Riemann Zeta Function and Its Applications

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Abstract. This paper focuses on the introduction and proof of the fundamental properties of $\boldsymbol{\zeta(z)}$, i.e. Riemann zeta function and explores its applications in algebra. We begin with a systematic derivation and proof of the basic characteristics of the zeta function. Following this, we examine its application in algebra, including the use of the Dirichlet L-function to prove Dirichlet's theorem. Furthermore, we show the classical result for the subgroup growth rate of $\boldsymbol{\mathcal{J}}$-groups and the enumeration of n-dimensional irreducible representations of Heisenberg groups.

Keywords: Riemann's zeta function, number theory, group zeta function.

1. Introduction

The discovery of zeta function can be traced back to around 1350, French mathematician Nicole Oresme identified the divergence of harmonic series.$\sum_{k=1}^{\infty}\frac{1}{k}$

Several methods can be employed to prove this result:

Riemann sum: Transform the series into $\int_0^1 \frac{1}{x}dx$

Scaling:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots\\ &\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\cdots=+\infty\end{aligned}$$

Scaling again:

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$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= \left(1+\frac{1}{2}+\cdots+\frac{1}{9}\right)+\left(\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{99}\right)+\cdots \\ &> \left(\frac{1}{10}+\frac{1}{10}+\cdots+\frac{1}{10}\right)+\left(\frac{1}{100}+\frac{1}{100}+\cdots+\frac{1}{100}\right)+\cdots \\ &= \frac{9}{10}+\frac{9}{10}+\cdots=+\infty\end{aligned}$$

Using the inequality $x > \ln(x+1)$:

In 1644, Pietro Mengoli posed the Basel problem: calculating the sum $\sum_{n=1}^{\infty}\frac{1}{n^2}$.It was first addressed by Euler in 1735 using identity

$$\frac{sinx}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots=\prod_{n=1}^{\infty}\left[1-\frac{x^2}{(n\pi)^2}\right]$$

Compare the coefficients of the term x^2, it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Moreover, Parseval's identity can be applied to solve this problem:

$$\sum_{n=-\infty}^{\infty}|C_n|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx$$

where $C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f\,(x)e^{-inx}dx$ denotes the Fourier coefficient.

$$f(x)=x=2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$$

We obtain $4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{2\pi^2}{3}$, leading to the solution $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

The Basel problem can also be addressed using complex analysis. Let

$$I=\int_0^{\frac{\pi}{2}}\ln(2\text{cosx})\,dx$$

Since $2\cos x=e^{ix}+e^{-ix}$,

$$I=\int_0^{\frac{\pi}{2}}\ln\left(e^{ix}+e^{-ix}\right)=\int_0^{\frac{\pi}{2}}\ln\left[e^{ix}\left(1+e^{-2(ix)}\right)\right]=\frac{i\pi^2}{8}+\int_0^{\frac{\pi}{2}}\ln\left(1+e^{-2(ix)}\right)dx$$

Expanding $\ln\left(1+e^{-2ix}\right)$ into a Taylor series, we have

$$\int_0^{\frac{\pi}{2}}\ln\left(1+e^{-2ix}\right)=\frac{1}{i}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{3}{4i}\zeta(2)$$

Consequently, we find

$$I=i\left(\frac{\pi^2}{8}-\frac{3}{4}\zeta(2)\right)$$

Since I is a real number, it follows that

$$\left(\frac{\pi^2}{8}-\frac{3}{4}\zeta(2)\right)=0\Rightarrow\zeta(2)=\frac{\pi^2}{6}$$

Building on the divergent harmonic series and the classical Bessel problem, we now turn our attention to the research of zeta function:

Definition 1.1. $\zeta(z)=\sum_{n=1}^{\infty}n^{-z}$

The zeta function not only generalizes the concept of series but also encapsulates profound connections between various mathematical fields, including number theory and algebra. In the following

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chapters, we will first prove the preliminary properties of the zeta function, which were initially proposed by Riemann and have since been rigorously formalized by generations of mathematicians. We will also explore the applications of the zeta function in series calculation and algebra.

2. Basic Property of Riemann Zeta function

When discussing the $\zeta(z)$, an essential characteristic is the Euler Product Formula. Euler initially discovered that $\zeta(z)$ can be represented as a product of factors corresponding to prime numbers.

Proposition 2.1. *For* $z > 1$, $\zeta(z) = \prod_{\text{p is prime}} \left(1 - \frac{1}{p^z}\right)^{-1}$

Proof.

$$\begin{aligned}\prod_{\text{p is prime}} \left(1 - \frac{1}{p^z}\right)^{-1} &= \left(1 - \frac{1}{2^z}\right)^{-1} \left(1 - \frac{1}{3^z}\right)^{-1} \cdots \\ &= \left(1 + \frac{1}{2^z} + \frac{1}{2^{2z}} + \cdots\right)\left(1 + \frac{1}{3^z} + \frac{1}{3^{2z}} + \cdots\right)\cdots \\ &= \prod_{\text{p is prime}} \sum_{i=0}^{\infty} \frac{1}{p^{iz}}\end{aligned}$$

$\forall N \in \mathbb{Z}$, let $N = p_1^{a_1} \cdots p_n^{a_n}$. Then $\frac{1}{N^z} = \prod_{i=1}^{n} \frac{1}{p_i^{a_i}}, \sum_{N\geq 1} \frac{1}{N^z} = \prod_{\text{p is prime}} \sum_{i=0}^{\infty} \frac{1}{p^{iz}}$

In fact, this formula can be generalized to apply to multiplicative functions:

Theorem 2.2. *(Smith, 1983) Consider* $L(x) \in \ell^1$ *that respects the property* $L(xy) = L(x)L(y)$*, then* $\sum_{x=1}^{\infty} L(x) = \prod_{\text{p is prime}} (1 + L(p) + L(p^2) + \cdots)$

Proposition 2.1 gives an approximate characterization for the roots of $\zeta(z)$. If ρ is a root for $\zeta(z)$, then for some prime q, $\left(1 - \frac{1}{q^{-z}}\right)^{-1} = 0$, which implies that $\mathrm{Re}(z) \leq 1$. In other words, the function $\zeta(z)$ is free of zeron in the half-plane where $\mathrm{Re}(z) > 1$.

