Proceedings of the 2nd International Conference on Mathematical Physics and Computational Simulation
DOI: 10.54254/2753-8818/41/20240504

Review on application of fractional Fourier transform in Linear Frequency Modulation signal and communication system

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Abstract. Traditional Fourier transform often apply to analyze and process stationary signals, however, it is weak for time-varying non-stationary signals, and fractional Fourier transform (FRFT) can better solve such problems. The FRFT can be comprehended as the expressive methods on the fractional Fourier domain constituted by the spinning coordinate axis of the signal anticlockwise about the origin at arbitrarily Angle in the time-frequency plane. In this paper, the improved fractional Fourier transform is combined with other calculation methods to achieve high precision estimation of chirp signal parameters. And the communication system built on weighted fractional Fourier transform and discrete fractional Fourier transform is studied and simulated respectively, which verifies the feasibility and improves the anti-jamming and anti-interception ability of the communication system.

Keywords: communication system, fractional Fourier transform, chirp signal.

1. Introduction

Namias first proposed the theory of fractional Fourier transforms in 1980. He came up with this idea from a mathematical point of view and applied it to the solution of differential equations. Then Mcbride et al. made a stricter definition based on Namias and expressed the fractional Fourier transform in integral form [1]. In 1993, Mendlovic and Ozaktas broke through the boundaries of mathematical research and implemented fractional Fourier transforms with optical methods, which have been widely used in optical signal processing [2]. However, because fractional Fourier transform has no strict physical meaning and fast implementation algorithm, it has a lot of potential in the area of signal processing, but it can not be fully utilized. In 1993, Almeida clarified its physical meaning, that is, fractional Fourier transform is the traditional Fourier transform to do a certain Angle rotation in the time-frequency plane, which essentially includes the information of the signal in the time domain and the frequency domain, so it is a time-frequency analysis method [3]. In 1996, Ozaktas and other scholars proposed a discrete algorithm of fractional Fourier transform, which has a very small computational load, only equivalent to that of Fast Fourier Transform Algorithm (FFT) [4]. Since then, fractional Fourier transform has drawn the attention of scholars in the area of signal handling at home and abroad, plenty of study results have gradually emerged. Compared with traditional Fourier transform, fractional Fourier transform is more flexible and has been applied in many aspects. Such as time-frequency

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analysis, time-frequency filtering, quantum mechanics, artificial neural networks, sweep filters, optical image processing, etc.

In 1981, French geophysicist Morlet found in the analysis of artificial seismic exploration signals that such signals should have a high resolution in the low frequency band, but the frequency resolution can be low in the high frequency band. It is because of this feature of seismic signals, Morlet proposed the concept of wavelet transform and gave a definition [5]. The traditional Fourier transform handle non-stationary signals has flaw. It can only know what frequencies a signal includes in usual instead of confirming when each component appears. So two signals which are very disparate in time domain may have the same spectral pattern. Since the signal can have different resolution in different positions of the time-frequency domain plane after wavelet transform, the signal can be analyzed by wavelet transform in multi-resolution. Because of its multi-resolution characteristics, the signal has very good time-frequency localization characteristics, which can make the signal from coarse to fine, more convenient for signal analysis and observation. This overcomes the shortcomings of the traditional Fourier transform. Thus, in recent years, wavelet transform not only has significant theoretical research results, but also has applied to lots of engineering fields, such as signal processing, speech recognition, image processing analysis, analytical chemistry, biomedicine, etc. Scientists' research on wavelet transform has not stopped because of the wide application, and the continuous in-depth study of its theory will bring new applications to various fields.

This paper presents some fundamental theorems about Fourier transform, and studies some recent applications of fractional Fourier transform in secure communication and parameter estimation of chirp signals.

2. Relevant theory

2.1. Definition

f(t) is a periodic function of *t* if *t* satisfies the Dirichlet condition: If *f(x)* is continuous or has only a finite amount of discontinuities of the first kind in a period of 2T, and f(x) is monotonic or can be separated into finite monotonic intervals, then the Fourier series of *F (x)* with period of 2T converges, the function *S (x)* is also a periodic function with period of *2T*, and it is finite at these discontinuities; It has a finite amount of extreme points in a period; Absolutely integrable.

