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Deformation and Maximum Quadrupole Moment for a Single Neutron Star

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Abstract: The emergence of gravitational-wave astronomy has enabled us to unveil events that were previously concealed: compact binary coalescences. Neutron stars have been recognized as significant sources of gravitational radiation, thus triggering a timely exploration of their deformations. Through gravitational waves, scientists are presented with a unique opportunity to study the interiors of neutron stars and deepen the understanding of the equation of state of ultra-dense nuclear matter. In this article, reflects the necessary condition for the generation of gravitational radiation, namely the time-varying quadrupole moment, and calculate the ellipticity resulting from a neutron star's simplistic model. Subsequently, this article examine Ushomirsky's research on the maximum quadrupole moment, in which this term does not show any explicit dependence on mass. Finally, the author makes a comparison between simplistic model and Ushomirsky, Haskell, gives a estimation on ellipticity and maximum quadrupole moment, expecting a potential improvement on the model would be incoporating dynamic terms.,such as accretion and star quakes..

Keywords: gravitational waves, quadrupole moment, neutron star deformation

1. Introduction

It is a well-known fact that with non-axisymmetric deformation, or "mountains," can neutron star produce gravitational waves. The neutron star mountain is widely regarded as a significant source of gravitational radiation. However, current observations are unable to directly detect gravitational radiation signals stimulated by a single neutron star. This article aims to discuss various models that illustrate the concept of a neutron star mountain through theoretical derivation. The idea that the crust of a rotating neutron star could potentially form a "mountain," thereby resulting in the emission of gravitational waves, has attracted substantial attention recently. This interest mainly stems from a previous proposition, which builds earlier theories [1,2]. Theoretically, this article reviews the generation conditions of gravitational waves and discusses neutron star mountain models, ranging from the simple to the more realistic ones. These discussions are based on the research by Ushomirsky [3] and Haskell Additionally, it highlights theoretical computation method of ellipticity caused by neutron star deformation and multipole expansion method [4]. This turns out to be meaningful uncovering the evolution of neutron star and identifying new source for gravitational waves.

2. Generation of gravitational wave

In the linearized theory of general gravity, the spacetime metric can be described as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ $h_{\alpha\beta}$ is the metric perturbation that small enough [5]. In the Lorentz gauge, Einstein's equation is expressed as $h_{\alpha\beta} = -16\pi G T_{\alpha\beta}$ wave operator and can be solved using radiative green's function, if only considering the spatial component, the equation is reduced to $h_{ij} = \frac{4G}{r}\int T_{ij}(t-r, x')d^3x'$, recall that $\partial_\beta T_{\alpha\beta} = 0$,it is easy to figure out that $\int T_{ij}d^3x' = \frac{1}{2}\frac{d^2}{dt^2}\int T_{00}(t-r, x')d^3x'$ and$= \frac{2G}{r}\frac{d^2}{dt^2}\int T_{00}(t-r, x')x_i'x_j'd^3x' = \frac{d^2 I_{ij}}{dt^2}$ is the quadrupole moment tensor and r is the distance between source and field point. Far from the source, this paper have $\Box h_{ij} = 0$ and it is obviously a wave equation in the vacuum and h_{ij} can be expanded as plane waves.

