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A study of sign-changing Poisson-type equations in two configurations

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Abstract. In this paper, We are interested in specific non-coercive issues concerning electromagnetic wave propagation in the presence of metals or particular metamaterials. We focus on some non-coercive problems which cannot be studied by a classical Lax-Milgram approach. We consider the sign-changing Poisson-type equations with the homogeneous Dirichlet boundary conditions in the circular configuration and in the three square configuration. We utilize a method called T-isomorphism, which allows the conversion of non-coercive problems into coercive ones, to examine specific Poisson-type equations with sign-changing parameters. We first use the classical method to transform the equations into the corresponding variational formulations, and then apply the T-isomorphism method to show the well-posedness of these variational formulations with some parameters. By applying appropriate isomorphisms, we can conclude the well-posedness of the related variational formulations with certain parameters, which constitute the main findings of this paper. After giving appropriate isomorphisms, we can deduce the well-posedness of the corresponding variational formulations with some parameters, which are our main results in this paper.

Keywords: T-isomorphism, non-coercive problems, Poisson-type equations.

1. Introduction

Partial Differential Equations (PDEs) are a fundamental mathematical tool used to describe various physical phenomena, such as heat conduction, fluid dynamics, electromagnetism, and wave propagation. Unlike ordinary differential equations (ODEs), which involve functions of a single variable and their derivatives, PDEs involve functions of multiple variables and their partial derivatives.

The variational formulation is a powerful mathematical framework used to solve Partial Differential Equations (PDEs), particularly in complex domains and for problems where direct methods may be challenging to apply. By using some functional analysis methods (e.g., the Lax-Milgram theorem), we can show the well-posedness of some variational formulations. Also, the variational formulation is the foundation of the Finite Element Method (FEM), a numerical technique widely used to solve PDEs in engineering and physics. This method is particularly effective for solving PDEs in complex geometries or with complicated boundary conditions (see in [1]).

We are interested in certain non-coercive problems related to electromagnetic wave propagation in the presence of metals or specific types of metamaterials (see also [8]). In fact, for non-coercive problems, we cannot directly apply the Lax-Milgram theorem to deduce well-posedness of the corresponding variational formulation. Instead, we adapt a technique known as T-isomorphism, which

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enables the transformation of non-coercive problems into coercive problems, to analyze certain Poisson-type equations with parameters which change sign.

In this paper, we study the sign-changing Poisson-type equations with the homogeneous Dirichlet boundary conditions in the circular configuration and in the three square configuration. The point lies in constructing appropriate isomorphisms. After giving appropriate isomorphisms, we can deduce the well-posedness of the corresponding variational formulations with some parameters, which are our main results in this paper.

2. Preliminaries

This section will introduce some standard notations and theorems used in this study.

2.1. Hilbert Space

Definition 2.1. For a real vector space V, a bilinear form a on V is a function of two variables $a: V \times V \to \mathbb{R}$ which satisfies the following conditions for any $scalar \alpha$ and any vectors $u, v, u_1, u_2, v_1, v_2 \in V$:

$$\begin{aligned} a(u_1 + u_2, v) &= a(u_1, v) + a(u_2, v), \\ a(\alpha u, v) &= \alpha a(u, v), \\ a(u, v_1 + v_2) &= a(u, v_1) + a(u, v_2), \\ a(u, \alpha v) &= \alpha a(u, v). \end{aligned}$$

Definition 2. 2. A real inner product space is a vector space V over R with a bilinearform

$$\langle \cdot,\cdot \rangle_V : V \times V \to \mathbb{R}$$

$such$ that for all vectors $x, y \in V$,

$$\begin{aligned} &\langle x, y \rangle_V = \langle y, x \rangle_V, \\ &\langle x, x \rangle_V > 0 \ for \ x \neq 0. \end{aligned}$$

$\langle \cdot,\cdot \rangle_V$ is called the inner product of V.

Definition 2. 3. A vector space V with an inner product $\langle \cdot,\cdot \rangle_V$ is called a Hilbert space if it is complete with respect to the norm $\|\cdot\|_V$, which is defined as

$$\| v \|_V = \sqrt{\langle v, v \rangle}, \quad for \ all \ v \in V.$$

Definition 2.4. For a real vector space V , the algebraic dual space V ∗ consists of alllinear functions"

$$\varphi: V \to \mathbb{R}.$$

Definition 2.5. Given a real vector space V and its dual space V ∗, the natural pairing between V ∗ and V is a non-degenerate bilinear map

$$\langle \cdot,\cdot \rangle : V^* \times V \to \mathbb{R}.$$

2.2. Lebesgue Space

Definition 2.6. If $u: \Omega \subset \mathbb{R}^N \to \mathbb{R}$ satisfies

$$\int_\Omega |u(x)|^2 \, dx < +\infty,$$

then we have $u \in L^2(\Omega)$. The space $L^2(\Omega)$ is equipped with the norm

$$\| u \|_{L^2(\Omega)} := \left(\int_\Omega |u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

We observe that $L^2(\Omega)$ is also a Hilbert space with the inner product

$$\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega u(x) v(x) \, dx$$

2.3. Sobolev Space

We have to define weak derivative before Sobolev Space

Definition 2.7 Let $v \in L^2(\Omega)$. We say that bv is weakly differentiable in $L^2(\Omega)$ if there exist functions $w_i \in L^2(\Omega)$ for $i \in \{1, \dots, N\}$ such that for any test function $\varphi \in C_c^\infty(\Omega)$, the following holds:

$$\int_\Omega v(x) \frac{\partial \varphi}{\partial x_i}(x) \, dx = -\int_\Omega w_i(x) \varphi(x) \, dx.$$

Each w_i is called the i -th weak partial derivative of v and is denoted by $\frac{\partial v}{\partial x_i}$

With definition of week derivative, now define the Sobolev space $H^m with\, m \in N^+$

Let Ω be an open subset of R^N. The Sobolev space $H^1(\Omega)$ is defined as

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \,\middle|\, \frac{\partial v}{\partial x_i} \in L^2(\Omega) \text{ for all } i \in \{1, \dots, N\} \right\},$$

where $\frac{\partial v}{\partial x_i}$ denotes the i -th weak partial derivative of v .