The fourier transform of *x(t):*

$$X(\omega) = X[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-i\omega t}\mathrm{d}t \tag{3.1.1}$$

Inverse transform:

$$x(t) = X^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{i\omega t}\mathrm{d}\omega \tag{3.1.2}$$

$F(\omega)$: the image function of $f(t)$

$f(t)$: the preimage function of $F(\omega)$

2.2. Deduction

(1) Fourier serise of Periodic function are defined as (3.2.2):

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{i2\pi nt/T} \tag{3.2.1}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right) \text{ (For real-valued functions)} \tag{3.2.2}$$

Fourier expansion coefficient:

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i2\pi nt/T}dt \tag{3.2.3}$$

periodic signals can be expanded into a Fourier series only if the Dirichlet condition is satisfied.

The Dirichlet condition is defined as follows:

①A continuous or finite amount of discontinuity points of the first kind during a period.

②The quantity of maximum and minimum values in a period should be finite.

③Within a period, the signal is absolutely integrable.

Now assume that a function *f(t) is made up* of a direct current(DC) component and several cosine functions, as shown in equation (3.2.4).

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \, cos(n\omega t + \varphi) \tag{3.2.4}$$

Using the sum difference product formula of trigonometric functions, the above equation can be deformed to (3.2.5):

$$f(t) = c_0 + \sum_{n=1}^{\infty} [c_n \, cos\, \varphi \, cos(n\omega t) - c_n \, sin\, \varphi \, sin(n\omega t)] \tag{3.2.5}$$

Assume a_n, b_n is:

$$\mathrm{a_n} = \mathrm{c_n} \cos \varphi \tag{3.2.6}$$

$$\mathrm{b_n} = -\mathrm{c_n} \sin \varphi \tag{3.2.7}$$

Then formula (3.2.4) can be written:

$$f(t) = c_n + \sum_{n=1}^{\infty} [a_n \, cos(n\omega t) + b_n \, sin(n\omega t)] \tag{3.2.8}$$

Formula (3.2.8) is actually an expansion of the Fourier series, and it can be seen that if you want to expand a periodic signal into the Fourier series form, you are actually determining the series $a_n b_n$.

Multiply both sides of equation (3.2.8) by an $\sin(k\omega t)$ and integrate them over one period.

$$\int_0^T f(t) \, sin(k\omega t) \, dt = \int_0^T c_0 \, sin(k\omega t) \, dt + \int_0^T sin(k\omega t) \sum_{n=1}^{\infty} [a_n \, cos(n\omega t) + b_n \, sin(n\omega t)] \, dt \tag{3.2.9}$$

Equation (3.2.9) can be further simplified as:

$$\int_0^T f(t) \, sin(k\omega t) \, dt = b_n \int_0^T sin(n\omega t)^2 \, dt = b_n \frac{T}{2} \tag{3.2.10}$$

So it can be concluded that:

$$b_n = \frac{2}{T} \int_0^T f(t) \, sin(n\omega t) \, dt \tag{3.2.11}$$

in the same way:

$$a_n = \frac{2}{T} \int_0^T f(t) \, cos(n\omega t) \, dt \tag{3.2.12}$$

(2) discrete-time Fourier transform (DTFT)

For a sequence of numbers with domain Z, let $\{x_n\}_{n=-\infty}^{\infty}$ be one of the series. DTFT can be defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x_n e^{-i\omega n} \tag{3.2.13}$$

Inverse transform:

$$x_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{i\omega n} d\omega \tag{3.2.14}$$

DTFT is discrete in time domain and periodic in frequency domain. It is usually applied to analyze the spectrum of discrete-time signals. DTFT is viewed as the inverse of Fourier series.

(3) Fractional fourier transform

In the time-frequency plane, the fractional Fourier transform actually represents the counterclockwise rotation of the coordinate axes to obtain the fractional Fourier domain

Equivalent relationship:

$$u = t\cos\alpha + f\sin\alpha \text{ (u is fractional fourier axis)} \tag{3.2.15}$$

$$v = -t\sin\alpha + f\cos\alpha \tag{3.2.16}$$

The fractional Fourier transform $X_p(u)$ of the signal *x(t)* is defined as:

$$X_p(u) = \int_{-\infty}^{\infty} K_p(u,t)x(t)dt = \begin{cases} \sqrt{1 - j\cot\alpha}\int_{-\infty}^{\infty} x(t)e^{j\cdot 2\pi(u^2\cot\alpha/2 - ut\csc\alpha + t^2\cot\alpha/2)}dt, \alpha \neq \pi \\ x(t), \alpha = 2n\pi \\ x(-t), \alpha = (2n \pm 1)\pi \end{cases} \tag{3.2.17}$$