Conversely, by taking the logarithm of both sides, we derive an alternate expression of the Euler Product Formula:

Corollary 2.3. Let $\Lambda(n) = \begin{cases}\ln p, n = p^k \\ 0, otherwise\end{cases}$, then$\sum_{n=1}^{\infty} \Lambda(n) \frac{1}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}$

Proof. Applying a logarithmic transformation followed by differentiation, it follows that

$$\begin{aligned}-\frac{\zeta'(z)}{\zeta(z)} = \sum_{\text{p is prime}} \frac{\ln p}{p^z - 1} &= \sum_{\text{p is prime}} \sum_{j=1}^{\infty} (\ln p) \frac{1}{p^{jz}} \\ &= \sum_{j=1}^{\infty} \sum_{\text{p is prime}} (\ln p) \frac{1}{p^{jz}} \\ &= \sum_{j=1}^{\infty} \Lambda(j) \frac{1}{j^z}\end{aligned}$$

Understanding $\zeta(z)$ near $z = 1$ is crucial, as $\zeta(1)$ diverges to infinity. Additionally,

Proposition 2.4. $\lim_{z\to 1} \frac{\zeta(z)}{(z-1)^{-1}} = 1$

Proof. Since the function t^{-z} decreases monotonically as t increases for a fixed z, it follows that $\frac{1}{(n+1)^z} < \int_n^{n+1} \frac{1}{t^z}\, dt < \frac{1}{n^z}$ By summing this inequality over 1 to ∞, $\zeta(z) - 1 < \int_1^{\infty} \frac{1}{t^z}\, dt < \zeta(z)$ Since $\int_1^{\infty} \frac{1}{t^z}\, dt = (z-1)^{-1} \Rightarrow 1 < \frac{\zeta(z)}{(z-1)^{-1}} < z$. As z approaches 1, we find that $\lim_{z\to 1}(z-1)\zeta(z) = 1$.

Through some transformations and calculations, we can obtain another form of this proposition:

Corollary 2.5. $\lim_{z\to 1} \frac{\ln\zeta(z)}{-\ln(z-1)} = 1$

Typically, the Riemann zeta function is not used with z as a real number where $z > 1$, but rather by treating z as a complex number and considering the region where $Re(z) \leq 1$. Consequently, extending $\zeta(z)$ analytically to cover the entire complex plane is crucial, as highlighted in:

Theorem 2.6. $\zeta(z)$ *can be extended to the entire* $\mathbb{C}$*. Its only singularity is at* $z = 1$*, where the residue is 1*

The elementary techniques employed in this proof are partial summation and the Stieltjes integral. The former transforms sums into more manageable sums or integrals, while the latter does the opposite by converting sums into integrals. One can find the proof in Book (lvic, 1949)

The analytic extension of $\zeta(z)$ is crucial for exploring the distribution of its zeros. J. Hadamard once proved that $\zeta(z)$ does not vanish on $\{z \in \mathbb{C} | \mathrm{Re}(z) = 1\}$:

Proposition 2.7. $\forall y \in \mathbb{R}, \zeta(1 + iy) \neq 0$

Proof.

Assuming that $1 + iy$ is a m-order zero. Theorem 7 implies that $1 + iy$ is a first-order pole with residue $m \geq 1$ for the function $\frac{\zeta'(z)}{\zeta(z)}$. Consequently, for $\alpha > 1$ but sufficiently close to 1, we obtain $\frac{\zeta'(\alpha+iy)}{\zeta(\alpha+iy)} = \frac{m}{\alpha-1} + o\left(\frac{1}{\alpha-1}\right)$ Given that 1 is a pole of $\zeta(z)$ with residue 1, we get $\frac{\zeta'(\alpha)}{\zeta(\alpha)} = \frac{-1}{\alpha-1} + o\left(\frac{-1}{\alpha-1}\right)$ Let $\frac{\zeta'(\alpha+2iy)}{\zeta(\alpha+2iy)} = \frac{k}{\alpha-1} + o\left(\frac{k}{\alpha-1}\right)$ Where $k = 0$ if $\zeta(1 + 2iy) \neq 0$, $k \geq 1$ if $\zeta(1 + 2iy) = 0$. Therefore, $Re\left(\frac{\zeta'(\alpha+2iy)}{\zeta(\alpha+2iy)} + \frac{3\zeta'(\alpha)}{\zeta(\alpha)} + \frac{4\zeta'(\alpha+iy)}{\zeta(\alpha+iy)}\right) = \frac{k-3+4m}{\alpha-1} + o\left(\frac{k-3+4m}{\alpha-1}\right) > 0$ On the other hand, by 4,

$$Re\left(\frac{\zeta'(\alpha+2iy)}{\zeta(\alpha+2iy)} + \frac{3\zeta'(\alpha)}{\zeta(\alpha)} + \frac{4\zeta'(\alpha+iy)}{\zeta(\alpha+iy)}\right) = -\sum_{j=1}^{\infty} \Lambda(j) \frac{1}{j^\alpha} \left(\cos(2t\ln j) + 3 + 4\cos(t\ln j)\right)$$
$$= -\sum_{j=1}^{\infty} 2\Lambda(j) \frac{1}{j^\alpha} (1 + \cos(t\ln j)^2) \leq 0$$

This is contradiction. Thus $\zeta(1 + iy) \neq 0$ for all $y \in \mathbb{R}$.

In the next part, we will introduce a symmetric form of the zeta function based on its functional equations. These are fundamental results in zeta function theory, originally proven by Riemann. The three equivalent functional equations are given by:

Proposition 2.8. *(cf. (Smith, 1983), Theorem 1.6) For all complex z, let* $Z(z) = \frac{1}{\pi^{z/2}} \Gamma\left(\frac{z}{2}\right) \zeta(z)$, *then* $Z(z) = Z(1 - z)$

This equation for the zeta function implies that $\forall s \in \mathbb{C}_{\mathrm{Re}(s)<0}, s$ is not a root for $\zeta(z)$. Because if ρ is a root of $\zeta(z)$, then $1 - \rho$ is also a root. In summary, we conclude that the strip $0 \leq \mathrm{Re}(\rho) < 1$ contain every nontrivial zero of $\zeta(z)$

Next, we will explore specific values of $\zeta(z)$ and examine how to compute series associated with it. The zeta function's values at positive even integers can be determined using the recurrence relation provided below:

Proposition 2.9. $\zeta(2n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}\pi^{2k}\zeta(2n-2k)}{(2k+1)!}$

The validity of this formula can be easily demonstrated with mathematical induction.Or we can just compare the coefficients of the terms in the Taylor series and product expansion of $\frac{\sin x}{x}$ and obtain it by inference.

