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Soliton Solutions of an Integrable Coupled Discrete Nonlocal Nonlinear Schrödinger Equation

Mengqi Lv

College of Science, University of Shanghai for Science and Technology, Shanghai, China
18555585025@163.com

Abstract: This study employs Hirota's bilinear method to derive exact solutions for the coupled discrete non-local nonlinear Schrödinger (NLS) equation. The equation under investigation is derived from the non-local reduction of the coupled discrete nonlinear NLS equation, which arises in various physical contexts such as nonlinear optics and Bose-Einstein condensates. Exact solutions of coupled discrete non-local NLS equations are obtained, including bright-bright one-soliton solutions, two-soliton solutions, and dark-dark soliton solutions. For the dark-dark soliton solution, the construction of the solution and the bilinear expansion are derived from the continuous system, but the continuous system solved in this way yields a breathing solution, however, in this coupled discrete non-local NLS equation, under specific parameters, we obtain coupled dark-dark soliton waves. In addition, periodic solutions, singular solutions and double spatial period solutions are obtained by taking different parameters. The soliton dynamics are visualized using mathematical software, providing insights into their behavior and interactions. This work enhances the understanding of soliton solutions in discrete non-local systems and provides a practical approach for analyzing similar nonlinear wave phenomena.

Keywords: Bilinear method, Coupled discrete nonlocal Schrödinger equation, Exact solutions.

1. Introduction

The nonlinear Schrödinger (NLS) equation

$$iq_t(x,t) = q_{xx}(x,t) \pm 2|q(x,t)|^2 q(x,t), \tag{1}$$

which is an important model in nonlinear optics, providing good observation conditions for detecting PT symmetry theory [1-4]. The nonlinear Schrödinger (NLS) equation is widely applicable in physics, such as plasma physics [5-7], deep water waves [8-9], etc. Ablowitz and Musslimani proposed a non-local NLS equation, which is a new PT symmetric equation [10].

$$iq_t(x,t) = q_{xx}(x,t) \pm 2q^2(x,t)q^*(-x,t), \tag{2}$$

Equation (2) is Lax integrable, in fact, the AKNS system, under the new non-local symmetric reduction $q(x,t) = q^*(-x,t)$, gives rise to Equation (2).

Ablowitz and Musslimani also studied the following integrable nonlocal discrete nonlinear Schrödinger equation [11].

$$i\frac{du_n}{dt} = u_{n+1} - 2u_n + u_{n-1} \pm u_n u^*_{-n}(u_{n+1} + u_{n-1}), \tag{3}$$

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where u_n is a time-dependent function of the integer n. Equation (3) has Lax pairs and infinite conservation law, therefore it is an integrable system. In reference [11], a discrete breathing soliton solution was obtained by establishing IST method of the decaying data. In [12], the N-soliton solution of integrable non-local discrete focused nonlinear non-linear Schrödinger (dNLS+) equation is constructed by bilinear method, and the asymptotic analysis of the double-soliton solution is given.

A generalization of the two-component discrete NLS [13], namely,

$$\begin{cases} i\dot{u}_n + (u_{n+1}+u_{n-1})(1+2\delta_1|u_n|^2+2\delta_2|v_n|^2) - 2u_n = 0, \\ i\dot{v}_n + (v_{n+1}+v_{n-1})(1+2\delta_1|u_n|^2+2\delta_2|v_n|^2) - 2v_n = 0. \end{cases} \tag{4}$$

where the superscript $*$ means the complex conjugate, $\delta_j = \pm 1, j = 1,2$, Equation (4) holds significant mathematical and physical relevance. This system was first solved using the IST method in references [14][15]. More recently, a general multi-soliton solution expressed in terms of Pfaffians was derived in [16], where bright soliton solutions emerge in the focusing-focusing regime ($\delta_1 = \delta_2 = 1$), while dark soliton solutions appear in the defocusing-defocusing case ($\delta_1 = \delta_2 = -1$). In [13], the Pfaffian form of the general bright dark soliton solution for the integrable semi discrete vector NLS equation was constructed using Hirota's bilinear method, and the bright-dark one-soliton solutions and two-soliton solutions of the two-component semi-discrete NLS equation were given. For Equation (4), with the reduction $u_n^*(n,t) \to u_{-n}^*(-n,t)$, $v_n^*(n,t) \to v_{-n}^*(-n,t)$, it can be obtained from the two-component non-local discrete NLS equation:

$$\begin{cases} i\dot{u}_n + (u_{n+1}+u_{n-1})(1+2\delta_1 u_n u_{-n}^*+2\delta_2 v_n v_{-n}^*) - 2u_n = 0, \\ i\dot{v}_n + (v_{n+1}+v_{n-1})(1+2\delta_1 u_n u_{-n}^*+2\delta_2 v_n v_{-n}^*) - 2v_n = 0. \end{cases} \tag{5}$$