For $m \geq 2$, the Sobolev space $H^m(\Omega)$ is defined as

$$H^m(\Omega) = \{ v \in L^2(\Omega) \mid \partial^\alpha v \in L^2(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \leq m \},$$

With

$$\partial^\alpha v(x) = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}(x),$$

where $\partial^\alpha v$ is understood in the weak sense. Here, $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index with $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^N \alpha_i$

Proposition 2.9 The Sobolev space $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle = \int_\Omega \sum_{|\alpha| \leq m} \partial^\alpha u(x) \partial^\alpha v(x) d\, x \qquad (2.1)$$

and the norm

$$\| u \|_{H^m(\Omega)} = \sqrt{\langle u, u \rangle} = \left(\int_\Omega \quad \sum_{|\alpha| \leq m} |\partial^\alpha u(x)|^2 dx\right)^{\frac{1}{2}}. \qquad (2.2)$$

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We now introduce a important subspace of $H^1(\Omega)$, noted as $H_0^1(\Omega)$.

Definition 2.10. Let Ω be a regular and bounded domain. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions which are null at the boundary $\partial\Omega$.

Remark 2.11. In fact, the Sobolev space $H_0^1(\Omega)$ is essentially the completion of $C_c^{\infty}(\Omega) in H^1(\Omega)$. But with the assumption of Ω and the trace theorems in (2.6) and (2.7), we can prove this statement is equivalent to Definition 2.10.

We hope to simplify the norms for easier calculation. In order to achieve this, we introduce the definition of equivalent norms.

Definition 2.12. Two norms $|\cdot|_{\alpha} and |\cdot|_{\beta}$ defined on X are called equivalent if there exist positive real numbers C and D such that for all $x \in X$

$$C|x|_{\alpha} \leq |x|_{\beta} \leq D|x|_{\alpha}.$$

We can now apply this definition to the norm of $H_0^1(\Omega)$.

Corollary 2.13. If Ω is a regular and bounded domain, then the norm of $H_0^1(\Omega)$can be simplified as

$$\| v \|_{H_0^1(\Omega)} = \left(\int_{\Omega} |\nabla v(x)|^2 dx\right)^{\frac{1}{2}}. \tag{2.3}$$

For $H_0^1(\Omega)$, the norm(2,3) is equivalent to the norm (2.2). To prove this, we introduce the Poincaré inequality.

Proposition 2.14. (*Poincar´e Inequality*) Let Ωbe a regular, bounded domain. There exists a constant $C > 0$ such that for any function $v \in H_0^1(\Omega)$,

$$\int_{\Omega} |v(x)|^2 dx \leq C \int_{\Omega} |\nabla v(x)|^2 dx. \tag{2.4}$$

With the Poincaré inequality, we can easily prove Corollary 2.13. Before that, we need to introduce the Rellich–Kondrachov theorem.

Theorem 2.15. (*Rellich–Kondrachov Theorem*) If Ω is a regular and bounded domain, then any bounded sequence in $H^1(\Omega)$ has a convergent subsequence in $L^2(\Omega)$.

We can use this theorem to prove the Poincar´e inequality.

Proof. We will prove by contradiction. Suppose there is no constant $C > 0$ such that for any function $v \in H_0^1(\Omega)$,

$$\int_{\Omega} |v(x)|^2 dx \leq C \int_{\Omega} |\nabla v(x)|^2 dx.$$

This implies that there exists a sequence$v_n \in H_0^1(\Omega)$such that

$$1 = \int_{\Omega} |v_n(x)|^2 dx > n \int_{\Omega} |\nabla v_n(x)|^2 dx. \tag{2.5}$$

From (2.5), we see that the sequence v_n is bounded in $H_0^1(\Omega)$. By the Rellich–Kondrachov theorem, there exists a subsequence$v_{n'}$ that converges in $L^2(\Omega)$. Additionally, (2.5) implies that the sequence

$\nabla v_{n'}$ converges to zero in $L^2(\Omega)$(component by component). Consequently, $v_{n'}$ forms a Cauchy sequence in $H_0^1(\Omega)$, which, being a Hilbert space, implies convergence in $H_0^1(\Omega)$to a limit v.

Furthermore, we have

$$\int_\Omega |\nabla v(x)|^2 dx = \lim_{n'\to+\infty} \int_\Omega |\nabla v_{n'}(x)|^2 dx \le \lim_{n'\to+\infty} \quad \frac{1}{n'} = 0,$$

which indicates that \$v\$ is constant in every connected component of Ω. Given that v is zero on the boundary $\partial\Omega$, v must be identically zero throughout Ω. Moreover,

$$\int_\Omega |v(x)|^2 dx = \lim_{n'\to+\infty} \int_\Omega |v_{n'}(x)|^2 dx = 1,$$

which contradicts $v = 0$.

Remark 2.16. In general, this proof by contradiction can be applied to demonstrate inequality (2.4)for functions within a subspace of $H^1(\Omega)$ that are zero at certain parts of the boundary.

Since $\partial\Omega$ has zero measure, defining the boundary value, or trace, of v on $\partial\Omega$ is not straightforward. Fortunately, the trace $v|_{\partial\Omega}$ of a function in $H^1(\Omega)$can still be defined. This is shown by the trace theorems.

Theorem 2.17. (*Trace Theorem H¹*) Let Ω be a regular, bounded domain. We define the trace operator γ_0 as follows:

$$H^1(\Omega) \cap C^1\left(\overline{\Omega}\right) \to L^2(\partial\Omega) \cap C\left(\overline{\partial\Omega}\right),$$

$$v \to \gamma_0(v) = v|_{\partial\Omega}.$$

This operatorγ_0 extends continuously to a linear map from $H^1(\Omega)$ into $L^2(\partial\Omega)$, still denoted as γ_0. In particular, there exists a constant $C > 0$ such that for any function $v \in H^1(\Omega)$, we have

$$|v|_{L^2(\partial\Omega)} \le C|v|_{H^1(\Omega)}. \tag{2.6}$$

Theorem 2.18. (*Trace Theorem H²*) Let Ω be a regular, bounded domain. We define the trace operator γ_1 as follows:

$$H^2(\Omega) \cap C^1\left(\overline{\Omega}\right) \to L^2(\partial\Omega) \cap C\left(\overline{\partial\Omega}\right),$$

$$v \to \gamma_1(v) = \frac{\partial v}{\partial n}\Big|_{\partial\Omega}$$

with $\frac{\partial v}{\partial n} = \nabla v \cdot n$. This operator γ_1 extends continuously to a linear map from $H^2(\Omega) to L^2(\partial\Omega)$. In particular, there exists a constant $C > 0$ such that for any function $v \in H^2(\Omega)$, then we have

$$|\frac{\partial v}{\partial n}|_{L^2(\partial\Omega)} \le C|v|_{H^2(\Omega)}. \tag{2.7}$$

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Using the trace theorems and the density of $C_c^\infty(\bar{\Omega})$ in $H^1(\Omega) and H^2(\Omega)$, we obtain Green's formulas.

