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A Mathematical Approach to Pipa String Vibration and Energy Damping

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Abstract: The role of stringed instruments in traditional music is of significant importance, with their vibrational characteristics exerting a direct effect on sound production and timbre alterations. This paper aims to mathematically analyze the vibration of a stringed instrument, such as the pipa, through the microelement method and the Fourier transform, to explore its sound generation mechanism and energy loss. The partial differential equations of the string vibration are derived in the ideal state and transformed into solvable ordinary differential equations using Fourier transform to obtain the analytical solutions for the displacement, velocity and acceleration of the string. In addition, the kinetic and potential energies of the string are investigated and an expression for the total energy in the ideal state is derived, thus demonstrating its periodic variation with time. In order to investigate the effects of air resistance and friction on the energy decay, the damping coefficient is further considered. The results reveal that under ideal conditions, the string's energy fluctuates periodically, whereas with damping, it decays exponentially, thus affecting sound continuity and timbre stability.

Keywords: String Vibration, Mathematical Model, Fourier Transform, Energy Dissipation

1. Introduction

The timbre and sound quality of the pipa are primarily determined by the vibration characteristics of its strings. Factors like string tension, mass density, and vibration modes directly influence sound generation and propagation [1]. The study of string vibrations typically relies on mathematical models, and commonly used analytical tools include Lagrange's equation, fluctuation equation and Fourier transform. Fourier transform, in particular, allows for the conversion of time-domain problems into frequency-domain analysis, simplifying the study of vibration characteristics and energy variations. Despite extensive research on the fundamental theories of string vibration, most studies focus on idealized models and overlook the energy dissipation effects present in real-world scenarios. Energy dissipation in string vibrations is primarily caused by damping effects, such as air resistance and friction, which are important in practical performance. The paper seeks to develop a mathematical model to investigate the vibration characteristics of pipa strings, analyze the energy dissipation process, and investigate the energy variation patterns in ideal and damped conditions. Through the infinitesimal element method and Fourier transform method, the partial differential equations of string vibration are derived and the mathematical model of vibration is established. In addition, damping factors are considered to analyze the energy dissipation process. This study provides a theoretical

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basis for optimizing the timbre of the pipa and other stringed instruments, as well as a new perspective on the vibration analysis of musical instruments.

2. Physical and Mathematical Modeling of String Vibrations

2.1. Fundamental Physical Mechanisms of String Vibration

The vibration of a string is induced by its tension and elastic restoring force. When the string is tightened and fixed at both ends, it is in a state of tension, which generates a restoring force that resists any displacement from its equilibrium position. This restoring force is fundamental to string vibration, ensuring that the string returns to its original position when disturbed by an external force. When the string is plucked or struck, it is displaced by the external force, and the elastic restoring force pulls it back to the equilibrium position, initiating vibration [1]. The vibration of the string is not just the motion of the string itself; it also interacts with the surrounding air. The periodic motion of the string pushes air molecules, generating sound waves [2]. These sound waves propagate through the air and are eventually amplified by the instrument's resonating body. The role of the resonator is to enhance the amplitude of sound waves at specific frequencies, concentrating their energy to produce a louder sound. The resonator works in tandem with the string vibration to amplify the sound waves produced by the string, thereby improving both volume and sound quality. Through mathematical modeling, string vibration can be described using the wave equation, taking into account factors such as string tension, mass density, and vibration modes.

2.2. Derivation of the Equation of Vibration

In the ideal string vibration model, the tension force plays a key role as the elastic restoring force, while the string's own stiffness is relatively small and can be neglected in comparison to the tension. Therefore, the vibration process of the string can be considered a highly sensitive and intuitive wave process. And the vibration of the string is considered to occur solely in the vertical direction, with no displacement in the horizontal direction, as seen in Figure 1. In this study, mathematics, physics, and music work together to simplify structural models in the analysis of the acoustic characteristics of instrument vibrations [3]. The strings vibrate up and down because they are out of balance. If the x-axis is the string's direction and the u-axis is its vibration direction, the string's vibration at different times and positions can be described as a function $u(x,t)$. As seen in Figure 1, the string vibrates only in the vertical direction with no displacement in the horizontal direction.

