also convergent, the original function and derivative being convergent under the L_2 norm. In this circumstance, the derivative approximates the *f* derivative. The squares of the two norms remain 0 and form the square of the integral. If *n* goes to infinity, the equation holds, which is easy to estimate later with inequalities. Finally, the author finds the specific function $\varphi(x)$ by solving an ODE, hence getting the results and the application condition.

3. Results and Application

3.1. proof in $\boldsymbol{C_{\downarrow}^{\infty}(R^1)}$space

The calculation and properties of complex numbers are very important in this paper. By doing which, author will do the contraction [6]

$$|A| \geq \frac{1}{2}(A + A^*) \qquad (7)$$

If $A = a + b$, then $A^* = (a + b)^* = a^* + b^*$. One can prove the inequalities in $R^1(C_\downarrow^\infty)$ space by using the calculation related to the Fourier transform and the inverse Fourier transform.

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \qquad (8)$$

$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t}d\omega \qquad (9)$$

By using the properties of rapidly decreasing function, one can know

$$\lim_{x\to\infty} xf(x) = 0 \qquad (10)$$

$$\int_{-\infty}^{+\infty} |\varphi(x)|^2 = 1 \qquad (11)$$

Then, let the author prove the Heisenberg inequality if $\boldsymbol{f} \in \boldsymbol{C}_\downarrow^\infty(\boldsymbol{R}^1)$, which is a rapidly decreasing function.

$$I = 4\pi^2 \int_{-\infty}^{+\infty} x^2|f(x)|^2dx \int_{-\infty}^{+\infty} \gamma^2 \left|\hat{f}(x)\right|^2 dx \qquad (12)$$

$$I = \int_{-\infty}^{+\infty} |xf(x)|^2\, dx \int_{-\infty}^{+\infty} \left|2\pi i\gamma \hat{f}(x)\right|^2 dx \qquad (13)$$

Then, the author will use Plancherel's identity. The proof is as follows.

$$J = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt \qquad (14)$$

$$= \int_{-\infty}^{+\infty} x(t)\frac{1}{2\pi}\int_{-\infty}^{+\infty} x^*(j\omega)e^{-j\omega t}d\omega dt \qquad (15)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} x^*(j\omega)\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\, d\omega \qquad (16)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} x^*(j\omega)\, x(j\omega)d\omega \qquad (17)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |x(j\omega)|^2\, d\omega \qquad (18)$$

In the steps 2 and 3, just take inverse Fourier transform and Fourier transform in order. Then, by using Plancherel's identity, the author can turn the formula into [7]

$$I = \int_{-\infty}^{+\infty} |xf(x)|^2dx \int_{-\infty}^{+\infty} \left|\frac{df(x)}{dx}\right|^2 dx \qquad (19)$$

Then, the author will use Cauchy-Schwarz's inequality: For any two elements x and y in the inner product space, Schwarz's inequality states that the square of the absolute value of their inner product is not greater than the product of their norms.

Here, the author is going to prove Schwartz's inequality. For functions φ, δ: $|(\varphi, \delta)| \leq |\varphi||\delta|$,

$$\vartheta = \delta - \frac{(\varphi, \delta)}{|\varphi|^2}\varphi, \qquad \left(|\varphi|^2 \geq 0\right) \qquad (20)$$

$$(\vartheta,\vartheta)=|\delta|^2-(\delta,\varphi)\frac{(\varphi,\delta)}{|\varphi|^2}-\frac{(\varphi,\delta)^*}{|\varphi|^2}(\varphi,\delta)+\frac{(\varphi,\delta)^*(\varphi,\delta)}{|\varphi|^4}|\varphi|^2=\varphi^2-\frac{|(\varphi,\delta)|^2}{|\varphi|^2} \quad (21)$$