Theorem 2.19. (*Green's Formula*) Let Ω be a regular, bounded domain. If u and v are functions in $H^1(\Omega)$, then we have

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)dx + \int_{\partial\Omega} u(x)v(x)n_i(x)ds.) \quad (2.8)$$

Moreover, if $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have

$$\int_\Omega \Delta u(x)v(x)dx = -\int_\Omega \nabla u(x)\cdot\nabla v(x)dx + \int_{\partial\Omega}\frac{\partial u}{\partial n}(x)v(x)ds. \quad (2.9)$$

For the variational formulations of PDEs, the Lax-Milgram theorem is a valuable tool for analyzing their well-posedness under three assumptions.

2.4. Lax-Milgram Theorem

Many PDEs can be reformulated using equations (2.8) and (2.9) into their vari-ational form. The well-posedness of the variational formulation of a PDE in a Hilbert space is generally equivalent to the well-posedness of the original PDE. The Lax-Milgram theorem offers significant insights into the well-posedness of PDEs.

Theorem 2.20. (*Lax-Milgram Theorem*) Let V be a real Hilbert space.

1. The bilinear form $a(\cdot,\cdot)$ is continuous on V, meaning there exists a constant $M > 0$ such that

$$|a(w,v)| \le M|w|_V|v|_V \text{ for all } w, v \in V. \quad (2.10)$$

2. The bilinear form $a(\cdot,\cdot)$ is coercive (or elliptic), meaning there exists a constant $\alpha > 0$ such that

$$a(v,v) \ge \alpha|v|_V^2 \text{ for all } v \in V. \quad (2.11)$$

Then, the variational problem

$$\text{Find } u \in V \text{ such that } a(u,v) = \langle L, v\rangle \text{ for all } v \in V. \quad (2.12)$$

has a unique solution in V, where $L: V \longrightarrow R$ is a bounded linear functional on V, and $\langle\cdot,\cdot\rangle$ denotes the pairing of V and V^*.

Remark 2.21. Different partial differential equations (PDEs) with their respective boundary conditions correspond to distinct variational formulations. To derive the appropriate variational formulation, we typically follow these steps: first, identify a Hilbert space V. Next, multiply both sides of the PDE by $v \in V$ and integrate. Then, apply Green's formula to reduce the differential order of the integral equation by one. This process yields the following formulation:

$$\text{Find } u \in V \text{ such that } a(u,v) = L(v) \text{ for all } v \in V, \quad (2.13)$$

where $a(\cdot,\cdot)$ is a bilinear form on V and $L(\cdot)$ is a linear form on V. The solution to the variational formulation is also known as the weak solution of the corresponding PDE.

2.5. Application of the T-isomorphism method

Lemma 2.22. Assume there exists an isomorphism T of $H_0^1(\Omega)$ such that the bilinear form $a(u,v) \mapsto a(u,Tv)$ is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$. In this case, equation (3.2) is well-posed.

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Proof. By definition, T is a continuous bijective operator from $H_0^1(\Omega)$ to $H_0^1(\Omega)$.

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, Tv) = L(Tv), \quad \text{for all } v \in H_0^1(\Omega). \tag{2.14}$$

If (2.14) is well-posed, since for all $u \in H_0^1(\Omega)$, there always exists $v \in H_0^1(\Omega)$ such that $Tv = u$, then (3.2) must also be well-posed. Moreover, by the definition of continuity, for all $u \in H_0^1(\Omega)$, there exists a constant $C > 0$ such that

$$\| Tu \|_{H_0^1(\Omega)} \leq C \| u \|_{H_0^1(\Omega)}.$$

Additionally, using similar methods as before, we prove that $a(\cdot, T\,\cdot)$ is continuous on $H_0^1(\Omega)$. Similarly, it is easy to see that $L(T\,\cdot)$ is a bounded linear functional, proving that (2.14) is indeed well-posed.

Remark 2.23. We wish for the operator T to compensate for the change in sign of σ. Specifically, we want to define $Tu = u$ in Ω_1 and $Tu = -u$ in Ω_2. This leads to

$$a(u, Tu) = \int_\Omega |\sigma| \, |\nabla u(x)|^2 \geq \min(\sigma_1, |\sigma_2|) \, |u|_{H_0^1(\Omega)}^2.$$

However, this choice of T is not an isomorphism. Hence, we cannot deduce the necessary properties for the well-posedness of the two problems to be equivalent.

3. Two non-coercive problems

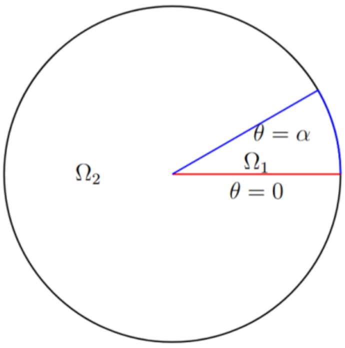

Figure 1. Circular Configuration

3.1. Configuration with circular boundary

We want to investigate the existence and uniqueness of this known non-coersived problems in the Figure 1 shown above. Firstly, we consider following variational formulation

$$\textit{Find } u \in H_0^1(\Omega) \textit{ such that } -\mathrm{div}(\sigma \nabla u) = f \quad \textit{in } \Omega. \tag{3.1}$$

Here, $f \in L^2(\Omega)$ denotes the source term. We also introduce the following notation

$$\kappa = \frac{\sigma_2}{\sigma_1}.$$

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The parameter κ will play a significant role in our problem analysis. We will examine the well-posedness of equation (3.1) within two distinct domains. The specific functions f and σ used will depend on the particular application of this equation.