In this paper, we construct the bilinear form of the Equation (5) using the Hirota method, derive its soliton solutions, and present their dynamical behavior through numerical simulations.

2. The interaction of bright-bright soliton solutions

In this chapter, we seek bright-bright soliton solutions of the Equation (5) by bilinear methods. To derive the bilinear form, we perform dependent variable transformation on Equation (5):

$$u_n = \frac{g_n}{f_n}, v_n = \frac{h_n}{f_n} \tag{6}$$

where g_n, h_n, f_n are complex functions, the corresponding bilinear equations for Equation (5) is:

$$\begin{gathered} (iD_t + 2(\cosh D_n - 1))g_n \cdot f_n = 0, \\ (iD_t + 2(\cosh D_n - 1))h_n \cdot f_n = 0, \\ f_{-n}^*(\cosh D_n - 1)f_n \cdot f_n = 2\delta_1 g_n g_{-n}^* f_n + 2\delta_2 h_n h_{-n}^* f_n. \end{gathered} \tag{7}$$

where D is the bilinear operator [17].

$$\begin{gathered} D_t^l D_n^m f_n(t) \cdot g_n(t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l \left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)^m f_n(t) g_{n'}(t)|_{t=t',n=n'}, \\ exp(\delta D_n) f_n \cdot g_n = f_{n+\delta} g_{n-\delta}. \end{gathered} \tag{8}$$

2.1. Bright-bright one-soliton solution

We will first focus on studying the one-soliton solutions of Equation (5). The following power series expansion is applied to f_n, g_n and h_n.

$$\begin{aligned} f_n &= 1 + f_n^{(2)}\varepsilon^2 + f_n^{(4)}\varepsilon^4 + f_n^{(6)}\varepsilon^6 + \cdots, \\ g_n &= g_n^{(1)}\varepsilon + g_n^{(3)}\varepsilon^3 + g_n^{(5)}\varepsilon^5 + \cdots, \end{aligned}$$

DOI: 10.54254/2753-8818/106/2025.22571

$$h_n = h_n^{(1)}\varepsilon + h_n^{(3)}\varepsilon^3 + h_n^{(5)}\varepsilon^5 + \cdots, \tag{9}$$

in which ε denotes a sufficiently small parameter. we cut off expression (9) to $f_n = 1 + f_n^{(2)}\varepsilon^2$, $g_n = g_n^{(1)}\varepsilon$, $h_n = h_n^{(1)}\varepsilon$, these expressions are then substituted into the bilinear formulation (7). By comparing terms order by order in powers of ε, we obtain the following system of equations:

$$A_1 g_n^{(1)} \cdot 1 = 0,\ A_1 h_n^{(1)} \cdot 1 = 0, \tag{10}$$

$$A_2(1 \cdot f_n^{(2)} + f_n^{(2)} \cdot 1) = 2\delta_1 g_n^{(1)} g_{-n}^{(1)*} + 2\delta_2 h_n^{(1)} h_{-n}^{(1)*}, \tag{11}$$

$$A_1 g_n^{(1)} f_n^{(2)} = 0, A_1 h_n^{(1)} f_n^{(2)} = 0, \tag{12}$$

$$A_2 f_n^{(2)} \cdot f_n^{(2)} + f_{-n}^{(2)*} A_2(1 \cdot f_n^{(2)} + f_n^{(2)} \cdot 1) = 2\delta_1 g_n^{(1)} g_{-n}^{(1)*} f_n^{(2)} + 2\delta_2 h_n^{(1)} h_{-n}^{(1)*} f_n^{(2)}, \tag{13}$$

$$f_{-n}^{(2)*} A_2 f_n^{(2)} \cdot f_n^{(2)} = 0, \tag{14}$$

where the bilinear operators A_1 and $A_2$are given by

$$\begin{aligned} A_1 &= iD_t + 2(\cosh D_n - 1) \\ A_2 &= \cosh D_n - 1 \end{aligned} \tag{15}$$