Zeta function's value at negative integers can be obtained using the following expression of zeta function(One can prove it readily by proving that the definite integral in the formula equals $\zeta(z)\Gamma(z)$):

Proposition 2.10. $\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x - 1} dx$

Corollary 2.11. $\zeta(-K) = (-1)^K \frac{B_{K+1}}{K+1}, \forall K \in \mathbb{Z}^+$ *B_K denotes the Bernoulli number*

Ramanujan also offered a simple method to calculate $\zeta(-K)$,that is: Let $A = 1^K + 2^K + \cdots$ and $B = 1^K - 2^K + \cdots$, then A adds B and B adds itself by dislocation for a few times,but in contrast, the aforementioned method is much more persuasive.

The following results from Aries in Zhihu shows the series related to zeta function:

Proposition 2.12. $\sum_{k=1}^{\infty} x^{2k} \zeta(2k) = \frac{1}{2} - \frac{\pi}{2} x\cot\pi x$

Proof.

$$\frac{\sin\pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \Rightarrow \ln\sin\pi x - \ln\pi x = \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^n}{n^2}\right)$$

Take the derivative, we have

$$\pi x\cot\pi x = 1 - 2\sum_{k=1}^{\infty} x^{2k}\,\zeta(2k) \Rightarrow \sum_{k=1}^{\infty} x^{2k}\,\zeta(2k) = \frac{1}{2} - \frac{\pi}{2}x\cot\pi x$$

Corollary 2.13. $\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^k} = \frac{1}{2}, \sum_{k=1}^{\infty}\frac{\zeta(2k)}{16^k} = \frac{1}{2} - \frac{\pi}{8}$

Proof. Just take $\mathrm{x} = \frac{1}{2}$ *and* $\mathrm{x} = \frac{1}{4}$ *in Proposition 2.13, we will arrive at the answer.*

Corollary 2.14. $\sum_{k=1}^{\infty}\frac{\zeta(4k)}{4^k} = \frac{\pi}{4\sqrt{2}}\left(\cot\frac{\pi}{\sqrt{2}} + \coth\frac{\pi}{\sqrt{2}}\right)$

Proof. Substitute ix for x in Proposition 13, we have

$$\sum_{k=1}^{\infty}(-1)^k x^{2k}\zeta(2k) = \frac{1}{2} - \frac{\pi}{2}x\coth\pi x$$

Adding 2.12 and [4], then taking $x = \frac{1}{\sqrt{2}}$, we have

$$\sum_{k=1}^{\infty}\frac{\zeta(4k)}{4^k} = \frac{\pi}{4\sqrt{2}}\left(\cot\frac{\pi}{\sqrt{2}} + \coth\frac{\pi}{\sqrt{2}}\right)$$

Corollary 2.15. $\sum_{k=1}^{\infty}[\zeta(2k) - 1] = \frac{3}{4}$ *(a)*

Proof. Using the identity

$$\sum_{k=1}^{\infty} x^{2k} = \frac{x^2}{1 - x^2}$$

Then By Proposition 13 and take the limit $x \to 1$, we can prove the result. □

Corollary 2.16. $\sum_{k=1}^{\infty}[\zeta(2k) - \zeta(2k+1)] = \frac{1}{2}$

$\sum_{k=1}^{\infty}[\zeta(2k+1) - 1] = \frac{1}{4}$

$\sum_{k=1}^{\infty}[\zeta(k) - 1] = 1$

Proof. Denote the three series with (b),(c) and (d) from top to bottom. (b) is quite simple, we just calculate it directly with $\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}$ (c)=(a)-(b) and (d)=(a)+(c)

Following are the amazing conclusions which demonstrate that zeta function is also related to trigonometric functions.

Proposition 2.17.

$$\sum_{n=1}^{\infty}\frac{\cos n}{n^{2k}} = (-1)^k\frac{\pi}{2(2k-1)!} + \sum_{a=0}^{\infty}(-1)^i\frac{\zeta(2k-2a)}{(2a)!}$$

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^{2k+1}} = (-1)^k\frac{\pi}{2(2k)!} + \sum_{a=0}^{\infty}(-1)^i\frac{\zeta(2k-2a)}{(2a+1)!}$$

Obviously, the formula of Fourier Series is a nice proof for the aforementioned two series. We can also take several fixed positive integer k and it will be quite easy to conclude the formula by inference.

3. Dirichlet Theorem

This section aims to prove the Dirichlet Theorem by the technique of zeta function:

Theorem 3.1. *Assume a and m are coprime integers. Define $P(a, m)$ as the set of prime numbers p that satisfy $p \equiv a \bmod m$. Then $P(a, m)$ contains infinitely many elements.*

Based on the generalization of the zeta function, we can approach the proof. In general, for an arithmetic function, the Dirichlet L-function associated with λ is defined by

$$L(z, \lambda) = \sum_{n=1}^{\infty}\lambda(n)n^{-z}$$

Proposition 2.1 holds for $L(z, \lambda)$ if λ is multiplicative, we have

Proposition 3.2. Let $L(z, \lambda)$ be the Dirichlet L-function with respect to λ, then $L(z, \lambda) = \prod_{\text{p is prime}}\frac{1}{(1-\lambda(p)p^{-z})}^{-1}$

More importantly, we investigate the properties of $\ln L(z,\lambda)$. By taking the logarithm of the above expression, we obtain:

Proposition 3.3. $\ln L(z,\lambda)=\sum_{\text{p is prime}}\lambda(p)\frac{1}{p^z}+R(z)$ Where $R(z)$ remains bounded as $z\to 1$

Proof. Only need to prove $\ln\zeta(z)=\sum_{\text{p is prime}}\frac{1}{p^z}+R(z)$ By proposition 2, Let $\zeta(z)=\prod_{p\le M}\left(1-\frac{1}{p^z}\right)^{-1}\lambda_M(z)$, $\lambda_M(z)\to 1$, $M\to\infty$