And because of $|\varphi|^2\geq 0$, then the formula can be written as [8]

$$|(\varphi,\delta)|\leq|\varphi||\delta| \quad (22)$$

The result is as required. According to the inequality which the author has proved yet, one can turn upwards formula into

$$I\geq\left[\int_{-\infty}^{+\infty}|xf'(x)f^*(x)|dx\right]^2 \quad (23)$$

Because the basic property of complex number (the norm of a complex number is greater than the norm of its real part), $|A|\geq\frac{1}{2}(A+A^*)$. Then, the author can change the result into

$$I\geq P=\left[\int_{-\infty}^{+\infty}x\frac{1}{2}\left(f'f^*+\left(f'f^*\right)^*\right)dx\right]^2 \quad (24)$$

The *properties* of complex function show that If $A=a+b$, then $A^*=(a+b)^*=a^*+b^*$. By using this properties, the author can rewrite result [9]

$$P=\frac{1}{4}\left[\int_{-\infty}^{+\infty}x\frac{1}{2}\left(f'f^*+f'^*f\right)dx\right]^2 \quad (25)$$

By using the derivative multiplication rule: $(uv)'=u'v+v'u$, the result $(f'f^*+f'^*f)$ can be written as $2(|f|^2)'$. Thus, the result can be transformed into

$$P=\frac{1}{4}\left[\int_{-\infty}^{+\infty}x\left(|f|^2\right)'dx\right]^2 \quad (26)$$

In the next step, the author will use integration by parts

$$P=\frac{1}{4}\left[\left(x|f|^2\right)_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}|f|^2dx\right]^2 \quad (27)$$

Because the function is a rapidly decreasing function, which means the first polynomial equals to 0. Then people can get the result

$$P=\frac{1}{4}\left[\int_{-\infty}^{+\infty}|f(x)|^2\,dx\right]^2=\frac{1}{4}\|f\|_2^4. \quad (28)$$

People now know in this space, the norm of f is 1. Thus, the article have got the result

$$\left(\int_{-\infty}^{+\infty}x^2|\varphi(x)|^2dx\right)\left(\int_{-\infty}^{+\infty}\varepsilon^2|\hat{\varphi}(x)|^2d\varepsilon\right)\geq\frac{1}{16\pi^2} \quad (29)$$

3.2. proof in $\boldsymbol{L^2(R^1)}$ *space*

Firstly, the author proved in a certain dense subspace with good properties. Then, the author will generalize to a more general function space, which is proving the equality in L^2space.

Because $\|xf\|_2>0$ (function f is a rapidly decreasing function in any space), one may assume that $\|\gamma\hat{f}\|_2<\infty$. If the opposite circumstance holds, there's nothing to prove because the result will be much greater. In this case, you can't measure accurately both the location and the momentum of a particle. This means the Plancherel's identity which the author has proved, $\widehat{f'}=2\pi i\hat{f}$, also holds in the $L^2(R^1)$ space. Thus, the proof for $C_\downarrow^\infty(R^1)$ also holds for this circumstance [10]

$$\int_{-\infty}^{+\infty}x\left(f'f^*+f'^*f\right)dx \quad (30)$$

Now, the author set a function $f_n \in C_{\downarrow}^{\infty}$in order to approximate f . Because the subspace is dense, there are continuous functional series. As for $n \to \infty$, the series converges uniformly to 0 in the integral, which is also analytically convergent, and the original function and derivative converge under the L^2 norm. The function meets the requirement

$$\lim_{n\uparrow\infty}\int_{-\infty}^{+\infty}(1+4\pi^2\gamma^2)\left|\widehat{f_n}+\hat{f}\right|^2 d\gamma \tag{31}$$

$$\lim_{n\uparrow\infty}\|f_n-f\|_2^2+\left\|f_n'-f'\right\|_2^2=\lim_{n\uparrow\infty}\int_{-\infty}^{+\infty}(1+4\pi^2\gamma^2)\left|\widehat{f_n}-\hat{f}\right|^2 d\gamma=0. \tag{32}$$

They can be proved simply by finding two equations

$$\|f_n-f\|_2^2=\int_{-\infty}^{+\infty}\left|\widehat{f_n}-\hat{f}\right|^2 d\gamma\,,\left\|f_n'-f'\right\|_2^2=\int_{-\infty}^{+\infty}4\pi^2\gamma^2\left|\widehat{f_n}-\hat{f}\right|^2 d\gamma \tag{33}$$