For our study, we consider the variational formulation corresponding to equation (3.1) as follows:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u,v) = L(v) \quad \text{for all } v \in H_0^1(\Omega), \tag{3.2}$$

where

$$a(u,v) = \sigma_1 \int_{\Omega_1} \nabla u(x) \cdot \nabla v(x) dx + \sigma_2 \int_{\Omega_2} \nabla u(x) \cdot \nabla v(x) dx, \backslash$$

$$L(v) = \int_{\Omega} f(x) v(x)\, dx$$

Theorem 3.1. (3.2) is well-posed if $\kappa \in \left(-\infty, -\max\left(\gamma, \frac{1}{\gamma}\right)\right) \cup \left(-\frac{1}{\max\left(\gamma,\frac{1}{\gamma}\right)}, 0\right)$.

In more sophisticated models, we can no longer assume the condition $u \in H_0^1(\Omega)$. Thus, we are interested in the wellposedness of a more general mathematical problem. Let Ω be the bounded and connected open of R^2 defined by

$$\Omega = \{(r,\theta) \in R^2 \text{ sc ta } 0 < r < 1 \text{ and } 0 < \theta < 2\pi\}.$$

Correspondingly, we introduce the following domains,

$$\Omega_1 = (0,0) \times (1,\alpha), \Omega_2 = (0,\alpha) \times (1, \alpha - 2\pi).$$

In considering the well-posedness of (3.2) on an open Ω. We now apply to use T-isomorphism, we choose the application R such that $(R\varphi)(r,\theta) = \varphi\left(r, -\frac{1}{\gamma}\theta\right)$ for all $\varphi \in H_0^1(\Omega)$. Respectively, we define $\varphi_1 \in \Omega_1$ and $\varphi_2 \in \Omega_2$. In this application, the value of γ have to be define. It is esay to find the value, since we could input the value for θ and we got $R\varphi_1(r,0) = \varphi(r,0)$ and $R\varphi_1(r,-2\pi+\alpha) = \varphi(r,\alpha)$

Therefore we could get that $-\frac{1}{\gamma} = \frac{\alpha - 2\pi}{\alpha}$. We introduce the operator given by:

$$T\varphi := \{ \begin{matrix} \varphi & in \Omega_1, \\ -\varphi + 2R\varphi_1 & in \Omega_2. \end{matrix} \tag{3.3}$$

We now present the following corollary:

Corollary 3.2. The operator T is an isomorphism of $H_0^1(\Omega)$.

Proof. We observe the properties of $(T \circ T)\varphi$,

$$(T \circ T)\varphi := \begin{cases} T\varphi & \text{in } \Omega_1, \\ -(T\varphi) + 2R(T\varphi)_1 & \text{in } \Omega_2, \end{cases}$$
$$= \begin{cases} \varphi & \text{in } \Omega_1, \\ \varphi & \text{in } \Omega_2. \end{cases}$$

Hence we see that $T \circ T = Id$, implying that T is bijective from $H_0^1(\Omega)$ to =. To prove that the weak partial derivative of $T\varphi$ exists on the interface, we take the limit as \$$x$\$ approaches \$0\$, for all $0 < r < 1$:

$$\lim_{(r,\theta)\to(r,a^+)} T\,\varphi = \lim_{(r,\theta)\to(r,a^+)} (-\varphi + 2R\varphi_1)$$

$$= \lim_{(r,\theta)\to(r,a^+)} -\varphi + \lim_{(r,\theta)\to(r,a^-)} 2\,\varphi_1$$

$$= \lim_{(r,\theta)\to(r,a^-)} \varphi$$

$$= \lim_{(r,\theta)\to(r,a^-)} T$$

The desired result immediately follows due to the continuity of φ.
In addition, for all $u \in H_0^1(\Omega)$,

$$\int_{\Omega} \left|\nabla\big(Tu(x)\big)\right|^2 d\,x = \int_{\Omega_1} |\nabla u(x)|^2 d\,x + \int_{\Sigma} |\nabla u(x)|^2\, dx \int_{\Omega_2} \left|\nabla\big(-u(x) + 2Ru_1(x)\big)\right|^2 d\,x\backslash$$

$$\leq C \int_{\Omega} |\nabla u(x)|^2 d\,x.$$

Hence, T is continuous on $H_0^1(\Omega)$, implying that T is an isomorphism of $H_0^1(\Omega)$.
Now, we may apply standard techniques to prove the well-posedness of the variational formulation. By definition, for all $u \in H_0^1(\Omega)$,

$$a(u,Tu) = \sigma_{1\int_{\Omega_1}|\nabla u_1(x)|^2 d} x + \sigma_{2\int_{\Omega_2}\nabla u_2(x)} \cdot \nabla\big(Tu_2(x)\big)dx$$

$$= \sigma_{1\int_{\Omega_1}|\nabla u_1(x)|^2 d} x + \sigma_{2\int_{\Omega_2}\nabla u_2(x)} \cdot \nabla\big(-u_2(x) + 2Ru_1(x)\big)dx$$

$$= \sigma_{1\int_{\Omega_1}|\nabla u_1(x)|^2 d} x - \sigma_{2\int_{\Omega_2}|\nabla u_2(x)|^2} dx + 2\sigma_{2\int_{\Omega_2}\nabla u_2(x)} \cdot \nabla\big(Ru_1(x)\big)dx.$$

Applying the Cauchy-Schwartz inequality backwards, we get

$$a(u,Tu) \geq \int_{\Omega} |\sigma||\nabla u(x)|^2 dx - 2|\sigma_2| \parallel \nabla u_2 \parallel_{L^2(\Omega_2)} \parallel \nabla(Ru_1) \parallel_{L^2(\Omega_2)}$$
$$\geq \int_{\Omega} |\sigma||\nabla u(x)|^2 dx - 2|\sigma_2| \parallel \nabla u_2 \parallel_{L^2(\Omega_2)} \parallel R \parallel \parallel \nabla u_1 \parallel_{L^2(\Omega_1)}.$$