Much of the treatment of gravitational radiation resembles the electromagnetic radiation case. In the electromagnetic theory, a typical radiation is produced by an accelerated electric dipole. However, a mass dipole cannot produce gravitational radiation as the result of the conservation of momentum. From the previous discussion, gravitational radiation requires a changing quadrupole moment. The wave equation $h_{ij} = 0$ can be expanded into the solution$h_{jl} = A_{jl}e^{i(kx-\omega t)}$. Furthermore, to find the solution at trasverse-traceless condition, then $h_{ij}^{TT} = \frac{2G}{r}\frac{d^2 I_l k}{dt^2}(P_{li}P_{kj} - \frac{1}{2}P_{lk}P_{ij})$where $P_{ij} = \delta_{ij} - n_i n_j$. there recognize that P_{ij} is a tensor that can project components into a subspace that is orthogonal to unit vector n Suppose a sphere with radius R, spinning around z axis[6],then the principal moments of inertia of the body in the comoving frame are calculated:$I_1 = \int d^3x'\rho(y^2 + z^2) = I_{yy} + I_{zz}$ $I_2 = \int d^3x'\rho(x^2 + z^2) = I_{xx} + I_{zz}$ $I_3 = \int d^3x'\rho(x^2 + y^2) = I_{xx} + I_{yy}$ Solving the linear equation, the quadrupole moment tensor are given by $I_{xx} = \frac{I_3 + I_2 - I_1}{2}$ $I_{yy} = \frac{I_3 - I_2 + I_1}{2}$ $I_{zz} = \frac{-I_3 + I_2 + I_1}{2}$ apply the tensor transformation principle $I^{jk} = \frac{\partial x^j}{\partial x^{a'}}\frac{\partial x^k}{\partial x^{b'}}I^{a'b'}$, if this paper define the ellipticity parameter $\epsilon = \frac{I_1 - I_2}{I_3}$,then $\frac{d^2}{dt^2}I^{xx} = 2\epsilon\Omega^2 I_3 \cos 2\Omega t$ And $\frac{d^2}{dt^2}I^{yy} = -2\epsilon\Omega^2 I_3 \cos 2\Omega t$ For a approximation of gravitational radiation amplitude, apply the quadrupole formula, results in $h_0 = \frac{4G}{c^4 R}\epsilon I_3 \Omega^2$.2.

As a consequence, spherical symmetric motion cannot produce gravitational radiation, which can be inferred by Birkhoff's theorem. It states that the schwartzschild solution $ds^2 = -(1-\frac{2M}{r})^{-1}dt^2 + (1-2M)^{-1}dr^2 + r^2 d\omega^2$is the only solution of Einstein's equation in the vacuum. The spacetime metric remains static when spherically symmetry is preserved.It means there is no gravitational radiation for a spherically radiating system. Only a deformed sphere would produce gravitational radiation. However, sometimes source with no time-varying quadrupole moment can produce gravitational radiation in the higher order. Let's consider a special case. Suppose there are four equal mass lying on the corner of a square, rotating with a angular velocityω. The initial location of the four corners are labeled as. $(a, 0,0), (0, a, 0), (-a, 0,0), (0, -a, 0)$. Then the quadrupole moment tensor for this system can be calculated. $I_{xx} = mr^2[\cos^2\omega t * 2 + \sin^2\omega t * 2] = 2mr^2 = I_{yy}, r = \sqrt{2}a$ $I_{xy} = I_{yx} = 0$. It shows that there is no time-varying quadrupole moment in the tensor. However, the quadrupole is only a leading term in the expression. Now, to expand h_{ij} in higher order.

Notice that in the previous discussion, the size of the source has been neglected as it is extremely small compared with the distance between the source and the point. Suppose the typical size of the source is a sphere with radius d, the actual integral is $h_{ij}(t, x) = 4G\int d^3x' \frac{T_{\alpha\beta}(t-|x-x'|, x')}{|x-x'|}$ sincer >>

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d, now can expand $|x - x'|$ in a taylor's series.$|x - x'| = r - x * \hat{n} + O(\frac{1}{r^2})$Moreover, in the non-relativistic scenario where the velocity of the source is much smaller than the speed of light, $v_s << c$, the low - velocity expansion method can be utilized. Consequently, the primary component of the spacetime perturbation h_{ij} is the quadrupole term. To obtain the higher order term, a Taylor's expansion of the stress - energy tensor is performed. From Maggiore's result [7], $h^{ij}(t,x) = \frac{4G}{c^4 r}\Lambda_{ij,kl}[S^{kl} + \frac{1}{c}n_m \frac{d}{dt}S^{kl,m} + \frac{1}{2c^2}n_m n_p \frac{d^2}{dt^2}S^{kl,mp} + \ldots]$, where $S^{ij}(t) = \int d^3x T^{ij}(t,x)$, $S^{ij,k}(t) = \int d^3x T^{ij}(t,x)x^k$.