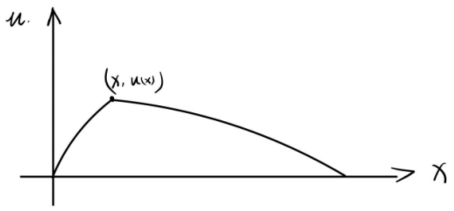

Figure 1: The Model of String Vibration

A small segment of the string is considered, and it is assumed that this segment can be regarded as a point mass [4]. As shown in Figure 2, let the mass of this small segment be m, and analyze the forces acting on the segment between points P and Q. The tensions at the two ends are F_1 and F_2, with angles α and β relative to the horizontal plane, respectively.

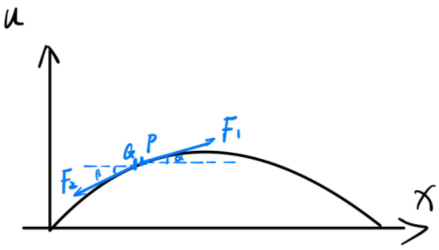

Figure 2: The Physical Model of String Vibration

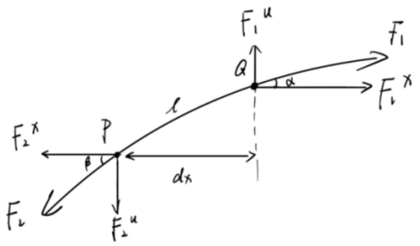

Figure 3: The Mathematical Model of String Vibration

Since no motion is assumed in the horizontal direction, the net horizontal component of the force is zero [5]. As a result, the force can be analyzed by decomposing them into horizontal and vertical components. For F_1 and F_2, they are decomposed into horizontal and vertical components, denoted as F_1^x and F_1^u, where x and u represent the directions of F_1 along the horizontal and vertical axes, respectively. Similarly, F_2 is decomposed into F_2^x and F_2^u. Based on this assumption, the net force in the vertical direction causes the acceleration of the segment of the string. Assuming the linear mass density of the string is μ, which is the mass per unit length, the corresponding mechanical relationships can be used to derive the equation of motion for the string's vibration [6]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

where c is the wave speed, which is related to the tension and linear mass density of the string. This equation describes the longitudinal vibration process of the string. In the derivation process, it is assumed that the vibration is linear, and the initial and boundary conditions of the string are as follows:

$$u\,(x,0) = 0, \frac{\partial u(x,t)}{\partial t}\bigg|_{t=0} = v_m \cos(kx)\ (0 \le x \le L) \tag{2}$$

where v_m is the maximum initial velocity, and k is the wave number of the string. The string's vibration satisfies the boundary conditions:

$$u(0,t) = u(L,t) = 0 \tag{3}$$

which means both ends of the string are fixed. By using the method of separation of variables, assume the solution has the form [7]:

$$u(x,t) = X(x)T(t) \tag{4}$$

where $X(x)$ and $T(t)$ are functions of space and time, respectively. Based on the boundary conditions, the solution for $X(x)$ is a sine function:

$$X_n(x) = sin(\frac{n\pi x}{L}) \tag{5}$$

Substituting this into the wave equation gives the solution for the time part:

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$$T_n(t) = A_n \cos(\omega_n t) + b_n \sin(\omega_n t) \quad (6)$$

where $\omega_n = \frac{n\pi c}{L}$ is the natural frequency for each mode of vibration. Finally, the solution for the vibration of the string can be expressed as:

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \sin(\frac{n\pi x}{L}) \quad (7)$$

where A_n and B_n are constants determined by the initial conditions.