Here, $(|R|^2 = \sup_{u_1 \in H_0^1, \Sigma(\Omega_1)} \frac{|\nabla(Ru_1)|_{L^2(\Omega_2)}}{|\nabla u_1|_{L^2(\Omega_1)}} = \max\left(\gamma, \frac{1}{\gamma}\right) \geq 1)$. Following is the calculaton of $|R|^2$and denote that $\theta' = -\gamma\theta\backslash implies d\theta = -\frac{1}{\gamma} d\theta'$:

$$\| R \|^2 = \frac{\int_{\Omega_2} |\nabla(Ru_1)|^2 dxdy}{\int_{\Omega_1} |\nabla u_1|^2 dxdy}$$
$$= \frac{\int_{-2\pi+\alpha}^{0} \int_0^1 rdr\left(\left|\frac{\partial u}{\partial r}(u(r,-\gamma\theta))\right|^2 + \left|\frac{1}{r}\frac{\partial u}{\partial \theta}(u(r,-\gamma\theta))\right|^2\right)d\theta}{\int_0^\alpha \int_0^1 rdr\left(\left|\frac{\partial u}{\partial r}(r,\theta)\right|^2 + \left|\frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta)\right|^2\right)d\theta}$$
$$= \int_0^\alpha \frac{1}{\gamma}d\theta \int_0^1 rdr\left(\left|\frac{\partial u}{\partial r}(r,\theta)\right|^2 + \gamma^2\left|\frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta)\right|^2\right)$$
$$= \int_0^\alpha d\theta \int_0^1 rdr\left(\left|\frac{\partial u}{\partial r}(r,\theta)\right|^2 + \left|\frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta)\right|^2\right)$$
$$\le \begin{Bmatrix} \gamma, & \gamma \ge 1 \\ \frac{1}{\gamma}, & \gamma < 1 \end{Bmatrix} = max(\gamma, \frac{1}{\gamma}) \ge 1.$$

Therefore, we have

$$a(u,Tu) \ge \int_\Omega |\sigma| \| \nabla u \|^2 \, dx - 2\sqrt{max(\gamma,\frac{1}{\gamma})}|\sigma_2| \|\| \nabla u_2 \|_{L^2(\Omega_2)} \| \nabla u_1 \|_{L^2(\Omega_1)}.$$

By application of Young's inequality, for all ($\eta > 0$),

$$2\sqrt{max\left(\gamma,\frac{1}{\gamma}\right)}|\sigma_2| \| \nabla u_2 \|_{L^2(\Omega_2)} \| \nabla u_1 \|_{L^2(\Omega_1)}$$
$$\le \eta\left(max\left(\gamma,\frac{1}{\gamma}\right)|\sigma_2 \| |\nabla u_2 \|^2_{L^2(\Omega_2)}\right) + \eta^{-1}|\sigma_2| \| \nabla u_1 \|^2_{L^2(\Omega_1)}.$$

Hence,

$$a(u,Tu) \ge (\sigma_1 - \eta^{-1}|\sigma_2|)\int_{\Omega_1} |\nabla u_1|^2 \, dx + |\sigma_2|\left(1 - \max\left(\gamma,\frac{1}{\gamma}\right)\eta\right)\int_{\Omega_2} |\nabla u_2|^2 \, dx.$$

In order to make it coercive, the coefficient should be positive: Therefore,

$$\begin{cases} \sigma_1 - \eta^{-1}|\sigma_2| > 0 & \Rightarrow \eta > \frac{|\sigma_2|}{\sigma_1}, \\ 1 - max\left(\gamma,\frac{1}{\gamma}\right)\eta > 0 & \Rightarrow \eta < \frac{1}{\sqrt{max(\gamma,\frac{1}{\gamma})}}. \end{cases}$$

Known ($\kappa = \frac{\sigma_2}{\sigma_1}$)and ($\gamma = \frac{\alpha}{2\pi-\alpha}$), so we have the result that the equation is well-posed when

$$\kappa < -\max\left(\gamma,\frac{1}{\gamma}\right). \tag{3.4}$$

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Finally, by switching the sign of σ_1 and σ_2, we could esaily found that, the equation is well-posed when

$$\kappa \in \left(-\infty, -\max\left(\gamma, \frac{1}{\gamma}\right)\right) \cup \left(-\frac{1}{\max\left(\gamma, \frac{1}{\gamma}\right)}, 0\right). \tag{3.5}$$

3.2. Three square configuration

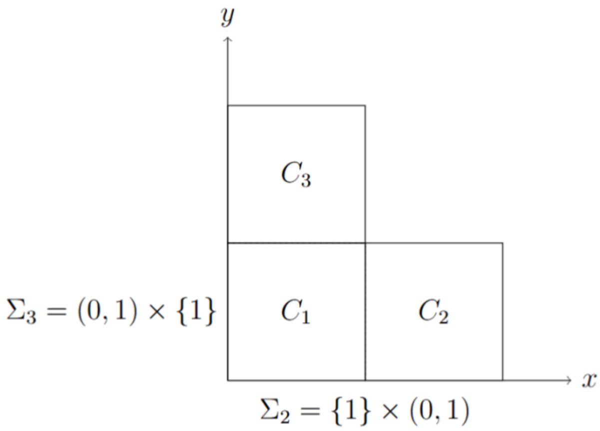

Figure 2. Three square configuration

In this Configuration, we have define boundary:

$$C_1 = (0,1) \times (0,1),$$

$$C_2 = (1,2) \times (0,1),$$

$$C_3 = (0,1) \times (1,2).$$

Let define the region:

$$C = \{(x, y) \in R^2 \mid 0 < x < 2 \text{ and } 0 < y < 1\},$$

$$C_1 = \{(x, y) \in R^2 \mid 0 < x < 1 \text{ and } 0 \le y < 1\},$$

$$C_2 = \{(x, y) \in R^2 \mid 1 < x < 2 \text{ and } 0 \le y < 1\},$$

$$C_3 = \{(x, y) \in R^2 \mid 0 < x < 1 \text{ and } 1 \le y < 2\}.$$

We have $\sigma_1 > 0$ in C_1, $\sigma_2 > 0$ in C_2, and $\sigma_3 < 0$ in C_3. Similarly we have variational form:

$$\boxed{\text{Find } u \in H_0^1(\Omega), \text{such that } a(u,v) = L(v) \quad \forall v \in H_0^1(\Omega).}$$

$$a(u, v) = \int_C \nabla u \cdot \nabla v \, dx \quad \text{for } u, v \in H_0^1(\Omega), \tag{3.6}$$

$$L(v) = \int_C f v \, dx \quad \text{for } v \in H_0^1(\Omega). \tag{3.7}$$

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In the same way, For ($u \in H_0^1(\Omega)$), and ($u_1 \in C_1$), ($u_2 \in C_2$), ($u_3 \in C_3$), let

$$Tu = \begin{cases} u_1 - 2S_3u_3 & \text{for}(x,y) \in C_1, \\ u_2 & \text{for}(x,y) \in C_2, \\ -u_3 & \text{for}(x,y) \in C_3. \end{cases} \tag{3.8}$$

We have to verify this T-isomorphism before we analyze the well-posedness.