P: the order of fractional Fourier transform

$\alpha = \frac{p\pi}{2}$: rotation angle

$K_p(u,t)$: kernel function

Inverse transform:

$$x(t) = \int_{-\infty}^{+\infty} X_p(u)K_{-p}(u,t)du \tag{3.2.18}$$

If the Fourier transform of a function $\varphi(t)$ can satisfy the following form:

$$F[\varphi(t)] = \mu\varphi(t) \tag{3.2.19}$$

F: Fourier transform operator

$\mu = exp(-jn\pi/2)$: eigenvalue

Hermite-Gaussian function (common fourier function):

$$\varphi(t) = H_n(t)exp(-t^2/2) \rightarrow F[\varphi(t)] = exp(jn\pi/2)H_n(f)exp(-f^2/2) \tag{3.2.20}$$

The normalized Hermite-Gaussian function can be expressed as:

$$\varphi_n(t) = \frac{\sqrt[4]{2}}{\sqrt{2^n n!}}H_n(\sqrt{2\pi}t)exp(-t^2) \tag{3.2.21}$$

$H_n = (-1)^n exp(t^2)\frac{d^n}{dt^n}exp(-t^2)$: A Hermite polynomial of order n

The signal *x(t)* can be promoted as a complete set of orthogonal functions composed of Hermite-Gaussion eigenfunctions:

$$x(t) = \sum_{n\epsilon Z} X_n\varphi_n(t) \tag{3.2.22}$$

Where the expansion coefficient is:

$$X_n = \int_{-\infty}^{\infty} x(t){\varphi^*}_n(t)dt \tag{3.2.23}$$

Let μ_n be the eigenvalue corresponding to the eigenfunction $\varphi_n(t)$.Take the Fourier transform of both ends of equation (3.2.22), we can get:

$$X(f) = \sum_{n\epsilon Z} \mu_n X_n\varphi_n(f) \tag{3.2.24}$$

Put (3.2.23) into (3.2.24):

$$X(f) = \sum_{n\epsilon Z} \mu_n \varphi_n(f) \int_{-\infty}^{\infty} x(t)\varphi^*{}_n(t)dt = \int_{-\infty}^{\infty} x(t)K(t,f)dt \quad (3.2.25)$$

The kernel of the Fourier transform:

$$K(t,f) = \sum_{n\epsilon Z} \mu_n \varphi^*{}_n(t)\varphi_n(f) = \sum_{n\epsilon Z} \exp\left(-\frac{jn\pi}{2}\right) \varphi^*{}_n(t)\varphi_n(f) = exp(-j2\pi ft) \quad (3.2.26)$$

The usual Fourier transform form is obtained by substituting equation(3.2.26) into eqution(3.2.25).

The eigenvalue of Fourier transform is generalized to fractional order, and the eigenvalue of fractional Fourier transform is defined as the fractional power of Fourier transform eigenvalue. So the kernel of fractional Fourier transform is:

$$K_p(t,\mu) = \sum_{n\epsilon Z}(-jnp\pi/2)\varphi_n(t)\varphi_n(\mu) \quad (3.2.27)$$

(4) Wavelet Transform

The theory of wavelet transform was first proposed in 1984. When handling the local features of earthquake waves, Morlet, a French geophysicist, found that it was difficult to satisfy the demand of the traditional time-frequency domain handling method of Fourier transform when observing the high and low frequency characteristics of signals in practical engineering applications. Therefore, wavelet transform was adopted for geophysical exploration, and thus the wavelet transform had its first practical application.

The fundamental theory of wavelet transform is as follows: 1, to expand and shift the original signal; 2, the original signal is divided into a series of sub-band signals with different spatial resolutions, different frequency characteristics and direction characteristics. The sub band signal obtained in this way has well local features of time domain and frequency domain. So, it can overcome the defect of Fourier analysis in handling non-stationary signals and complex images.

The signal representation of wavelet transforms and Fourier transform is a linear combination of basis functions. The difference is that Fourier transform adopts a harmonic function with time belonging to $(-\infty, +\infty)$ and its basis function is e^{inx}, while the basis function of wavelet transform is a generating function $\varphi(t)$ with compact support set, and the wavelet sequence is acquired by stretching and shifting the generating function $\varphi(t)$.The concrete formula is as follows:

$$\varphi_{a,b}(x) = \frac{1}{|a|^{\frac{1}{2}}} \varphi\left(\frac{x-b}{a}\right) \quad (3.2.28)$$

$$a, b \in R; a \neq 0$$

a: the scaling factor

b: the translation factor.