So $\ln\zeta(z)=\sum_{p\le M}-\ln\left(1-\frac{1}{p^z}\right)+\ln\lambda_M(z)$ Since $p^{-z}<1$, we have $\ln\left(1-\frac{1}{p^z}\right)=-\sum_{n=1}^{\infty}\frac{1}{np^{-nz}}$ Thus as M approaches infinity, $\ln\zeta(z)=\sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{p^{kz}}=\sum_p\sum_{k=1}^{\infty}\frac{1}{kp^{kz}}=\sum_p\frac{1}{p}+\sum_p\sum_{k=2}^{\infty}\frac{1}{kp^{kz}}$
$=\sum_p\frac{1}{p^{2z}}\left(1-\frac{1}{p^z}\right)^{-1}\le(1-2^{-z})^{-1}\sum_p\frac{1}{p^{2z}}\le\frac{\pi^2}{3}$ Therefore, $\ln\zeta(z)\sim\sum_p\frac{1}{p^z}$ Since λ is a character of $\mathbb{Z}/m\mathbb{Z}$, for $p\mid m$, $\lambda(p)=0$. Therefore the above product is equals to

$$\ln L(z,\lambda)=\sum_{p\nmid m}\lambda(p)p^{-z}+R(z)$$

From Proposition 3.2 and Corollary 2.5, we can define a meaningful quantity to characterize the density of a specific set of prime numbers:

Definition 3.4. *Assume $\mathcal{P}$ is a set of positive primes. If the limit $d(\mathcal{P}):=\lim_{z\to 1}\frac{\sum_{p\in\mathcal{P}}p^{-z}}{-\ln(z-1)}$ exists, we called $\mathcal{P}$ has Dirichlet density.*

The basic fact of Dirichlet density is

$$d(\mathcal{P})=\begin{cases}0, & \mathcal{P}\text{ is limited}\\ 1, & \mathcal{P}\text{ consists of all but finitely many positive primes}\end{cases}$$

In that cases, Theorem 3.1 is equivalent to $d(\mathcal{P}(a,m))>0$.

Now we can prove Theorem 19:

Proof. Let $L(z,\lambda)$ be the Dirichlet L-function with respect to λ, where λ is a character of $\mathbb{Z}/m\mathbb{Z}$. Since $|\lambda(n)|=1\Rightarrow|\lambda(n)n^{-s}|\le n^{-s}$, $L(z,\lambda)$ converges and continuous for $z>1$. Now define $G(z,\lambda)=\sum_{\text{q is prime}}\sum_{k=1}\left(\frac{1}{k}\right)\lambda(q^k)\frac{1}{q^{kz}}$ It also converges and continuous for $z>1$. Additionally, for any complex number s with $|s|<1$, we have $\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}s^k\right)=(1-s)^{-1}$ Substituting $s=\lambda(q)\frac{1}{q^z}$, we obtain $\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}\lambda(q^k)\frac{1}{q^{kz}}\right)=\left(1-\lambda(q)\frac{1}{q^z}\right)^{-1}$ Therefore, $\exp G(z,\lambda)=L(z,\lambda)$. By Proposition 21, $G(z,\lambda)=\sum_{q\nmid m}\lambda(q)\frac{1}{q^z}+R_\lambda(z)$ where $\lim_{z\to 1}R_\lambda(z)<\infty$. Through some transformations and calculations, we have $\sum_\lambda\overline{\lambda(a)}\,G(z,\lambda)=\sum_{q\nmid m}\frac{1}{q^s}\sum_\lambda\overline{\lambda(a)}\,\lambda(q)+\sum_\lambda\overline{\lambda(a)}\,R_\lambda(z)$ By utilizing the orthogonality of the characters(cf. (Kenneth Ireland, 2000) Proposition 16.3.2) $\sum_\lambda\overline{\lambda(a)}\,G(z,\lambda)=\psi(m)\sum_{q\equiv a\bmod m}q^{-z}+R_{\lambda,a}(z)$ Dividing $-\ln(z-1)$ on both sides, $\frac{R_{\lambda,a}(z)}{\ln(z-1)}-\frac{G(z,\lambda_0)}{\ln(z-1)}-\frac{\sum_{\lambda\ne\lambda_0}\overline{\lambda(a)}G(z,\lambda)}{\ln(z-1)}=\psi(m)d(\mathcal{P}(a,m))$ Since $R_{\lambda,a}(z)$ is bounded as $z\to 1$, $\lim_{z\to 1}\frac{R_{\lambda,a}(z)}{\ln(z-1)}=0$ Notice that $G(z,\lambda_0)=\sum_{\lambda|m}\ln(1-\lambda^{-z})+\ln\zeta(z)$. By Proposition 5, $\lim_{z\to 1}\frac{G(z,\lambda_0)}{\ln(z-1)}=-1$ It remains to prove that $G(z,\lambda)$ remains bounded as $z\to 1$ forall nontrival character λ. It's equivalent to prove $L(z,\lambda)$ can be extended to half complex plane and $L(1,\lambda)\ne 0$ forall nontrival character. The first assertion is similar to Theorem 7. For the second assertion, summing over λ on both sides of $G(z,\lambda)=\sum_{\text{q is prime}}\sum_{k=1}\left(\frac{1}{k}\right)\lambda(q^k)q^{-ks}$ we have $\sum_\lambda G(z,\lambda)=\psi(m)\sum_{q^k\equiv 1\bmod m}\frac{1}{k}q^{-kz}$ Since RHS ≥ 0, we take the exponential of both sides: $F(z):=\prod_\lambda L(z,\lambda)\ge 1,\forall z>1$. Now Firstly we prove that if λ is a nontrivial complex character of $\mathbb{Z}/m\mathbb{Z}$, then $L(1,\lambda) \neq 0$:Assumed $L(1,\lambda) = 0$, then $\overline{L(z,\lambda)} = L(z,\lambda)$ implies $L(1,\overline{\lambda}) = 0$. Therefore $F(1) = 0$, contradiction!. Next we prove the nontrivial real character λ also satisfies $L(1,\lambda) \neq 0$: Since $|\lambda(n)| = 1, \lambda(n) \in \mathbb{R} \Rightarrow \lambda(n) = \pm 1, \forall n \in \mathbb{Z}$. Assume $L(1,\lambda) = 0$, We construct a function as follows:$\phi(z) = \frac{L(z,\lambda)L(z,\lambda_0)}{L(2z,\lambda_0)}$ $\phi(z)$ is analytic on $Re\ z > \frac{1}{2}$ and $\phi(z) \to 0$ as $z \to \frac{1}{2}$. Supposed z is real and $z > 1$, then by Proposition 2, $\phi(z) = \prod_{q \nmid m} \frac{(1-q^{-2z})}{(1-q^{-z})(1-\lambda(q)q^{-z})} = \prod_{\lambda(q)=1} \frac{1+q^{-z}}{1-q^{-z}}$ $= (1+q^{-z})(\sum_{k=0}^{\infty} q^{-kz})$ $= 1 + 2q^{-z} + 2q^{-2z} + \cdots +$ Using Theorem 3, $\phi(z) = \sum_{n=1}^{\infty} a_n n^{-z}$ converges for $z > 1$. Expanding $\psi(z)$ into power series at $z = 2$, we have $\phi(z) = \sum_{m=0}^{\infty} b_m (z-2)^m$Since $\phi(z)$ is analytic for $Re\ z > \frac{1}{2}$, the radius of convergence of it is at least $\frac{3}{2}$. Since $b_m = \frac{\phi^{(m)}(2)}{m!}$, differentiate $\phi(s)$ m times it follows that $\phi^{(m)}(2) = \sum_{n=1}^{\infty} a_n (-\ln n)^m n^{-2}$ $:= (-1)^m c_m, c_m \geq 0$ Thus $\phi(z) = \sum_{n=0}^{\infty} c_m (2-z)^m$ with the coefficient c_m nonnegative and $c_0 = \phi(2) = \sum_{n=1}^{\infty} a_n n^{-2} \geq a_1 = 1$ It follows that for $\frac{1}{2} < z < 2$, $\phi(z) \geq 1$. This contradicts $\phi(z) \to 0$ as $z \to \frac{1}{2}$! Therefore, forall the nontrival character λ, we have $L(1,\lambda) \neq 0$, so that $G(z,\lambda)$ remains bounded as $z \to 1$. Taking $z \to 1$ in [1], we obtain $1 = \psi(m) d(\mathcal{P}(a,m)) \Rightarrow d(\mathcal{P}(a,m)) = \frac{1}{\psi(m)} > 0$ Where $\psi(n)$ is the Euler's totient function. Thus we finally prove the Theorem 19.