2.3. Applications of the Fourier Transform to Restringing Vibrations

An essential technique for periodic signal analysis, the Fourier transform helps simplify vibration problems in the frequency domain from their more complicated time domain counterparts. The displacement of the string in a string vibration problem is often a periodic function, which may be represented as a superposition of frequency components through the Fourier series. To simplify the solution process, particularly for nonlinear systems, the string vibration equation can be converted from the time domain to the frequency domain after performing the Fourier transform.

Assume the displacement *u(x,t)* of the string is a periodic function. In the time domain, it can be represented as a Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} \hat{u}_n(x)\, e^{in\omega_0 t} \quad (8)$$

where $\hat{u}_n(x)$ is the Fourier coefficient for the *n-th* harmonic, denoting the frequency component at position x, $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency, and T is the period. The integer n represents different harmonic frequencies.

By applying the Fourier transform, the original time-domain vibration equation is converted into a set of independent equations for each frequency component in the frequency domain. This enables the analysis of each frequency component separately, which is especially useful when addressing nonlinear string vibration systems. Nonlinear effects often lead to interactions between frequency components, and the Fourier transform effectively translates these interactions into corresponding terms in the frequency domain, simplifying the analysis.

In the frequency domain, the string vibration equation is expressed as:

$$\hat{u}_n(x,t) = \mathcal{F}\{u(x,t)\} \quad (9)$$

where $\hat{u}_n(x,t)$ is the Fourier transform of *u(x,t)* in the frequency domain, and $\mathcal{F}$ denotes the Fourier transform operation.

In string vibration systems with nonlinear stiffness or damping, the Fourier transform facilitates the decomposition of the system's behavior into distinct frequency components, thereby elucidating the contributions of each component, especially in the presence of significant high-order harmonics and frequency mixing effects. The Fourier transform provides an effective means of analyzing these effects. For example, in nonlinear string vibrations, the system's nonlinear behavior can lead to interactions between frequency components. Fourier transform allows these nonlinear interactions to be represented in the frequency domain, making the analysis more straightforward. In nonlinear systems, the cross terms in the equations may cause originally independent frequency components to interact with each other, and the Fourier transform provides an intuitive way to analyze this process. Therefore, by transforming the string vibration problem into the frequency domain using the Fourier transform, the solution process is simplified, and it provides a clearer understanding of the interactions between frequency components, especially when nonlinear effects are involved.

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3. Energy Analysis: Vibration under Ideal and Damped Conditions

3.1. Energy Calculations in the Ideal State

In the ideal string vibration model, the total energy is composed of kinetic and potential energy, and remains conserved in the absence of external damping. The kinetic energy of the string arises from the velocity of each point mass, and is given by:

$$E_{kinetic} = \int_0^L \frac{1}{2}\mu\left(\frac{\partial u}{\partial t}\right)^2 dx \tag{10}$$

where μ represents the linear mass density of the string, and $\frac{\partial u}{\partial t}$ is the time derivative of the displacement, representing the velocity of each point. And the kinetic energy is proportional to the velocity and changes periodically with time. The potential energy of the string comes from the tension and the deformation of the string. It can be expressed as:

$$E_{potential} = \int_0^L \frac{1}{2}T\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{11}$$

where T denotes the tension in the string, and $\frac{\partial u}{\partial x}$ is the spatial derivative of the displacement, representing the deformation of the string.

During the vibration process, kinetic and potential energy are periodically exchanged. Due to the orthogonality of sine and cosine functions, the cross terms between them can be neglected. The total energy of the string vibration system is the sum of the kinetic and potential energies:

$$E_{total} = \int_0^L \left(\frac{1}{2}\mu\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial u}{\partial x}\right)^2\right) dx \tag{12}$$

In the ideal case, the total energy is conserved. However, in real systems, damping effects lead to gradual energy loss. In nonlinear string vibration systems, nonlinear stiffness or damping can cause interactions between different frequency components. Tools like Fourier transforms can effectively convert the system from the time domain to the frequency domain, hence simplifying the analysis, especially when frequency mixing and higher-order harmonics are significant. Through frequency domain analysis, nonlinear effects can be represented as interactions between different frequency components, providing a clearer understanding of the behavior of the string vibration system.