And because of$|f_n(x)-f(x)|\le\left\|\widehat{f_n}-\hat{f}\right\|_1$. The author can use Cauchy-Schwartz inequalities to zoom the formula

$$I\le\left[\int_{-\infty}^{+\infty}(1+4\pi^2\gamma^2)^{-1}d\gamma\right]^{1/2}\left[\int_{-\infty}^{+\infty}(1+4\pi^2\gamma^2)\left|\widehat{f_2}-\hat{f}\right|^2 d\gamma\right]^{1/2} \tag{34}$$

By using Schwartz's inequality. For any fixed x, and one can have

$$\begin{aligned}\int_{-\infty}^{+\infty}x\left(f'f^*+f'^*f\right)dx&=\lim_{l\uparrow\infty}\lim_{n\uparrow\infty}\int_{|x|\le l}x\left(f_n'f_n^*+f_n'^*f_n\right)dx\\&=\lim_{l\uparrow\infty}\lim_{n\uparrow\infty}\left[x(|f_n|^2)_{-l}^{l}-\int_{|x|\le l}|f_n|^2dx\right]\\&=\lim_{l\uparrow\infty}l\left[|f(l)|^2+|f(-l)|^2\right]-\|f\|_2^2\end{aligned} \tag{35}$$

The proof in this step is with the same logic with the one in 2.1, because the function is a rapidly decreasing function, one can rewrite the result into $-\|f\|_2^2$. Then, the author has finished the whole proof in this space.

For the next step, the article will focus on the specific wave function φ . By observing the proof, author finds that for specific β, which always satisfies a differential equation $\varphi'(x)=\beta x\varphi(x)$. Then, the author solves the ODE by separating variables, the author gets the solution, which is $\varphi(x)=Ae^{\beta x^2/2}$, and $|A|^2=\sqrt{2B/\pi}, \beta=-2B<0$.

3.3. Results and Application

The exact expression of Heisenberg's inequality first appeared in the study of quantum mechanics when researchers were trying to determine the position and momentum of an example at the same time. Suppose that there is a electron moving along a line and there are laws of physics that can be described by a state function φ.

The position of the electron is described by the probability that the particle located in (*a,b*). Function $|\varphi(x)|^2dx$ is the density function, and the expectation function is

$$\bar{x}=\int_{-\infty}^{+\infty}x|\varphi(x)|^2 \tag{36}$$

Then the author can discuss the value of the *x* that minimizes the error, which is a great significance in quantum mechanics. The error is

$$\int_{-\infty}^{+\infty}(x-\bar{x})^2|\varphi(x)|^2dx \tag{37}$$

And the error of the momentum is

$$\int_{-\infty}^{+\infty}(\varepsilon-\varepsilon_0)^2|\hat{\varphi}(\varepsilon)|^2d\varepsilon \tag{38}$$

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4. Conclusion

According to the Heisenberg inequality the author has proved, the result is just the product of the error of the position and momentum is greater than $1/16\pi^2$. There are plenty of applications of Heisenberg inequality. For example, magnetic resonance imaging is a medical imaging technique used to observe the internal structure of biological tissues. In magnetic resonance imaging (MRI), the resonance signal of the atomic nuclear can be obtained by applying the enhanced magnetic field and electromagnetic pulse to the object under test. According to Heisenberg's uncertainty principle, doctors cannot accurately measure the position and momentum of an atomic nucleus at the same time, so in MRI, people can only get position or momentum information to a certain extent, which is why MRI images are often blurry. The Heisenberg Uncertainty Principle is a fundamental principle in modern physics that has profoundly changed people's understanding of the natural world. Although this principle prevents people from accurately determining the position and momentum of an elementary particle at the same time, it has not stopped people from using this principle to perform some important calculations and analysis. In the future, with the development of science and technology, people may find more opportunities to use the uncertainty principle to better understand and apply the fundamental laws of the natural world.

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