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An Overview of the Basic Theory of Probability Theory and its Commercial Application

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Abstract. Probability theory is a key field of mathematics that focus on developing theoretical models and mathematical frameworks to describe and analyze random events or processes. As a part of basic science, probability theory holds a central place in pure mathematics while also serving a crucial function across various domains, including natural science, social science and engineering technology. This paper summarizes the main concepts, basic theories and commercial applications of probability theory. Probability theory encompasses ideas such as random experiment, sample space, event, probability, conditional probability, independence, Bayes' Theorem, random variable, probability distribution, expected value and variance, law of large numbers and central limit theorem. This paper provides an overview of key concepts and theorems from probability theory, along with their practical applications in everyday life, in order to make people deeply recognize the broad and profound contents and research fields of probability theory and its wide application in various fields as an academic tool, which has a great relation with people.

Keywords: probability theory, Probability space, Random variable, Bayes' theorem.

1. Introduction

Probability theory is a field that examines randomness phenomena and their mathematical models. It equips individuals with methods to model and forecast uncertain events by defining probability spaces and random variables. Probability theory is utilized across diverse areas, ranging from basic dice-rolling scenarios to sophisticated financial risk management. The study of probability theory has been very complete recently. In machine learning, especially deep learning, probability theory plays a key role. Researchers have further promoted the development of this field through probabilistic graphical models, Bayesian methods, and generative models (such as generative adversarial networks). In the field of random processes, especially stochastic differential equations and stochastic control, researchers have made important progress in system theory and application. However, there are still many gaps in the study of probability theory that are waiting to be explored. For example, in many application areas, uncertainty quantification is still an important research direction. For example, in climate modeling and risk assessment, how to accurately quantify and propagate uncertainty still faces challenges. This paper explains and reviews key aspects of probability theory and its applications across various specialized fields by citing relevant literature and summarizing it.

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2. Probability space

Probability space is the basic framework of probability theory, which contains three main components: sample space, events, and probability measures. Among them, the sample space Ω denotes the set of all possible outcomes. For example, in the case of rolling a dice, the possible outcomes are represented by the set $\{1, 2, 3, 4, 5, 6\}$. An event refers to any subset of the sample space, which may consist of a single sample point or a collection of several outcomes. An event represents a certain situation or result of interest. For example, in the example of rolling a dice, "rolling an even number" is an event, and the corresponding subset is $\{2,4,6\}$. The probability measure assigns a likelihood to each event contained within the sample space, following a precise mathematical rule:

A. Non-negativity: For any event E, there is $\mathrm{P(E)} \geq 0$

B. Normalization: The total probability across all possible outcomes in the sample space equals 1, that is, P(Ω)=1

C. Additivity: If two events, E1 and E2, are mutually exclusive, their combined probability is simply the addition of the probabilities of each event, as expressed by:

$$\mathrm{P(E_1 \cup E_2) = P(E_1) + P(E_2)\ (if E_1 \cap E_2 = \emptyset)} \quad (1)$$

These elements together constitute a probability space (Ω,F,P), where F represents the collection of events associated with the sample space Ω. Typically, F must meet the conditions of a σ-algebra, which defines a collection of events that adheres to specific properties.

3. Random variables and distribution

A random variable assigns a real number to each outcome within the sample space. Its distribution function describes how probabilities are allocated across different values of the random variable.

3.1. Random variables

A random variable associates each point in the sample space with a real number. More specifically, random variables are categorized into discrete and continuous types.

Discrete random variables: one that can be enumerated in a specific order, with of its values falling within one or more finite or infinite intervals. The distinction between discrete and continuous random variables lies in the nature of their value ranges: discrete random variables are limited to specific, countable values, typically natural numbers, whereas continuous random variables can take on any value within a given range. An example of a discrete random variable is the outcome of rolling a die, where the possible values range from 1 to 6. Discrete random variables are represented by a probability mass function (PMF), which assign a probability to each specific outcome.

Continuous random variables: Its values are continuous, usually within a certain interval. For example, measuring a person's height produces such a variable. These variables are characterized by a probability density function (PDF), whose integral over an interval gives the probability.