Theorem 3.3. The operator T is an isomorphism of $H_0^1(\Omega)$.

Theorem 3.4. By application of the T-Isomophoism, the variational formulation$a(u,Tu)$(3.6) is coercive if

$$-1 < \frac{\sigma_1}{\sigma_3} < 0.$$

Theorem 3.5. When σ_1,$\sigma_2 > 0$,and $\sigma_3 < 0$, problem (3.6) is well-posed if

$$\left\{\frac{\sigma_1}{\sigma_3} > -1\right\} \cup \left\{\frac{\sigma_3}{\sigma_1} + \frac{\sigma_3}{\sigma_2} > -1\right\}.$$

4. Proof of main results

4.1. Proof of Theorem 3.2

We need to prove $T: H_0^1(\Omega) \to H_0^1(\Omega)$ is bijective first.

Known that on C_1 boundary $u_1 = 0$ except on Σ_1, Σ_2. We found that $Tu = u_1 - 2u_3(x, 2-y) = 0$,and $u_2 = 0$. On C_2 boundary $Tu = u_2 = 0$,and $u_3 = 0$. On C_3 boundary $Tu = -u_3 = 0$. Therefore, we get that $u \in H_0^1(\Omega)$,and $Tu \in H_0^1(\Omega)$.

After that, we need to prove Tu is continuous on Σ_2 and Σ_3, we use limit approach on both two boundary, on boundary Σ_2, we need to prove that [$\lim_{x\to 1^-} T\,u = \lim_{x\to 1^+} T\,u.$]

Firstly for all $0 < y < 1$,

$$\lim_{(x,y)\to(1^-,y)} T\,u = \lim_{(x,y)\to(1^-,y)} (u_1 - 2S_3u_3)$$

$$= \lim_{(x,y)\to(1^-,y)} u_1 - 2 \lim_{(x,y)\to(1^-,y)} S_3\,u_3$$

$$= \lim_{(x,y)\to(1^-,y)} u_1 - 2 \lim_{(x,y)\to(1^-,y)} u_3\,(x, 2-y)$$

$$= \lim_{(x,y)\to(1^-,y)} u_1 - 2 \lim_{(x,y)\to(1^-,y)} u_3\,(x,y) \quad \text{in } u_3, 1 < y < 2$$

$$= \lim_{(x,y)\to(1^-,y)} u_1$$

Therefore,

$$\lim_{\backslash underset(x,y)\to(1^+,y)} T\,u = \lim_{\backslash underset(x,y)\to(1^+,y)} u_2 \quad \text{for } 0 < y < 1$$

Since u is continuous at Σ_2 for all $0 < y < 1$,

$$\lim_{(x,y)\to(1^-,y)} u_1 = \lim_{(x,y)\to(1^+,y)} u_2$$

$$\lim_{(x,y)\to(1^-,y)} T\,u = \lim_{(x,y)\to(1^+,y)} T\,u$$

Similarly, in continuity on Σ_3 for all $0 < x < 1$

$$\lim_{(x,y)\to(x,1^-)} T\,u = \lim_{(x,y)\to(x,1^-)} (u_1 - 2S_3u_3)$$

$$= \lim_{(x,y)\to(x,1^-)} u_1\,(x,y) - 2\lim_{(x,y)\to(x,1^-)} u_3\,(x, 2-y)$$

$$= \lim_{(x,y)\to(x,1^-)} u_1\,(x,y) - 2\lim_{(x,y)\to(x,1^-)} u_3\,(x,y)$$

Then,

$$\lim_{\backslash underset(x,y)\to(x,1^+)} T\,u = -\lim_{\backslash underset(x,y)\to(x,1^+)} u_3\,(x,y)$$

Since u is continuous at Σ_3 for $0 < x < 1$,

$$\lim_{y\to 1^-} u_1\,(x,y) = \lim_{y\to 1^+} u_3\,(x,y)$$

$$\lim_{y\to 1^-} T\,u = \lim_{y\to 1^+} T\,u$$

Next, we have to prove that

$$|Tu|_{H_0^1(\Omega)} \le C|u|_{H_0^1(\Omega)}$$

$$\begin{aligned}
|Tu|^2_{H_0^1(\Omega)} &= |Tu|^2_{H_0^1(C_1)} + |Tu|^2_{H_0^1(C_2)} + |Tu|^2_{H_0^1(C_3)} \\
&= \int_{C_1} |\nabla(u_1 - 2S_3u_3)|^2 dx + \int_{C_2} |\nabla u_2|^2 dx + \int_{C_3} |\nabla(-u_3)|^2 dx \\
&\le \int_{C_1} |\nabla u_1|^2 dx + 4\int_{C_1} |\nabla u_3|^2 dx + \int_{C_2} |\nabla u_2|^2 dx + \int_{C_3} |\nabla u_3|^2 dx \\
&\le 5\int_{C_1} |\nabla u_1|^2 dx + \int_{C_2} |\nabla u_2|^2 dx + \int_{C_3} |\nabla u_3|^2 dx \le 5\int_{C} |\nabla u|^2 dx \le 5|u|^2_{H_0^1(C)}
\end{aligned}$$

Meanwhile, through the application of T-isomophism, we should know that whether $T \circ T = Id$. We have

$$(T \circ T)u = \begin{cases} (Tu)_1 - 2S_3(Tu)_3 & \text{in } C_1, \\ (Tu)_2 & \text{in } C_2, \\ (Tu)_3 & \text{in } C_3, \end{cases}$$
$$= \begin{cases} u_1 - 2S_3(Tu_3) + 2S_3(Tu_3) & \text{in } C_1, \\ u_2 \text{in } C_2, \\ -Tu_3 \text{in } C_3, \end{cases}$$
$$= \begin{cases} u_1 & \text{in } C_1, \\ u_2 & \text{in } C_2, \\ u_3 & \text{in } C_3. \end{cases}$$

So we verify that $T \circ T = Id$, and $T: H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is bijective. Finally, we know that the application T is an isomorphism of $H_0^1(\Omega)$.