4. Counting Problem For Groups

Zeta-function is suitable for some counting questions: Let Γ be a mathematical object, $a_\Gamma(n)$ is a non-negative integers sequence which encoded with some information about Γ. Thus $n \mapsto a_\Gamma(n)$ is an arithmetic functions. It's natural to consider the Dirichlet L-function of $a_\Gamma(n)$, which gives rise to the following definition:

Definition 4.1. *The zeta function of* $(\Gamma, a_\Gamma(n))$ *is the Dirichlet generating series* $\zeta_\Gamma(z) = \sum_{n=1}^{\infty} a_\Gamma(n) n^{-z}$

Firstly let's consider the number of subgroup with finite index of an infinite group. Let Γ be a infinite group. For a given $n \in \mathbb{Z}$, denoted $a_\Gamma(n) = \#\{H \leq \Gamma \mid |\Gamma : H| = n\}$. Therefore the arithmetic function $n \mapsto a_\Gamma(n)$ is called the subgroup number function. The partial sum $S_\Gamma(n) = \sum_{m \leq n} a_\Gamma(n)$ is called the subgroup growth functions. The subgroup zeta function is given by

$$\zeta_\Gamma(z) = \sum_{n=1}^{\infty} a_\Gamma(n) n^{-z} = \sum_{H \leq_f \Gamma} |\Gamma : H|^{-z}$$

where the notation $H \leq_f \Gamma$ indicates that the summation is over all subgroups with finite index. By finding an explicit expression for this function, we can determine the properties of $a_\Gamma(n)$ through coefficient comparison or asymptotic analysis.

It's better that $n \mapsto a_\Gamma(n)$ has the following properties:

1. Polynomial growth: $S_n(\Gamma) = O(n^a)$ for some $a \in \mathbb{R}$.
2. Multiplicativity: If $n = \prod_i p_i^{e_i}$, then $a_\Gamma(n) = \prod_i a_\Gamma(p_i^{e_i})$.

Property 1 ensures the convergence of $\zeta_\Gamma(z)$. Generally, the abscissa of convergence is a useful tools to characterize the region of convergence, defined as

$$\alpha(a_\Gamma(n)) = \limsup_{n \to \infty} \frac{\ln S_n(\Gamma)}{\ln n}$$

Marcus du Sautoy Shows the relation between abscissa of convergence and region of convergnece:(cf. (Sautoy & Grunewald, 2000))

Theorem 4.2. *Consider a $\mathcal{J}$ −group Γ. The abscissa of convergence $\alpha\big(a_\Gamma(n)\big) \in \mathbb{Q}$. And $\zeta_\Gamma(z)$ is analytically continued to the half plane $Re\,(z) > \alpha\big(a_\Gamma(n)\big)$. Moreover, the line $\{z \in \mathbb{C} | Re(z) = \alpha\big(a_\Gamma(n)\big)$ can have at most one pole of $\zeta_\Gamma(z)$ located at $z = \alpha\big(a_\Gamma(n)\big)$*

Property 2 shows that Proposition 2 can be generalized to $\zeta_\Gamma(z)$:

$$\zeta_\Gamma(z) = \prod_{p \text{ prime}} \sum_{i=0}^{\infty} a_\Gamma\left(p^i\right) p^{-iz}$$

The factor $\zeta_{\Gamma,p}(z) := \sum_{i=0}^{\infty} a_\Gamma\left(p^i\right) p^{-iz}$ is called the local factor of $\zeta_\Gamma(z)$ at the prime p.

The next question is to identify the types of groups that satisfy these properties. Grunewald, Fritz J and Scharlau, Rudolf has proved in (Grunewald & Scharlau, 1979) that torsion-free, finitely generated nilpotent group, denoted as $\mathcal{J}$-group, exhibits polynomial subgroup growth and multiplicativity. Consequently, the focus shifts to finding a general formula for the zeta function of$-J$ group or calculating specific cases.