For the introduction of the concept of wavelet transform, we must first briefly introduce the classical convolution theorem in advance, that is:

$$f(t) * g(t) = \int_{+\infty}^{-\infty} f(\tau)g(t-\tau)d\tau = \langle f(\cdot), g^*(t-\cdot)\rangle \quad (3.2.29)$$

Where $*$ represents the classical convolution operator, the superscript$*$ represents the conjugation operation, and $\langle\cdot,\cdot\rangle$represents the inner product operation.

Thus, for any signal $x(t)\epsilon L^2(R)$, the wavelet transform is defined by the classical convolution operation as:

$$f^{\varphi}(a,b) = f(t) * (a^{-\frac{1}{2}}\varphi^*(-t/a)) = \langle f(\cdot), \varphi_{a,b}(\cdot)\rangle \quad (3.2.30)$$

3. Review

3.1. Signal parameter estimation

Limin Liu, Haoxin Li, Qi Li, huangzhi Han and Zhenbin Gao mentioned in A Fast Signal Parameter Estimation Algorithm for Linear Frequency Modulation (LFM) Signal under Low Signal Noise Ratio (SNR) Based on Fractional Fourier Transform [6]. The initial rotation order and interval of the LFM signal can be determined by the efficient fractional Fourier transform algorithm, however, the estimation error of parameters is large when SNR is low because the variation of the normalized fractional frequency spectrum amplitude no longer shows obvious distribution law. On this basis, by using the good anti-noise performance of the 4-order origin moment of fractional order spectrum, the defects of the efficient FRFT algorithm can be removed, and the optimal order can be quickly estimated under the condition of low SNR. Thus, the parameters of LFM signal with low SNR can be quickly calculated.

In Parameter Estimation of Linear Frequency Modulation Signal Based On Interpolated Short-time Fractional Fourier Transform and Variable Weight Least Square Fitting [7], Weihao Cao, Zhixiang Yao, Wenjie Xia, Su Yan proposed a variable weight least square fitting (VWSF)-interpolation short-time fractional Fourier transform (ISTFRFT) method to estimate the parameters of chirp signals. Short-time Fourier transform (STFT) is a generally used way for time-frequency analysis of LFM signals, but its effect is not ideal for frequency estimation of wideband signals, so it can be extended to calculate the instantaneous frequency of LFM signals more accurately. The VWSF method is used to reduce the error caused by the conventional least square fit method and better calculate the initial frequency and modulation frequency of signal. Finally, by studying CRLB of initial frequency and modulated frequency estimation, it can be obtained that VWSF-ISTFRFT method has the high accuracy of LFM signal parameter estimation.

3.2. Safety communication

It proposed the Fractional Fourier Transform Frequency Hopping with Variable Time Wide and Fixed Bandwidth (FrFT-FH-VTFB) system in Two-dimensional Frequency Hopping Communication System and Performance Analysis Based on Discrete Fractional Fourier Transform [8] which is write by Xiaoyan Ning, Dongxu Zhao, Yunfei Zhu and Zhenyi Wang. The traditional frequency hopping communication is easy to be intercepted because of the single dimension of signal parameter hopping. The FrFT-FH-VTFB system obtains Chirp signals with different start frequencies and time widths through discrete fractional inversion, and realizes the 2-dimensional jump of time widths and start frequencies. In addition, Chirp's natural spread spectrum gain in the FrFT-FH-VTFB system can not only effectively break the signal periodicity and improve the anti-interception capability of the system, but also has concealability in the frequency domain and can resist energy detection. Moreover, due to the time-width parameter hopping of FrFT-FH-VTFB system, the energy of some code elements will increase, which makes the system have better anti-fading performance and reduces the influence of fading on system performance.

Secure communication of IRS based on weighted fractional Fourier transform [9] of Shengfeng Li, Xin Yang and Ling Wang studies MIMO scenes with general channel Settings by introducing IRS into MIMO communication systems assisted by artificial noise and fourth-order weighted fractional Fourier transform (WFRFT). WFRFT can make the complex plane of the signal show different states, so that the processed signal has strong anti-interception ability, so it is widely used in the wireless physical layer security transmission. On this basis, the intelligent reflective surface technology can support the secure communication of direction modulation technology based on artificial noise superposition,and improve the security of physical layer. Because the whole signal model is difficult to solve, the block coordinate descent (BCD)-majorization-minimization (MM) algorithm is introduced to reduce the complexity. The Lagrange multiplier method is used to get the optimal transmission precoding matrix matrix and covariance matrix, and an effective MM algorithm is used to get the optimal phase shift. And the performance simulation and analysis of the algorithm verify its feasibility and good safety performance.