3.2. Energy Decay When Damping is Considered

In the presence of damping, the total energy of a mechanical system, including kinetic and potential energy, decreases over time. However, in the real world, it is impossible for the total energy to stay the same because of the air resistance and friction. The relationship between the object's speed and the damping force is directly related to the properties of these resistive forces. In a simple damped system, such as a mass-spring system or a vibrating string, the damping force $F_{damping}$ is often modeled as being proportional to the velocity of the moving object. This can be expressed as:

$$F_{damping} = -\gamma v(t) \tag{13}$$

where γ means the damping coefficient, which is a measure of the resistance experienced by the system. And the negative sign signifies that the damping force counteracts the direction of motion, thereby diminishing the amplitude of oscillation. In this circumstances, the damping force is:

$$F_{damping} = -2\gamma \frac{\partial u(x,t)}{\partial t} \tag{14}$$

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The rate of energy loss is a useful metric for describing the dynamic energy state of a system. In a damped system, the rate of energy decay is directly proportional to both the damping coefficient and the current energy of the system.

$$\frac{dE(t)}{dt} = -2\gamma E(t) \tag{15}$$

By separating variables, the equation becomes:

$$\begin{gathered} \frac{1}{E(t)} dE = -2\gamma dt \\ \int \frac{1}{E(t)} dE = \int -2\gamma dt \\ \ln E(t) = -2\gamma t + C \end{gathered} \tag{16}$$

According to $E(0) = E_{total}$, the following can be obtained:

$$\begin{gathered} C = \ln E_{total} \\ E(t) = E_{total} e^{-2\gamma t} \end{gathered} \tag{17}$$

As such, the following can be obtained:

$$E(t) = e^{-2\gamma t} \sum_{n=1}^{\infty} \frac{FL}{4} \frac{n^2 \pi^2}{L^2} \left[\frac{4 v_m d L^2}{n\pi^2 (L^2 - n^2 d^2)} \sqrt{\frac{\rho}{F}} \sin(\frac{n\pi x_0}{L}) \cos(\frac{n\pi d}{2L}) \right]^2 \tag{18}$$

The total energy decays exponentially with time, and the rate of decay is determined by the damping coefficient γ. Actually, in the real life, the resistance exerted by air, which is proportional to the velocity. Therefore, the existence of resistance should be considered. Air resistance is caused by the collision between air molecules and the surface of an object. Plucking the lute string causes the air at each location to vibrate at a high frequency. Air acts as a resistive force against the string's vibration, causing its kinetic energy to be transformed into heat energy. Thus, the string's amplitude decreases, and the vibration eventually comes to a halt. For example, when an object is shaken in water, the resistance from the water causes the object to come to a stop, similar to how air resistance dissipates the vibrational energy of the string.

4. Limitations of the Model and Future Improvements

In practical string vibration systems, the vibration of a string is often influenced by many nonlinear factors, such as large displacements of the string, changes in material properties, and the impact of the external environment. These factors may change the string vibration from linear to non-linear, impacting model accuracy [9]. Therefore, future research can improve the accuracy of the model by introducing these nonlinear effects.

4.1. Introduction of Nonlinear Effects

In existing models, string vibration is usually assumed to be a linear system, i.e., the string tension is assumed to be constant and the string displacement is small. However, in practice, especially when the displacement of the string is large, the tension of the string changes with the degree of bending, which makes the vibration behavior of the string more complicated. To illustrate, when the string is subjected to a substantial external force, the tensions at the two ends of the string become unequal, and the string's vibration is characterized by nonlinearity. Future research might add tension and displacement to the model and examine how tension variation affects vibration. This improvement helps describe the vibration behavior of strings under large deformation. In practical applications, the

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parameters of the model can be adjusted gradually through numerical simulations or experimental data to make it more relevant to the actual situation. For example, the tension change rule of the string at large displacements can be measured experimentally, and then these data can be introduced to optimize the vibration equations and make the model more predictive.