Applying the tensor virial theorem, the first term can be determined to be the quadrupole moments. In the same manner, the latter terms contain octupole and more multi-poles. In the previous example, it was demonstrated that the four - mass system does not produce quadrupole radiation. However, higher - order terms emerge when expanding to hexadecapole terms. Therefore, examining higher - order expansions is essential when addressing radiation.

3. Neutron star mountain models, simplistic and practical

Neutron star is a typical astrophysical source of gravitational waves on account of neutron star is the densest compact source in the universe. Getting a time-varying quadrupole moment requires the deviation from the spherical symmetry, which is entitled neutron star mountain. Firstly ,suppose a simple neutron star mountain model. Suppose a regular sphere, endowed with a mass M, radius r. Then the principal moments of inertia tensor can be easily calculated: $I_1 = I_2 = I_3 = \frac{2}{5}Mr^2$ Suppose there is a mountain with mass m, locating at the spherical coordinate (θ, ϕ) on the surface [6].As a result, this paper will figure out that it can make a change on the moments of inertia.If this paper review the ellipticity formula $\epsilon = \frac{I_1 - I_2}{I_3}$ then the mountain would contribute to the ellipticity.Next, there can omit the change within the denominator, since the deviation in the denominator is negligible Furthermore, try to extend the model to ellipsoid. Suppose a ellipsoid with uniform density ρ,the length of principle axes are a, b, cwhich satisfies equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

To handle the intergration inside the star, they can invoke the ellipsoidal coordinates $x' = ar\sin\theta\cos\phi$ $y' = br\sin\theta\sin\phi$ $z' = cr\cos\theta$r is a quantity ranging from 0 to 1. If construct a small mountain again on the surface of star,locating at the ellipsoidal coordinate at (θ, ϕ)again. Repeating the same procedure as in the sperical case, ignoring the variation in the denominator, $\delta I_1 - \delta I_2 = \delta m r^2 \sin^2\theta(b^2\sin^2\phi - a^2\cos^2\phi)$ the resulting ellipticity is$\epsilon = \frac{5\delta m r^2 \sin^2\theta(b^2\sin^2\phi - a^2\cos^2\phi)}{M(a^2+b^2)}$

From which this paper make use of this elliipticity to estimate typical amplitude for a source. A simplistic model of a neutron star mountain has been inspected, which involves the asymmetry in the distribution of mass while ignoring the deformation of the shape. Actually, a neutron star can have "mountains" on its surface because of the existence of a solid crust. The crusts can be subjected to shear stress, which perturbs the star from axis - symmetry. As the stress increases, the crusts deviate more and eventually break up. Since the deformation of symmetry causes gravitational radiation, the maximum deformation for a neutron star is of interest. Therefore, it is necessary to review the perturbation theory. For convenience, only the perturbation of the fluid variables, density and pressure, will be investigated. There are different approaches to perturbation. On the one hand, those denoted by δ are Eulerian perturbation. On the other hand, those marked by $\triangle$ are Lagrangian perturbation[8]. Eulerian perturbation refers to a change of a fixed point $\delta f = f(x,t) - f_0(x,t)$. On the contrary, Lagrangian perturbation marks a change that the point comoves with the flow $\triangle f = f(x + \xi x, t) -$

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$f_0(x,t)$. And first there need to analyze the problem in Newtonian gravity.the conservation of mass leads to the continuous equation $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0$.