Example 4.3. $\zeta_{\mathbb{Z}^n}(z) = \prod_{i=0}^{n-1} \zeta\,(z - i)$

Proof.

Lemma 26. Let G be a group, $G = A \times B, Q \leq B \leq L \leq G$, then there is $H, H \leq G, s.t. HB = L, H \cap B = Q$

Proof. Let $H = L \cap QA$, then

$$H \cap B = (L \cap QA) \cap B = QA \cap B = Q(A \cap B) = Q$$
$$HB = (L \cap QA)B = L \cap QAB = L \cap G = L$$

Lemma 27. *Subject to the conditions of Lemma 26 and further supposes that $Q \trianglelefteq B$, then number of distinct choices for H is $|Der(L/B, B/Q)|$*

Proof. In L/Q we have the classical configuration for which the result is well konwn. Note that in order to define derivations we need to prescribe an action of L/B on B/Q. This is done by distinguishing an H arbitrarily $(H/Q) \cong L/B)$, and defining the action as conjugation by elements of H/Q. □

Let $G \cong \mathbb{Z}^n \cong \mathbb{Z} \oplus \mathbb{Z}$. Denote the first component o this direct sum by B. Suppose $H \leq G, |G{:}H| < \infty$. H give rise to $L = HB$ and $Q = H \cap B$, and $|G{:}L| \cdot |B{:}Q| = |G{:}H| < \infty$. Conversely, if we are given Q and L, $Q \leq_f B \leq L \leq_f G$, then by Lemma 26, there exists $H \leq_f G$ such that $HB = L$ and $H \cap B = Q$.Suppose $|G{:}L| = n/d$ and $|B{:}Q| = d$ then $|G{:}H| = n$. Lemma 27 enables us to count there are $|Der(L/B, B/Q)$ of H associated with the pair (L, Q). Now G is abelian so L/B acts trivially on B/Q. Thus we have $|Der(L/B, B/Q)| = |Hom(\mathbb{Z}^{n-1}, c_d)| = d^{n-1}$. Let

$$\zeta_{\mathbb{Z}^n}(z) = \sum_{n=1}^{\infty} a_n\, n^{-z}, \zeta_{\mathbb{Z}^{n-1}}(z) = \sum_{n=1}^{\infty} b_n\, n^{-z}$$

B has a unique subgroup of each specific index, so

$$a_n = \sum_{d|n} b_{n/d}\, d^{n-1}$$

Therefore

$$\zeta_{\mathbb{Z}^n}(z) = \zeta_{\mathbb{Z}^{n-1}} \zeta(z - n + 1)$$

Finally, by induction we can prove this result. □

Corollary 4.6. *Define $a_n = \#\{H \leq \mathbb{Z}^2 | |\mathbb{Z}^2{:}H| = n\}$, then $\sum_{i=1}^{n} a_i \sim \frac{\pi^2}{12} n^2$*

Proof. :

Lemma 4.7. *(Tauberian Theorem (Marcus du Sautoy, 1925)): Let $f(z) = \sum_{n=1}^{\infty} a_n n^{-z}$be a Dirichlet series, $a_n \geq 0\ \forall n \in \mathbb{Z}$. It converges for $Re(z) > \alpha > 0$, If within the domain of convergence $f(z)$ satisfies $f(z) = a(z)(z - \alpha)^{-w} + b(z)$, where $a(z), b(z)$ are analytic on $\mathbb{C}_{Re(z) \geq \alpha}$, $a(\alpha) \neq 0, w > 0$, then $\lim_{x \to \infty} \sum_{n \leq x} a_n = \left(\frac{a(\alpha)}{\alpha\Gamma(w) + o(1)}\right) x^\alpha (lnx)^{w-1}$*

Since $\zeta_{\mathbb{Z}^2}(z) = \zeta(z)\zeta(z-1)$, it converges at $Re(z) > 2$, and $\zeta(2) = \frac{\pi^2}{6}$. Expanding $\zeta(z-1)$ at $z = 2$:

$$\zeta(z-1) = \frac{1}{z-2} + \gamma + o(z-2)$$

By Tauberian theorem, take $\alpha = 2, w = 1, g(x) = 1$, then

$$\lim_{x\to\infty} \sum_{n\le x} a_n = \frac{\pi^2 x^2}{12} \sim \frac{\pi^2 n^2}{12}$$

Example 4.8. *Let F_n^c be the free nilpotent group of rank n and class c, then* $\zeta_{F_2^2}(z) = \frac{\zeta(z)\zeta(z-1)\zeta(2z-2)\zeta(2z-3)}{\zeta(3z-3)}$

Proof. It only suffices to prove that

$$\zeta_{F_2^2,p}(z) = \frac{\zeta^p(z)\zeta^p(z-1)\zeta^p(2z-2)\zeta^p(2z-3)}{\zeta^p(3z-3)}$$

Let $S = \gamma_2(G) \cong \mathbb{Z}$, then $G/S \cong \mathbb{Z}^2 and S = Z(G)$. Suppose $Q \le G, |G:Q| = p^n$. Let $|G:QS| = p^a$ and $|S:Q\cap S| = p^b$, we have $n = a + b$. $Q/Q\cap S \cong QS/S$ is abelian and $S/Q\cap S$ is central in $SQ/Q\cap S$ so $QS/Q\cap S$ is abelian.Conversely, suppose that K/L is abelian, then since K/S is torsion free and S/L is finite, $\tau(K/L) = S/L$. We choose a complement to this torsion subgroup of K/L, denoteds as Q/L. Then $QS = K$ and $Q \cap S = L$.Moreover, from Lemma 27 applied to the abelain group K/L, the number of distinct choices for Q is exactly $|Hom(K/S,S/L)| = \left|Hom\left(\mathbb{Z}^2, c_{p^b}\right)\right| = p^{2b}$. Suppose (x,y) are free generators of G, and $z = [x,y]$. Then $K = \langle x^\alpha, x^\gamma y^\beta, z : \alpha > 0, 0 < \gamma < \alpha, \beta \ge 0\rangle$ uniquely. Now, $p^a = \alpha\beta, \gamma_2(K) = \langle z^{p^a}\rangle$, so K/L is abelian $\Leftrightarrow b \le a \Leftrightarrow b \le \left[\frac{n}{2}\right]$Suppose

$$\zeta_{F_2^2,p}(z) = \sum_{i=0}^{\infty} g_i\, p^{-iz},\ \zeta_{\mathbb{Z}^2,p}(z) = \sum_{i=0}^{\infty} g_i'\, p^{-iz}$$