4.2. Improvement of the Damping Model

At present, most analyses assume that the damping force of a string is linear, i.e., proportional to the velocity. In practice, the damping effect of a string may vary with increasing amplitude, especially when vibrating at large amplitudes, and the change in the damping force may no longer be a simple linear relationship. When the string vibrates with a large amplitude, the change in the damping force may be proportional to the square of the velocity or related to other factors. Thus, future models may add a nonlinear damping element to better characterize energy loss at large amplitudes. This improvement can be verified with experimental data. For example, the damping model can be further optimized by measuring the energy decay rate for vibrations of different amplitudes, which will make the model more accurate in practical applications, especially in scenarios involving large amplitude vibrations or prolonged vibrations.

4.3. Impact of Material Properties on String Vibration

Strings of different materials have a significant effect on vibration. Factors like the density, Young’s modulus, and damping coefficient of the material directly affect the frequency of vibration, wave speed, and decay rate of the string. For example, a steel wire string is heavier and has higher tension than a nylon string, thus producing a higher vibration frequency along with a slower decay, while a nylon string may have a lower vibration frequency and higher damping, resulting in the vibration disappearing more quickly. In order to improve the practicability of the model, future studies can include material-specific parameters in the model to consider the effects of different materials on the vibration characteristics of strings. For example, by experimentally measuring the density and Young’s modulus of strings of different materials, the wave velocity and vibration frequency of the strings can be calculated and these parameters can be applied to the vibration equations. In this way, the vibration characteristics of strings of different materials can be described more accurately and the predictive ability of the model can be improved. In addition, the vibration modes and frequency responses of specific materials need to be analyzed in detail in order to optimize the model and improve its application in practical scenarios.

5. Conclusion

The paper explored the vibration characteristics of pipa strings, energy changes and damping effects on the sound quality by establishing a mathematical model of pipa string vibration. And the results demonstrated that under ideal conditions, the energy of pipa strings varies periodically, while when the damping effect is considered, the energy of the strings decays exponentially, greatly affecting the continuity of the sound and the stability of the timbre. The Fourier transform method simplifies the study of vibration characteristics and energy variations by transforming complex problems in the time domain into frequency domain analysis. It has been shown that damping effects in string vibration, such as air resistance and friction, play a key role in sound quality in actual performance. These effects affect the decay rate of the sound, determining the continuity and stability of the lute's timbre. Thus, an in-depth understanding of the effects of damping on string vibration is important for optimizing the sound quality of the pipa and other stringed instruments.

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References

[1] Rossing, T.D. (2010) The science of string instruments. Springer, New York.
[2] Smith, J.O. (1992) Physical Modeling Using Digital Waveguides. Computer Music Journal, 16: 74-91.
[3] Chaigne, A. and Askenfelt, A. (1994) Numerical simulations of piano strings. I. A physical model for a struck string using finite difference methods. J. Acoust. Soc. Am., 95: 1112-1118.
[4] Desvages, C. (2018). Physical modelling of the bowed string and applications to sound synthesis.
[5] Evans, L.C. (2022) Partial Differential Equations: Second Edition. University of California, Berkeley, Berkeley, CA
[6] Myint-U, T. (1973) Partial differential equations of mathematical physics. American Elsevier Pub. Co., New York.
[7] Narasimha, R. (1968) Non-linear vibration of an elastic string. Journal of Sound and Vibration, 8(1): 134-146.
[8] Amabili, M. (2018) Nonlinear damping in large-amplitude vibrations: modelling and experiments. Nonlinear Dynamics, 93(1): 5-18.
[9] Bernal, D. (1994) Viscous Damping in Inelastic Structural Response. Journal of Structural Engineering-asce, 120: 1240-1254.