If the momentum conservation is invoked, the Euler equation for a perfect fluid is attained, and that is the paradigm needed to analyze the neutron star mountain. Ushirmirsky [3] first calculated the maximum mountain a neutron star can support. At the beginning, the shear tensor is included and a perturbation in the Euler equation is produced. $-\nabla_i\delta p - \rho\nabla_i\delta\Phi - \delta\rho\nabla_i\Phi + \nabla^j t_{ij} = 0$. In Ushirmirsky's computation, he utilized the Cowling approximation δΦ=0, which neglects potential perturbations.Next,perform the tensor decomposition in spherical harmonics: $t_{ij} = t_{rr}(\nabla_i r\nabla_j r - 1/2e_{ij})Y_{lm} + t_{r\perp}f_{ij} + t_\Lambda(\Lambda_{ij} + 1/2e_{ij}Y_{lm})$ recall the sperical harmonic expansion of the multipole moment,then the perturbation of the quadrupole moment$Q_{22} = -\int_{r0}^{r}\frac{r^3}{d\Phi/dr}[\frac{3}{2}(4-U)t_{rr} + \sqrt{\frac{3}{2}}(8 - 3U + \frac{1}{3}U^2 - \frac{r}{3}\frac{dU}{dr})t_{r\perp} + \frac{1}{3}(6-U)t_\Lambda]dr$ r0 and r are radius that bottom and top of the crust, and this paper have the as- sumption that the shear modulus μ is zero at the bottom and the top,and $U = \frac{d\ln(\frac{d\Phi}{dr})}{d\ln r} + 2$.In order to attain maximum quadrupole, it input von Mises criterion $\sigma \geq \sigma_{max}$ Due to the lack of precise detection and direct observation, neutron star formation can only be conjestured. As one proceed towards the center, neutron star layer may be divided into five pieces, surface, outer crust,inner crust, neutron liquid and finally a core region. The density increase as going from surface to the center and whether the core exists depends on if there is pion condensation, which tends to contract neutron stars.

Since there are to focus on elastic crust on neutron star, several factors which influences shear modulus. Ogata and ichimaru found the associate equation governing the shear modulus[9] $\mu = 0.1194\frac{3}{4\pi}(\frac{1-X_n}{4}n_b)^{\frac{4}{3}}Ze^2$ and $X_n$is the fraction of neutrons outside nuclei, n_b is the baryon density, Z is the proton number and A is the atomic number.From previous research it showed that an accreted crust sustains smaller Q than non-accreted ones [10].However, when incoporating perturbation of the core, with more mass comes greater quadrupole moment.

From Ushomirsky's formula this paper find that there are no explicit terms of the mass of the star involved [3].However, according to Haskell's work, the mass dependence on the maximum quadrupole moment must be considered especially when extending the perturbation to the core[4].The dependence of the maximum quadrupole moment on mass is likely related to the internal structure and distribution of mass within neutron stars. This relationship may influence the extent to which neutron stars deform, thereby affecting their capacity to produce gravitational waves and the characteristics of these waves. This mass dependence can be proposed as providing essential physical parameters crucial for understanding the source model of neutron star mountain deformation including some historical evolutionary information. Now, to compute the maximum mass for obtaining the maximum quadrupole moment. Inside the neutron star, the gravitational pressure must be balanced by other forces; otherwise, the star would collapse into a black hole. Recall the TOV equation [10]. $\frac{dp}{dr} = \frac{(\rho+p)(m+4\pi r^3 p)}{r(r-2m)}$ or a relativistic star, the mass function is defined $\frac{dm(r)}{dr} = 4\pi r^2\rho(r)$ and there need the relationship between pressure and energy density. From the first law of thermodynamics $dU = -pdV + TdS$ when dealing with the neutron star case, the temperature is far below the fermi temperature so that ignoring the entropy term is reasonable. Then it obtain the equation of state p = p(ρ).A simple model assumes that the density is constant, which implies the boundary condition can be treated as $\rho = \rho_0$, $m(r) = \frac{4}{3}\pi r^3\rho_0, r < R$ $\rho = 0$, $m(r) = M, r > R$ integrate the TOV equation, this paper get the solution $p(r) = \rho_0 R\frac{\sqrt{R-2GM} - \sqrt{R^3-2GMr^2}}{\sqrt{R^3-2GMr^2} - 3R\sqrt{R-2GM}}$ it is obvious