Assuming $g_n^{(1)} = \alpha e^{\xi}$, $h_n^{(1)} = \beta e^{\xi}$, $f_n^{(2)} = \theta e^{\xi + \xi_{-n}^*}$ with $\xi = \kappa n + \omega t$, $\xi_{-n}^* = -\kappa^* n + \omega^* t$, and α, β are any complex parameters. Equation (10) and (11) yield the dispersion relation $\omega = 2i(\cosh\kappa - 1)$ and the value of $\theta = \frac{\delta_1|\alpha|^2 + \delta_2|\beta|^2}{2} sinh^{-2}\frac{\kappa - \kappa^*}{2}$. Besides, the remaining equations are inherently fulfilled without additional constraints, the one-soliton solution for Equation (5) can be expressed in the following form:

$$\begin{cases} u_n = \frac{\varepsilon\alpha e^{\kappa n + \omega t}}{1 + \varepsilon^2 \theta e^{(\kappa - \kappa *)n + (\omega + \omega *)t}} \\ v_n = \frac{\varepsilon\beta e^{\kappa n + \omega t}}{1 + \varepsilon^2 \theta e^{(\kappa - \kappa *)n + (\omega + \omega *)t}} \end{cases} \tag{16}$$

Setting $\kappa = a + ib(b \neq 0)$, Equation (16) becomes

$$\begin{cases} u_n = \frac{\varepsilon\alpha e^{an - 2t \sinh a \sin b} e^{i(bn + 2t(\cosh a \cos b - 1))}}{1 - \varepsilon^2 \frac{(\delta_1|\alpha|^2 + \delta_2|\beta|^2)\csc^2 b}{2} e^{2ibn - 4t \sinh a \sin b}} \\ v_n = \frac{\varepsilon\beta e^{an - 2t \sinh a \sin b} e^{i(bn + 2t(\cosh a \cos b - 1))}}{1 - \varepsilon^2 \frac{(\delta_1|\alpha|^2 + \delta_2|\beta|^2)\csc^2 b}{2} e^{2ibn - 4t \sinh a \sin b}} \end{cases} \tag{17}$$

Setting $\varepsilon = 1$, Equation (16) can be written as

$$\begin{cases} u_n = \frac{\alpha}{\sqrt{2(\delta_1|\alpha|^2 + \delta_2|\beta|^2)}} e^{\frac{\kappa + \kappa *}{2}n + \frac{\omega - \omega *}{2}t} sech(\frac{\kappa - \kappa *}{2}n + \frac{\omega + \omega *}{2}t - \phi) \sinh(\frac{\kappa - \kappa *}{2}), \\ v_n = \frac{\beta}{\sqrt{2(\delta_1|\alpha|^2 + \delta_2|\beta|^2)}} e^{\frac{\kappa + \kappa *}{2}n + \frac{\omega - \omega *}{2}t} sech(\frac{\kappa - \kappa *}{2}n + \frac{\omega + \omega *}{2}t - \phi) \sinh(\frac{\kappa - \kappa *}{2}). \end{cases} \tag{18}$$

in which $\phi = \ln(\sqrt{\frac{2}{\delta_1|\alpha|^2 + \delta_2|\beta|^2}} \sinh\frac{|\kappa - \kappa^*|}{2})$. As $a \neq 0$, and $b \neq l\pi$, $l \in Z$, the singular point of one-soliton occurs at $(n, t) = (\frac{k\pi}{b}, \frac{-4}{sinha\, sinb} \ln\frac{2sin^2 b}{\varepsilon^2(\delta_1|\alpha|^2 + \delta_2|\beta|^2)})$, $k \in Z$. When $\varepsilon = 1$ and $\kappa = a + ib, (b \neq 0)$, we can get:

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$$\begin{cases} |u_n| = \dfrac{2|\alpha| e^{an-2t\sinh a \sin b}}{\sqrt{4-4e^{-4t\sin b \sinh a}\cos(2bn)\csc^2 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)+e^{-8t\sin b\sinh a}\csc^4 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)^2}}, \\ |v_n| = \dfrac{2|\beta| e^{an-2t\sinh a \sin b}}{\sqrt{4-4e^{-4t\sin b \sinh a}\cos(2bn)\csc^2 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)+e^{-8t\sin b\sinh a}\csc^4 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)^2}}. \end{cases} \tag{19}$$

Specifically, when $a = 0$, Equation (19) can be expressed in the following form:

$$\begin{cases} |u_n| = \dfrac{2|\alpha|}{\sqrt{4-4\cos(2bn)\csc^2 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)+\csc^4 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)^2}}, \\ |v_n| = \dfrac{2|\beta|}{\sqrt{4-4\cos(2bn)\csc^2 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)+\csc^4 b(\delta_1|\alpha|^2+\delta_2|\beta|^2)^2}}. \end{cases} \tag{20}$$

with the period $M = \frac{\pi}{b}$. Using Mathematica software, the soliton diagrams are given as follows. In Figure 1, using parameter $\kappa = 0.3i$, $\delta_1 = \delta_2 = 1, \alpha = 1, \beta = 0.7$, we can obtain that one-soliton solutions are periodic solutions. In Figure 2, using parameter $\kappa = 0.2 + 0.8i$, $\delta_1 = \delta_2 = 1, \alpha = -1 + 0.5i$, and $\beta = -0.1 - 0.2i$, we can obtain the one-soliton solutions are singular solutions.

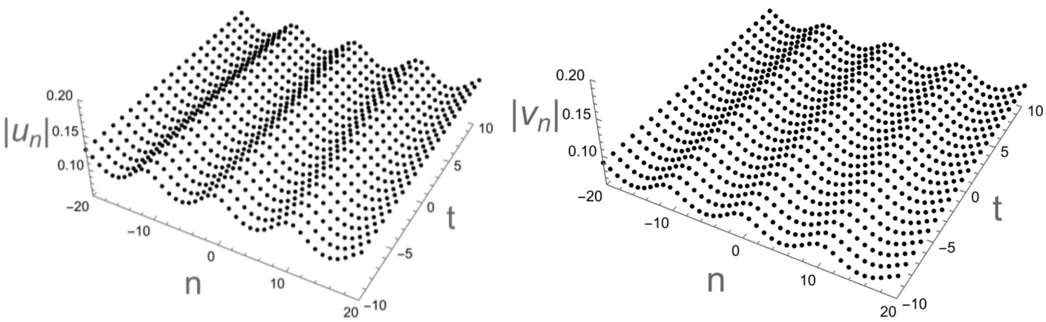

Figure 1: Period solutions $u_n$and $v_n$with parameter as: $\kappa = 0.3i, \delta_1 = \delta_2 = 1, \alpha = 1, \beta = 0.7$.

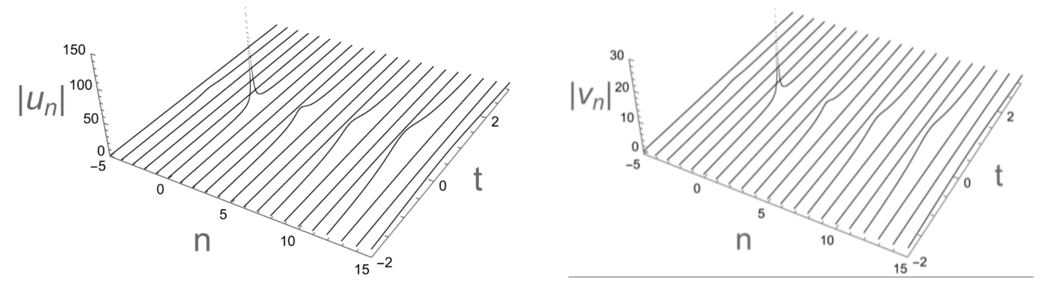

Figure 2: Singular solutions with parameters as:
$\kappa = 0.2 + 0.8i, \delta_1 = \delta_2 = 1, \alpha = -1 + 0.5i, \beta = -0.1 - 0.2\mathrm{i}$