From the previous discussion, we deduce

$$g_n = \sum_{i=0}^{[n/2]} g_{n-i}'\, p^{2i}$$

Since

$$\begin{aligned}\zeta_{\mathbb{Z}^2,p}(z) = \zeta^p(z)\zeta^p(z-1) &= (1^{-z} + p^{-z} + p^{-2z} + \cdots +)(1^{-z} + p\cdot p^{-z} + p^2\cdot p^{-2z} + \cdots +)\\ &= 1^{-z} + (1+p)p^{-z} + (1+p+p^2)p^{-2z} + \cdots\end{aligned}$$

Thus

$$g_i' = \frac{p^{i+1}-1}{p-1}, \forall i \in \mathbb{Z}$$

$$g_n = \frac{p^{n+1}-1}{p-1} + \frac{p^n-1}{p-1}p^2 + \cdots + \frac{p^{n+1-[n/2]}-1}{p-1}\cdot p^{2[n/2]}$$

$$(p-1)g_n = p^{n+1}\sum_{i=0}^{[n/2]} p^i - \sum_{i=0}^{[n/2]} p^{2i}$$

Let

$$a_n = \sum_{i=0}^{[n/2]} p^i, b_n = \sum_{i=0}^{[n/2]} p^{2i}$$

then

$$a_n = \sum_{i=0}^{n} f_1\left(p^i\right), \text{where } f_1\left(p^{2i}\right) = p^i, f_2\left(p^{2i+1}\right) = 0$$
$$b_n = \sum_{i=0}^{n} f_2\left(p^i\right), \text{where } f_2\left(p^{2i}\right) = p^{2i}, f_2\left(p^{2i+1}\right) = 0$$

From the multiplication of p-Dirichlet series, we deduce

$$\begin{aligned}\sum_{n=0}^{\infty} a_n\, p^{-nz} &= \zeta^p(z)(1 + p \cdot p^{-2z} + p^2 \cdot p^{-4z} + \cdots) = \zeta^p(z)\zeta^p(2z-1),\\ \sum_{n=0}^{\infty} b_n\, p^{-nz} &= \zeta^p(z)(1 + p^2 \cdot p^{-2z} + p^4 \cdot p^{-4z} + \cdots) = \zeta^p(z)\zeta^p(2z-2).\end{aligned}$$

Now, from [2] we know

$$\begin{aligned}(p-1)\zeta^p_{F_2^2}(z) &= \sum_{i=0}^{\infty} p^{i+1}\, a_n p^{-iz} - \sum_{i=0}^{\infty} b_i\, p^{-iz}\\ &= p\sum_{i=0}^{\infty} a_n\, p^{-i(z-1)} - \sum_{i=0}^{\infty} b_i\, p^{-iz}\\ &= p\zeta^p(z-1)\zeta^p(2z-3) - \zeta^p(z)\zeta^p(2z-2)\\ &= \frac{p}{(1-p^{-z+1})(1-p^{-2z+3})} - \frac{1}{(1-p^{-z})(1-p^{-2z+3})}\\ &= \frac{p(1-p^{-z}-p^{-2z+2}+p^{-3z+2}) - (1-p^{-z+1}-p^{-2z+3}+p^{-3z+4})}{(1-p^{-z+1})(1-p^{-2z+3})(1-p^{-z})(1-p^{-2z+2})}\\ &= (p-1)(1-p^{-3z+3})\zeta^p(z)\zeta^p(z-1)\zeta^p(2z-2)\zeta^p(2z-3).\end{aligned}$$

Therefore,

$$\zeta^p_{F_2^2}(z) = \frac{\zeta^p(z)\zeta^p(z-1)\zeta^p(2z-2)\zeta^p(2z-3)}{\zeta^p(3z-3)}$$

The Heisenberg group $H_3(\mathbb{Z}) = \left\{\begin{pmatrix}1 & p & q\\ 0 & 1 & r\\ 0 & 0 & 1\end{pmatrix} : p, q, r \in \mathbb{Z}\right\}$ is a F_2^2 groups. So using the same techinique from Corollary 28 we obtain:

Corollary 4.9. *Let* $a_n = \#\{H \leq H_3(\mathbb{Z}) : |H_3(\mathbb{Z}) : H| = n\}$*, then* $\sum_{i=1}^{n} a_n \sim \frac{\zeta(2)^2 n^2 \ln n}{2\zeta(3)}$

Grunewald F J provides a concrete formula for subgroup zeta function of $\mathcal{J}$-group with Hirsch length 3(cf. (Grunewald F J, 1988)). Here are some notations might be used: for a fixed prime p, let

$$X_b^a = p^{b-az}, P_b^a = (1 - X_b^a)^{-1}, Z_n = P_0^1 P_1^1 \cdots P_{n-1}^1$$

Theorem 4.10. *Let* J *be a* $\mathcal{J}$*-group with Hirsch length* 3*, number* m *satisfies* $p^m \mid |Z(J) : \gamma_2(J)|$*, then* $\zeta_{G,p}(z) = (1 - X_{2m}^m)Z_3 + X_{2m}^m(1 - X_3^3)P_2^2 P_3^2 Z_2$

They also prove that forall $\mathcal{J}$-group Γ, the p −local factor of $\zeta_\Gamma(z)$ is a rational function of p^{-z}:

Theorem 4.11. *Consider a* $\mathcal{J}$*-group* Γ*. For any prime number* p*, there exists polynomials* $\phi_p(x)$ *and* $\psi_p(x)$ *over* $\mathbb{Z}$*, s.t.* $\zeta_{\Gamma,p}(z) = \frac{\phi_p(p^{-z})}{\psi_p(p^{-z})}$ *Additionally,* $deg\phi_p, deg\psi_p < \infty$

Using this theorem we can have a formal generalization of corollary 28 annd corollary 31, which is proved by Grunewald and duSautoy in (Sautoy & Grunewald, 2000):

Theorem 4.12. *Let* Γ *be a* $\mathcal{J}$ *-group, then* $\exists b(\Gamma) \in \mathbb{Z}, b(\Gamma) \geq 0, \exists c, c' \in \mathbb{R}$ *, such that* $S_\Gamma(n) = \sum_{m \leq n} \alpha_\Gamma(m) \sim c n^{\alpha(\Gamma)} (ln n)^{b(\Gamma)}$,
$S_\Gamma(n)^{\alpha(\Gamma)} \sim c' (ln n)^{b(\Gamma)+1}$.