3.2. Distribution of random variables

The probability distribution of a random variable outlines the likelihood of it assuming various possible values. Distributions can be represented in two main ways: probability quality function and probability density function.

Probability mass function (PMF) describes the probability that a discrete random variable X take a specific value x. For example, in the case of tossing a fair coin, the change of X being "heads" or "tails" is 0.5.

Probability density function (PDF) describes how the values of a continuous random variable are distributed. The integral of the PDF $\int \mathrm{abxf(x)\,dx}$ over the interval [a,b] gives the likelihood that the variable falls within that range. It's important to note that the PDF itself represents a density rather than a direct probability, and the probability is obtained by integrating the density over the desired range.

Cumulative distribution function (CDF) describes the way distributions are represented for both discrete and continuous random variables. The CDF, denoted as FX(x), shows the probability that a random variable X is less than or equal to a particular value x, written as $FX(x) = P(X \leq x)$. In discrete cases, the CDF takes the form of a step function, while for continuous cases, it forms a smooth curve.

3.3. Numerical characteristics of random variables

Expected value (mean): This is the "center" or "average" value of the random variable. For discrete random variables, the expected value is calculated as i); For continuous random variables, the expected value is calculated as

Expected value method: The concept of expected value is the mathematical expectation of a random variable in probability. The target variable of each scheme is regarded as a discrete random variable, and its value is the profit and loss value corresponding to each action scheme. The advantage of the expected value method is that it considers the problem comprehensively and takes into account various possible scenarios. Disadvantages of the expected value method: The expected value is determined in the future occurrence of several states and the probability of each state of occurrence (probability), but also know the profit and loss value in each state, to calculate the expected value of each scheme. The expected value is a weighted average, the weight is the probability, and the probability is divided into subjective probability and objective (based on historical facts) probability, the results of which are subject to different interpretations [1].

Variance: The average of the square of the difference between each sample value and the average of all sample values.

Covariance and correlation coefficient: Covariance and correlation coefficient are statistical judges of the intensity of the relationship between variables. Covariance describes the degree of correlation between the errors of two variables, and its positive and negative values indicate the consistency or opposites of the variation trend of the variables. The correlation coefficient is the standardized form of covariance, and the value range is between -1 and 1, which more intuitively expresses the degree of linear correlation between variables. A correlation coefficient of 1 indicates a perfect positive correlation, -1 indicates a perfect negative correlation, and 0 indicates no correlation.

4. Main theories in probability theory

4.1. Law of Large Numbers

According to the principle of the Law of Large Numbers, as the sample size grows, the average of the sample gets closer to the population mean. This concept forms the foundation of statistical inference. In the insurance industry, this principle is extensively applied. In fact, it can be argued that without this fundamental concept, the insurance industry as we know it would exist today. Nowadays, more and more families own cars, and may wish to start with car insurance. If a company obtains through data analysis that the probability of a car accident is 1/500, it means that the more the number of insurance policies, the actual probability of a car accident will be closer and closer to this value, so that the insurance company can reasonably formulate insurance rates and payment plans. Thus, the insurance premium and the total amount of claims collected by the insurance company during the insurance period reach a relatively stable state, which can ensure that the insurance company can obtain relatively stable profits. Applying the law of large numbers, chance in individual cases also tends to stabilize in large numbers. For insurance companies, the more people buy insurance, the more stable the company's profits will be, which is why the company puts a lot of energy into the promotion of the reason. Above we only consider a simple situation, for the sake of convenience, the probability of the accident to occur by default to 1 in 500, which is not the case. Determining the probability of this event is the most difficult problem for insurance companies, which tests the ability of insurance companies to collect data [2].

4.2. Conditional probability and Bayes' theorem

Conditional probability describes the probability that one event will happen if another event is known to occur. Bayes' theorem provides a way to update probabilities and is widely used in statistical inference and machine learning.

Definition of conditional probability: The probability of H based on E is defined as

$$P(H) = P(H \& E)/P\ (E) \quad (2)$$

Which provided that both terms of the ratio exist and P(E) >0 Here are some direct corollaries about the definition: a, Probability. PE represents a probability function. b. Logical result.

If E implies H, then $P\ E\ (H) = 1$. c. Be sure.