4.2. Proof of Theorem 3.3

In order to prove that $a(u, Tu)$ is coercive when $-1 < \frac{\sigma_1}{\sigma_3} < 0$, similarly we have

$$a(u, Tu)$$
$$= \sigma_1 \int_{C_1} \nabla u_1 \cdot \nabla(u_1 - 2S_3u_3)dxdy + \sigma_2 \int_{C_2} |\nabla u_2|^2 dxdy - \sigma_3 \int_{C_3} |\nabla u_3|^2 dxdy$$
$$= \sigma_1 \int_{C_1} |\nabla u_1|^2 dxdy + \sigma_2 \int_{C_2} |\nabla u_2|^2 dxdy - \sigma_3 \int_{C_3} |\nabla u_3|^2 dxdy - 2\sigma_1 \int_{C_1} \nabla u_1 \cdot \nabla S_3 u_3 dxdy.$$

Applying the Cauchy-Schwartz inequality backwards, we get

$$a(u, Tu) \geq \int_C |\sigma||\nabla u(x)|^2 dx - 2|\sigma_1| \parallel \nabla u_1 \parallel_{L^2(C_1)} \parallel \nabla(S_3U_3) \parallel_{L^2(C_1)}$$
$$\geq \int_\Omega |\sigma||\nabla u(x)|^2 dx - 2|\sigma_1| \parallel \nabla u_1 \parallel_{L^2(C_1)} \parallel S_3 \parallel \parallel \nabla u_1 \parallel_{L^2(C_1)}.$$

Then we need to know the value of $|S_3|$

$$\parallel S_3 \parallel^2 = \frac{\int_{C_1} |\nabla S_3 u_3|^2 dxdy}{\int_{C_3} |\nabla u_3|^2 dxdy} = 1$$

$$a(u, Tu) \geq \int_\Omega |\sigma||\nabla u(x)|^2 dx - 2|\sigma_1| \parallel \nabla u_3 \parallel_{L^2(C_3)} \parallel \nabla u_1 \parallel_{L^2(C_1)}.$$

By application of Young's inequality, for all $\eta > 0$,

$$2|\sigma_1| \parallel \nabla u_3 \parallel_{L^2(C_3)} \parallel \nabla u_1 \parallel_{L^2(C_1)} \leq \eta|\sigma_3 \parallel |\nabla u_3 \parallel_{L^2(C_1)}^2) + \eta^{-1}|\sigma_1| \parallel \nabla u_1 \parallel_{L^2(\Omega_1)}^2$$

Hence,

$$a(u, Tu) \geq \sigma_1(1-\eta) \parallel \nabla u_1 \parallel_{L^2(C_1)}^2 + \sigma_2 \parallel \nabla u_2 \parallel_{L^2(C_2)}^2 + (\sigma_3 - \eta^{-1}\sigma_1) \parallel \nabla u_3 \parallel_{L^2(C_3)}^2$$

In order to make it coercive, the coefficient should be positive

$$\begin{cases} 1-\eta>0 & \Rightarrow \eta<1, \\ -\sigma_3-\eta^{-1}\sigma_1>0 & \Rightarrow \eta>-\dfrac{\sigma_1}{\sigma_3}. \end{cases}$$

From the result of the equation group, we could get

$$\begin{cases} -\dfrac{\sigma_1}{\sigma_3}<\eta<1, \\ -\dfrac{\sigma_1}{\sigma_3}<0. \end{cases}$$

Finally, we have $a(u,Tu)$ is coercive when

$$-1<\frac{\sigma_1}{\sigma_3}<0.$$

4.3. Proof of Theorem 3.4

We have to create a new application of T-Isomorphism T' for this configuration,

$$T'u=\begin{cases} u_1 & inC_1, \\ u_2 & inC_2, \\ -u_3+2\sigma_3u_1-2S_3S_2u_2 & inC_3. \end{cases}$$

Before analyze well-posedness, we have to define a T-Isomorphism and prove that exisied. Similarly, we should prove existence of application of T' Isomorphism. Firstly, T' should be continuous form $H_0^1(\Omega)\to H_0^1(\Omega)$. Secondly, T' is bijective.And thirdly,T' need to be identical.It is easy to find that $T'u\equiv 0\ on\ \partial\Omega$, since in each C_1 and C_2, $u_1,u_2\equiv 0$, so $T'u\equiv 0$. From geometric analysis, we can also easily find that $S_3{u_1}_{|_\Sigma}=S_3S_2{u_2}_{|_\Sigma}$.

We then need to check that $T'u$ is continuous at both Σ_2 and Σ_3. And it is obvious that $T'u$ is continuous at Σ_2. So we just need to consider whether it is continuous at Σ_3. Similarly, we use limit appoarch to prove for all $0<x<1$

$$\lim_{y\to 1^-} T'u\,(x,y)=\lim_{y\to 1^+} T'u\,(x,y).$$

Initially, for $0<x<1$, we can get

$$\begin{aligned} \lim_{(x,y)\to(x,1^+)} T'u &= -\lim_{(x,y)\to(x,1^+)} u_3(x,y)+2\lim_{(x,y)\to(x,1^+)} u_1(x,2-y) \\ &= -\lim_{(x,y)\to(x,1^+)} u_3(x,y)+2\lim_{(x,y)\to(x,1^+)} u_2(2-x,2-y) \\ &= -\lim_{(x,y)\to(x,1^+)} u_3(x,y)+2\lim_{(x,y)\to(x,1^+)} u_1(x,y), \end{aligned}$$

now we have,

$$\lim_{(x,y)\to(x,1^-)} T'u=\lim_{(x,y)\to(x,1^-)} u_1\,(x,y).$$

Since u is continuous at Σ_3,

$$\lim_{y\to 1^-} u_1(x,y) = \lim_{y\to 1^+} u_3(x,y).$$

Finally,

$$\lim_{y\to 1^-} T'u(x,y) = \lim_{y\to 1^+} T'u(x,y).$$

Next to check $|T'u|^2_{H_0^1(\Omega)}$ is controlled by $C|u|_{H_0^1(\Omega)}$, then we have,

$$|T'u|^2_{H_0^1(\Omega)} = |T'u|^2_{H_0^1(C_1)} + |T'u|^2_{H_0^1(C_2)} + |T'u|^2_{H_0^1(C_3)}$$

$$= \int_{C_1} |\nabla u_1|^2\, dx + \int_{C_2} |\nabla u_2|^2\, dx + \int_{C_3} |\nabla(u_3 + 2S_3u_1 - 2S_3S_2u_2)|^2\, dx$$

$$\le 5|u|^2_{H_0^1(C)}.$$

Lastly, in order to prove T' is bijective, we should check $T' \circ T'^{-1} = \mathrm{Id}$. We have

$$(T \circ T)u = \begin{cases} u_1 & \text{in} C_1, \\ u_2 & \text{in} C_2, \\ -T'u_3 + 2S_3(T'u)_1 - 2S_3S_2(T'u)_2 = u_3 & \text{in} C_3. \end{cases}$$

Therefore, \$T'\$ is Isomorphism. Then we need to prove that is well-posed when $\left\{\frac{\sigma_1}{\sigma_3} > -1\right\} \cup \left\{\frac{\sigma_3}{\sigma_1} + \frac{\sigma_3}{\sigma_2} > -1\right\}$.