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that $p(r)$increases with smaller r. If $GM > \frac{4R}{9}$,the central pressure would exceed infinity, which is not physically allowed. Suppose this paper set standard radius for a neutron star $R = 10km$, then $Mmax = 4R/9G = 3M_{\odot}$.However, this is a simplistic model ignoring the interior structure which determines the equation of state inside the star.If them specify the equation of state $P = k\rho^{\Gamma}$, $\Gamma = 2$,which is a good approximation of a neutron star.By integrating the TOV equation under the equation of state mentioned above,this article reach a maximum mass of around $M = 2M_{\odot}$ Haskell figures out the formula for maximum quadrupole moment under a mass dependence condition[4],when $\rho = 1.6 * 10^{14}$ and $R = 10km$, $Q_{max} = 1.6 * 10^{37}(M/1.4M_{\odot})^{-0.3} g * cm^2$. If there insert the result into the formula will get the maximum quadrupole of$1.44 * 10^{37} g * cm^2$.By the way, there is no lower bound for a neutron star's mass since it will inevitably go through β-decay as a result of low density.However, if the mass is negligible, it will be reasonable enough to ignore the perturbation from the core.

4. Comparison between simplistic model and the two practical models

The first calculation excludes the internal structure and crust strength, focusing on asymmetry on mass distribution. The simplistic model, Ushomirsky's model, and Haskell's model provide increasingly refined approaches to estimating the maximum quadrupole moment Q_{max} and ellipticity ϵ_{max} of a neutron star. In this model, the quadrupole moment is directly proportional to the ellipticity via $Q_{max} \sim \epsilon I_0$, where I0 is the moment of inertia of the neutron star. Using some typical neutron star parameters $M = 1.4M_{\odot}, R = 10km, m_{mountain} = 10^{-4}M_{\odot}$, the simplistic model predicts $Q_{max} \sim 10^{41} g * cm^2, \varepsilon_{max} \sim 10^{-4}$. Ushomirsky's model introduces the crust's elasticity and breaking strain into the calculations, significantly refining the predictions, and can estimate $Q_{max} \sim 10^{38} g * cm^2, \varepsilon_{max} \sim 10^{-7}$.Haskell's model further extends Ushomirsky's approach by incorporating the mass dependence of the quadrupole moment, from his model, $\varepsilon_{max} \sim 10^{-8}, Q_{max} \sim 10^{37} g * cm^2$ .The simplistic model assumes a large-scale deformation caused by a mass asymmetry (a "mountain") without considering physical constraints like the crust's elasticity.These high estimates arise because the simplistic model neglects any constraints on how much stress the crust can sustain. Besides, in the simplistic model, the article simply "adding a mountain" on the surface of the star, ignoring the phase transition of the structure, and assume a static distribution of the mass, which are sources of error.

5. Conclusion

This article has examined the prerequisite for the production of gravitational radiation, which requires a varying quadrupole moment. Nonetheless, the absence of a quadrupole term does not guarantee the nonexistence of radiation of higher-order radiation. Since deformed neutron stars are significant sources of gravitational waves, this article has analyzed a simplified model of a neutron star focusing on asymmetry on the mass distribution, and calculated ellipticity of an ellipsoid model of a neutron star, providing a means to estimate the amplitude of the gravitational wave caused by the mountain. This article has also estimated the maximum quadrupole moment under mass dependence. The mass dependence of the maximum quadrupole moment may be related to the internal structure and mass distribution of the neutron star. This mass dependence could affect the star's deformability, thereby influencing the likelihood and characteristics of gravitational wave generation. Finally, this article compares the simplistic model with the complex model derived by Ushomirsky and Haskell, figured out the overestimation may come from ignoring of the stress and phase transition inside the neutron star. Since there are many refinement can be done for the simplistic ellipsoidal model, the next step

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will be adding temporal and dynamic effects, transient effects like accretion, starquakes, or rotational instabilities can influence the quadrupole moment over time.

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