Studying the abscissa of convergence is practical and useful, especially when the zeta function is not explicitly known. The following results from (Smith, 1983) show that the abscissa of convergence is bounded by the Hirsch length and is invariant under group commensurability:

Theorem 4.13. *Let Γ be a finitely generated nilpotent group, then* $\alpha_\Gamma \leq h(\Gamma)$

Particularly, for $\mathcal{J}$-group of class 2, we can find the exact upper and lower for the abscissas of convergence:

Theorem 4.14. *Let G be a $\mathcal{J}$-group of rank n and class* 2. *Denoted* $h = h(G)$,*then* $\frac{3n}{2} - 1 \leq \alpha_G \leq h - 1$

Secondly, we can enumerate the n-dimensional irreducible complex representations of a group Γ, considering isomorphism classes, which we will denote as $Irr_\Gamma(n)$. The number of such representations is denoted by $r_\Gamma(n) := \#Irr_\Gamma(n)$. If Γ is rigid, i.e. $\forall n \in \mathbb{N}, r_\Gamma(n)$ is finite, then we can define the representation zeta function:

$$\zeta_\Gamma^{irr}(z) = \sum_{n=1}^{\infty} r_\Gamma(n) n^{-z}$$

Our main purpose is to study the cases for $\mathcal{J}$-group. The problem is unless Γ is trivial, Γ has infinitely many one-dimensional representations. Thus it's valuable to consider the twists equivalence of finite-dimensional representations

Definition 4.15. *Let* $\rho_1, \rho_2 \in Irr_n(\Gamma)$, *if* $\exists \lambda \in Irr_1(\Gamma)$, *s.t.* $\rho_1 \cong \rho_2 \otimes \lambda$,*then* ρ_1 *and* ρ_2 *are twist-equivalent.*

We denoted $\tilde{r}_\Gamma(n)$ to be the number of n-dimensional representations up to twists equivalent. Alexander Lubotzky, Andy R Magid, and Andy Roy Magid has proved that the $\forall n \in \mathbb{Z}$, $\tilde{r}_\Gamma(n)$ is finite(cf. (Lubotzky et al., 1985)). Thus

$$\tilde{\zeta}_\Gamma(z) = \sum_{n=1}^{\infty} \tilde{r}_\Gamma(n) n^{-z}$$

is well defined

Ehud Hrushovski demonstrated in (Hrushovski et al., 2018) that the representation zeta function shares many properties with the subgroup zeta function.

Example 4.16. *For* $H_3(\mathbb{Z})$, $\tilde{\zeta}_{H(\mathbb{Z})}^{\widetilde{irr}}(z) = \frac{\zeta(z-1)}{\zeta(z)}$

Since

$$\sum_{n=1}^{\infty} \psi(n) n^{-z} = \frac{\zeta(z-1)}{\zeta(z)}$$

We have

$$\zeta_{H(\mathbb{Z})}^{\widetilde{irr}}(z) = \sum_{n=1}^{\infty} \widetilde{r_n}(\Gamma) n^{-z} = \frac{\zeta(z-1)}{\zeta(z)} = \sum_{n=1}^{\infty} \psi(n) n^{-z}$$

So that the number of irreducible representations of Γ up to twists equivalence is $\psi(n)$

5. Conclusion

This paper investigates various important aspects of the zeta function and its applications in number theory and group theory. We started by proving that the nontrivial zeros of $\zeta(z)$ lie within the critical strip, specifically between $Re\ z = 0$ and $Re\ z = 1$ on $\mathbb{C}$. This result, closely related to the Riemann Hypothesis, is a central issue in analytic number theory."

Furthermore, we utilized the asymptotic behavior of $\zeta(z)$ and its generalization, the Dirichlet L-function, as z approaches 1 to provide a proof of Dirichlet's theorem. This demonstration highlights the application of analytic techniques to address problems in number theory, illustrating how such methods can be leveraged to uncover deep connections and solve complex issues within the field.

Finally, we extended our study to the realm of group theory by introducing the concept of the zeta function of a group. Using $H_3(\mathbb{Z})$ as a case study, we calculated both its subgroup zeta function and its representation zeta function. These computations illustrate the utility of zeta functions in understanding the structure and representation theory of infinite groups.

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References

[1] Ezzat, S. (2014). Counting irreducible representations of the heisenberg group over the integers of a quadratic number field. *Journal of Algebra*, *397*, 609–624.
[2] Grunewald F J, S. G. C., Segal D. (1988). Subgroups of finite index in nilpotent groups. *Inventiones Mathematicae*, *93*, 185–223.
[3] Grunewald, F. J., & Scharlau, R. (1979). A note on finitely generated torsion-free nilpotent groups of class 2. *Journal of Algebra*, *58*(1), 162–175.
[4] Hrushovski, E., Martin, B., Rideau, S., & Cluckers, R. (2018). Definable equivalence relations and zeta functions of groups. *J. Eur. Math. Soc.(JEMS)*, *20*(10), 2467–2537.
[5] Kenneth Ireland, M. R. (2000). *A classical introduction to modern number theory*. Springer.
[6] Lubotzky, A., Magid, A. R., & Magid, A. R. (1985). *Varieties of representations of finitely generated groups* (Vol. 336). American Mathematical Soc.
[7] lvic, A. (1949). THE RIEMANN ZETA-FUNCTION: THE THEORY OF THE RIEMANN ZETA-FUNCTION WITH APPLICATIONS. A Wlley-Intersclence Publication.
[8] Marcus du Sautoy, L. W. (1925). *Zeta functions of groups and rings*. Spinger.
[9] Sautoy, M. du, & Grunewald, F. (2000). Analytic properties of zeta functions and subgroup growth. *Annals of Mathematics*, *152*(3), 793–833.
[10] Smith, G. C. (1983). *Zeta-functions of torsion-free finitely generated nilpoten groups*. The University of Manchester (United Kingdom).
[11] Sury, B. (n.d.). Subgroup growth and zeta functions.