We proceed to examine the continuum limit approximation of the discrete one-soliton solution (16). Set $u_n(t) = \varepsilon u(x,\tau)$, $x = n\varepsilon$, $\tau = \varepsilon^2 t$, and $\kappa = \lambda\varepsilon$, when $\varepsilon \to 0$, the one-soliton solution (16) converges to

$$\begin{cases} u(x,\tau) = \dfrac{\alpha e^{\lambda x + i\lambda^2\tau}}{1+\frac{2(\delta_1|\alpha|^2+\delta_2|\beta|^2)}{(\lambda-\lambda *)^2}e^{(\lambda-\lambda *)x+i(\lambda^2-\lambda *^2)\tau}}, \\ v(x,\tau) = \dfrac{\beta e^{\lambda x + i\lambda^2\tau}}{1+\frac{2(\delta_1|\alpha|^2+\delta_2|\beta|^2)}{(\lambda-\lambda *)^2}e^{(\lambda-\lambda *)x+i(\lambda^2-\lambda *^2)\tau}}. \end{cases} \tag{21}$$

This represents a novel soliton solution to the integrable coupled discrete nonlocal nonlinear Schrödinger equation. Set $\lambda = \mu_1 + i\mu_2$, then:

$$\begin{cases} u(x,\tau) = \dfrac{\alpha e^{i(\mu_2 x + ({\mu_1}^2 - {\mu_2}^2)\tau)} e^{\mu_1 x - 2\mu_1\mu_2\tau}}{1 - \frac{\delta_1|\alpha|^2 + \delta_2|\beta|^2}{2{\mu_2}^2} e^{2i\mu_2 x - 4\mu_1\mu_2\tau}}, \\ v(x,\tau) = \dfrac{\beta e^{i(\mu_2 x + ({\mu_1}^2 - {\mu_2}^2)\tau)} e^{\mu_1 x - 2\mu_1\mu_2\tau}}{1 - \frac{\delta_1|\alpha|^2 + \delta_2|\beta|^2}{2{\mu_2}^2} e^{2i\mu_2 x - 4\mu_1\mu_2\tau}}. \end{cases} \tag{22}$$

The singularity of this solution appears at $(x,\tau) = \left(\frac{k\pi}{\mu_2}, \frac{\ln(4{\mu_2}^2)}{4\mu_1\mu_2}\right), k \in Z$. The solution (22) does not exhibit breather characteristics in either the temporal or spatial domain.

2.2. Bright-bright two-soliton solution

It should be emphasized that multiple solutions exist for Equation (5). For deriving the coupled bright-bright two-soliton solution, we solve Equation (5) as

$$g_n = g_n^{(1)}\varepsilon + g_n^{(3)}\varepsilon^3, h_n = h_n^{(1)}\varepsilon + h_n^{(3)}\varepsilon^3, f_n = 1 + f_n^{(2)}\varepsilon^2 + f_n^{(4)}\varepsilon^4. \tag{23}$$