Ping Gao and Yuxiao Yang studies the application of three-layer weighted Fourier transform in secure communication in A Safe Communication System Based on Three-layer Weighted Fractional Fourier Transform [10]. Compared with the traditional Weighted fractional Fourier transform (WFRET) signal, Multiple Parameters Weighted Fractional Fourier Transform (MPWFRFT) signal has stronger anti-interception capability and can better ensure the safety of signal transmission. The communication system based on three-layer WFRFT divides the initial data into three layers by Quadrature Phase Shift Keying (QPSK) baseband mapping, and then processes and transmits it, which can effectively improve the confidentiality of signal transmission. On this basis, genetic algorithm is imported for iterative optimization, and the optimal control parameter set for the simulation of three-layer WFRFT signal modulation characteristics is obtained. The communication performance, simulation performance and security performance of the system are simulated respectively, which verifies that the system has good anti-parameter scanning characteristics and high security.

4. Conclusion

Linear Frequency Modulation (LFM) signal is a signal whose frequency changes linearly with time, widely used in radar and sonar technology. In this paper, chirp signals show different energy aggregation on the fractional Fourier domain of different orders, and the continuous Fourier transform of signals is carried out to obtain the parameter estimation of chirp signal. With the progress of electronic technology, the security of communication system has become one of the hot topics. By studying the application of discrete fractional Fourier transform and weighted fractional Fourier transform in secure communication system, the original periodicity of system signal is broken, and the problem of poor anti-interference and anti-interception capability of traditional communication system can be solved.

References

[1] MCBRIDE A. C. On Namias's fractional Fourier transform[J]. IMA Journal of Applied Mathematics,1987,Vol.39(2): 159-175

[2] David Mendlovic; Haldun M. Ozaktas. Fractional Fourier transforms and their optical implementation. [J]. Journal of the Optical Society of America. A, Optics, Image Science, & Vision, 1993, Vol.10(9): 1875-1881

[3] Almeida, L. B. Product and Convolution Theorems for the Fractional Fourier Transform[J]. Signal Processing Letters, IEEE,1997, Vol.4 (1): 15-17

[4] M.Fatih Erden, Haldun M. Ozaktas, David Mendlovic. Synthesis of mutual intensity distributions using the fractional Fourier transform. Optics Communications. 1996 Apr;125(4–6):288–301.

[5] ARENS, G; FOURGEAU, E; GIARD, D; MORLET, J. SIGNAL FILTERING AND VELOCITY DISPERSION THROUGH MULTILAYERED MEDIA[J]. GEOPHYSICS,1981, Vol.46: 419-420

[6] LIU Limin, LI Haoxin, LI Qi, HAN Zhuangzhi, GAO Zhenbin. A Fast Signal Parameter Estimation Algorithm for Linear Frequency Modulation Signal under Low Signal-to-Noise Ratio Based on Fractional Fourier Transform. Journal of Electronics & Information Technology. 2021 Oct;43(10).

[7] CA0 Weihao, YAO Zhixian, XIA Wenjie, YAN Su. Parameter Estimation of Linear Frequency Modulation Signal Based On InterpOlated ShOrt-time Fractional Fourier Transform and Variable Weight Least Square Fitting. ACTA ARMAMENTARII. 2020 Jan; 41(1).

[8] NING Xiaoyan, ZHAO Dongxu, ZHU Yunfei, WANG Zhenduo. Two-dimensional Frequency Hopping Communication System and Performance Analysis Based on Discrete Fractional Fourier Transform. Journal of Electronics & Information Technology. 2023 Feb;45(2).

[9] LI Shengfeng, YANG Xin, WANG Ling. Secure communication of IRS based on weighted fractional Fourier transform. J Huazhong Univ of Sci & Tech (Natural Science Edition). 2023 Mar;51(3).

DOI: 10.54254/2753-8818/41/20240504

[10] GAO Ping, YANG Yuxiao. A Safe Communication System Based on Three-layer Weighted Fractional Fourier Transform. Telecommunication Engineering. 2022 Nov; 62(11).