Start with,

$$a(u,T'u) = \sigma \int_C \nabla u \cdot \nabla(T'u)$$
$$= \sigma_1 \int_{C_1} \left|\nabla u_1\right|^2 dx + \sigma_2 \int_{C_2} \left|\nabla u_2\right|^2 dx$$
$$-\sigma_3 \int_{C_3} |\nabla u_3|^2 dx + 2\sigma_3 \int_{C_3} \nabla u_3 \cdot \nabla(S_3u_1) dx$$
$$-2\sigma_3 \int_{C_3} \nabla u_3 \cdot \nabla(S_3S_2u_2) dx.$$

Then, the value of $|S_3|$ and $|S_2|$ are both equal to 1,

$$\| S_3 \|^2 = \frac{\int_{C_3} |\nabla(S_3u_1)|^2 dx}{\int_{C_3} |\nabla(S_3S_2u_2)|^2 dx} = 1, \| S_2 \|^2 = \frac{\int_{C_3} |\nabla(S_3u_1)|^2 dx}{\int_{C_3} |\nabla(S_3S_2u_2)|^2 dx} = 1.$$

Next, we have to use Cauchy-Schwartz inequality and Young inequality twice:

$$2\int_{C_3} \nabla u_3 \cdot \nabla(S_3 u_1)dx \leq 2 \parallel \nabla u_3 \parallel_{L^2(C_3)} \parallel \nabla(S_3 u_1) \parallel_{L^2(C_3)}$$
$$\leq 2 \parallel \nabla u_3 \parallel_{L^2(C_3)} \parallel \nabla u_1 \parallel_{L^2(C_1)}$$
$$\leq \eta_1^{-1} \parallel \nabla u_3 \parallel_{L^2(C_3)}^2 + \eta_1 \parallel \nabla u_1 \parallel_{L^2(C_1)}^2$$

$$2\int_{C_3} \nabla u_3 \cdot \nabla(S_3 S_2 u_2)dx \leq 2 \parallel \nabla u_3 \parallel_{L^2(C_3)} \parallel \nabla(S_3 S_2 u_2) \parallel_{L^2(C_3)}$$
$$\leq 2 \parallel \nabla u_3 \parallel_{L^2(C_3)} \parallel \nabla u_2 \parallel_{L^2(C_2)}$$
$$\leq \eta_2^{-1} \parallel \nabla u_3 \parallel_{L^2(C_3)}^2 + \eta_2 \parallel \nabla u_2 \parallel_{L^2(C_2)}^2,$$

and now, we get

$$a(u, T'u) \geq (\sigma_1 + \sigma_3\eta_1)\int_{C_1} |\nabla u_1|^2 dx$$
$$+(\sigma_2 + \sigma_3\eta_2)\int_{C_2} |\nabla u_2|^2 dx$$
$$-\sigma_3(1 - \eta_1^{-1} - \eta_2^{-1})\int_{C_3} |\nabla u_3|^2 dx.$$

In order to make it coercive, the coefficient should be positive:

$$\begin{cases} \sigma_3\eta_1 > -\sigma_1 & \Rightarrow \eta_1 < -\dfrac{\sigma_1}{\sigma_3}, \\ \sigma_3\eta_2 > -\sigma_2 & \Rightarrow \eta_2 < -\dfrac{\sigma_2}{\sigma_3}, \\ 1 - \eta_1^{-1} - \eta_2^{-1} & > 0. \end{cases}$$

Througth the equation set, we get

$$\eta_1^{-1} \succ \frac{\sigma_3}{\sigma_1},$$

$$\eta_2^{-1} \succ \frac{\sigma_3}{\sigma_2},$$

$$1 + \frac{\sigma_3}{\sigma_1} + \frac{\sigma_3}{\sigma_2} > 0.$$

Finally, we obtain

$$\frac{\sigma_1}{\sigma_3} \succ 1,$$

$$\frac{\sigma_3}{\sigma_1} + \frac{\sigma_3}{\sigma_2} > -1.$$

In conclusion, problem (3.6) is well-posed when

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$$\left\{\frac{\sigma_1}{\sigma_3} > -1\right\} \cup \left\{\frac{\sigma_3}{\sigma_1} + \frac{\sigma_3}{\sigma_2} > -1\right\}.$$

5. Conclusion

In conclusion, this paper has demonstrated the well-posedness of sign-changing Poisson-type equations through the use of T-isomorphisms in both circular and three-square configurations. Theorem 3.1 establishes that for the circular configuration, the variational problem is well-posed when the parameter κ satisfies certain conditions, specifically $\kappa \in (-\infty, -\max(\gamma_1, \gamma)) \cup (-1, \max(\gamma_1, \gamma))$. This result underscores the effectiveness of the T-isomorphism in converting non-coercive problems into coercive ones, leading to well-posedness under the given parameter constraints.

Furthermore, Theorem 3.5 extends these findings to the three-square configuration, showing that the problem is well-posed when $\sigma_1\sigma_3 > -1$ or $\sigma_3\sigma_1 + \sigma_3\sigma_2 > -1$, even when σ_3 is negative. This reinforces the flexibility of the T-isomorphism approach in handling complex configurations with sign-changing parameters, confirming the broader applicability of this method to other non-coercive problems.

These results demonstrate that the T-isomorphism technique is a powerful tool for addressing variational formulations with sign-changing parameters, providing new insights into the well-posedness of non-coercive problems across different configurations.

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