If $P\ (H) = 1$, then $P\ E\ (H) = 1$. 4. Mix. $p(H) = p(E)pE(H)+p(\sim E)p\sim E(H)$ [3]

One of the most significant concepts in conditional probability is Bayes' theorem, whose importance was initially identifies by the English clergyman Thomas Bayes, and introduced in his renowned work, The Solution of a Problem in the Doctrine of Probability (Bayes 1764), published after his death. Bayes' theorem relates the "positive" probability P E (H) of a hypothesis given a set of data to the "inverse" probability P H (E) of the data given that hypothesis.

$$\text{Bayes theorem:} PE(H) = P(H)/P€P(H)E \quad (3)$$

Statisticians have adopted the term "likelihood" to describe the inverse probability P H (E), indicating how well the assumptions forecast the data based on the background information summarized by probability P. Although Bayes' theorem is mathematically trivial, it proves highly effective for determining conditional probabilities, as inverse probabilities are often simpler to calculate and less biased compared to direct probabilities. Individuals with differing opinions on the prior probabilities of E and H tend to disagree on the significance of E as an indicator of H. However, if they possess any of the following verifiable information, they can agree on the strength of their assumptions about the predicted data: (a) the objective probability of E given H, (b) the frequency of events similar to E if H is true, or (c) the fact that H logically implies E. Scientists frequently design experiments to derive possibilities through one of these "objective" approaches. Bayes' theorem guarantees that any disagreement over the significance of experimental outcomes stems from differing opinions regarding the prior probabilities of H and E.

Once P H (E) and P ~ H (E) are established, the researcher can use Baye's theorem to determine the value of P E(H) without requiring knowledge of the exact probability of.

$$\text{Bayes' Theorem (second form): } PE(H) = [P(H)/P(E)]\ PH(E) \quad (4)$$

Bayes' theorem is highly effective for determining a cause based on observed outcomes, as the likelihood of an outcome given a known cause is often easier to evaluate. For example, doctors often use diagnostic tests with known sensitivity and specificity to screen for diseases with known prevalence. The response rate of the test, known as the "true positive" rate, is the number of times a sick patient tests positive. The specificity, or the "true negative" rate, is the proportion of healthy individuals who test negative. Let H represent a patient having the disease and E represent a positive test result. The sensitivity and specificity are represented by P H (E) and P ~ H (~E), respectively, with the overall disease prevalence in the population denoted as P (H). With this informationabout the disease influences the test result, you can use Bayes' Theorem to determine the probability that a patient has the disease given a positive test result.

5. Application area

The use of probability theory in working world, as the mathematical foundation of statistics, probability theory is used in many real-life activities, including the quantitative analysis of data. Probability theory may be useful in business, banking, insurance, governance, sports, social media, weather reporting, and more.

In business, probability theory is applied to improve decision-making when dealing with uncertainty. It helps make decisions that are goal-driven and data-driven, rather than based on instinct. For example, cheese sales at beach shops depend on two things - good weather and lots of tourists. So, one week from now, if the probability of good weather is 84% and the expected tourist rate is 26%, then the probability of expected sales is 0.84×0.26 = 21.85%. That number alone can improve the boss's decision on how many employees to put on the day. Integrating probability theory into educational curricula has been challenging. Property insurers also use probability theory to determine the likelihood of an insurance claim from a group of customers and set prices. Similarly, life insurance companies use probability to estimate customers' expected lifespan and calculate premiums [4].

At the same time, probability can be used to calculate the expected benefit or cost of various decision schemes. This helps people by comparing the risks and benefits of various decision-making options. We make smarter decisions in uncertain environments. For example, during product development, probabilistic models can be used to predict market acceptance of new products. Probabilistic analysis can compare the benefits and risks of various schemes and choose the optimal solution [5].

6. Conclusion

This paper mainly summarizes the probability theory by making an outline and citing literature. As an important tool to describe and predict random phenomena, probability theory has a wide influence on scientific research and practical application, and its basic theory and method provide a solid foundation for solving uncertainty problems. However, the application of probability theory in engineering and medicine has not been deeply discussed in this paper. Today, as technology advances, probability theory will continue to evolve to support applications in various fields.

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