We obtain a series of bilinear equations

$$\begin{aligned}
&A_1 g_n^{(1)} \cdot 1 = 0,\ \ A_1 h_n^{(1)} \cdot 1 = 0, \\
&A_2(1 \cdot f_n^{(2)} + f_n^{(2)} \cdot 1) = 2\delta_1 g_n^{(1)} g_{-n}^{(1)*} + 2\delta_2 h_n^{(1)} h_{-n}^{(1)*}, \\
&A_1(g_n^{(1)} \cdot f_n^{(2)} + g_n^{(3)} \cdot 1 = 0,\ \ A_1(h_n^{(1)} \cdot f_n^{(2)} + h_n^{(3)} \cdot 1) = 0, \\
&A_2(1 \cdot f_n^{(4)} + f_n^{(2)} \cdot f_n^{(2)} + f_n^{(4)} \cdot 1) + f_{-n}^{(2)*} A_2(1 \cdot f_n^{(2)} + f_n^{(2)} \cdot 1) \\
&\quad = 2\delta_1(g_n^{(1)} g_{-n}^{(1)*} f_n^{(2)} + g_n^{(3)} g_{-n}^{(1)*} + g_n^{(1)} g_{-n}^{(3)*}) + 2\delta_2(h_n^{(1)} h_{-n}^{(1)*} f_n^{(2)} + h_n^{(3)} h_{-n}^{(1)*} + h_n^{(1)} h_{-n}^{(3)*}), \\
&A_2(f_n^{(2)} \cdot f_n^{(4)} + f_n^{(4)} \cdot f_n^{(2)} + f_{-n}^{(2)*} A_2(1 \cdot f_n^{(4)} + f_n^{(2)} \cdot f_n^{(2)} + f_n^{(4)} \cdot 1) + f_{-n}^{(4)*} A_2(1 \cdot f_n^{(2)} + f_n^{(2)} \\
&\qquad\qquad \cdot 1) \\
&\quad = 2\delta_1(g_n^{(1)} g_{-n}^{(1)*} f_n^{(4)} + g_n^{(3)} g_{-n}^{(1)*} f_n^{(2)} + g_n^{(1)} g_{-n}^{(3)*} f_n^{(2)} + g_n^{(3)} g_{-n}^{(3)*}) \\
&\qquad + 2\delta_2(h_n^{(1)} h_{-n}^{(1)*} f_n^{(4)} + h_n^{(3)} h_{-n}^{(1)*} f_n^{(2)} + h_n^{(1)} h_{-n}^{(3)*} f_n^{(2)} + h_n^{(3)} h_{-n}^{(3)*}), \\
&A_1 g_n^{(3)} \cdot f_n^{(4)} = 0, A_1 h_n^{(3)} \cdot f_n^{(4)} = 0, \\
&A_2 f_n^{(4)} \cdot f_n^{(4)} + f_{-n}^{(2)*} A_2(f_n^{(2)} \cdot f_n^{(4)} + f_n^{(4)} \cdot f_n^{(2)}) + f_{-n}^{(4)*} A_2(1 \cdot f_n^{(4)} + f_n^{(2)} \cdot f_n^{(2)} + f_n^{(4)} \cdot 1) \\
&\quad = 2\delta_1(g_n^{(3)} g_{-n}^{(1)*} f_n^{(4)} + g_n^{(1)} g_{-n}^{(3)*} f_n^{(4)} + g_n^{(3)} g_{-n}^{(3)*} f_n^{(2)}) \\
&\quad + 2\delta_2(h_n^{(3)} h_{-n}^{(1)*} f_n^{(4)} + h_n^{(1)} h_{-n}^{(3)*} f_n^{(4)} + h_n^{(3)} h_{-n}^{(3)*} f_n^{(2)}), \\
&f_{-n}^{(2)*} A_2 f_n^{(4)} \cdot f_n^{(4)} + f_{-n}^{(4)*} A_2(f_n^{(2)} \cdot f_n^{(4)} + f_n^{(4)} \cdot f_n^{(2)}) = 2\delta_1 g_n^{(3)} g_{-n}^{(3)*} f_n^{(4)} + 2\delta_2 h_n^{(3)} h_{-n}^{(3)*} f_n^{(4)}, \\
&\qquad\qquad f_n^{(4)*} A_2 f_n^{(4)} \cdot f_n^{(4)} = 0.
\end{aligned} \tag{24}$$

to solve these equations, we let

$$\begin{aligned}
g_n^{(1)} &= \alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2}, g_n^{(3)} = a_{1,2,1*} e^{\xi_1 + \xi_2 + \xi_{1,-n}^*} + a_{1,2,2*} e^{\xi_1 + \xi_2 + \xi_{2,-n}^*}, \\
h_n^{(1)} &= \beta_1 e^{\xi_1} + \beta_2 e^{\xi_2}, h_n^{(3)} = b_{1,2,1*} e^{\xi_1 + \xi_2 + \xi_{1,-n}^*} + b_{1,2,2*} e^{\xi_1 + \xi_2 + \xi_{2,-n}^*}, \\
f_n^{(2)} &= a_{1,1*} e^{\xi_1 + \xi_{1,-n}^*} + a_{1,2*} e^{\xi_1 + \xi_{2,-n}^*} + a_{2,1*} e^{\xi_2 + \xi_{1,-n}^*} + a_{2,2*} e^{\xi_2 + \xi_{2,-n}^*}, \\
f_n^{(4)} &= a_{1,2,1*,2*} e^{\xi_1 + \xi_2 + \xi_{1,-n}^* + \xi_{2,-n}^*}.
\end{aligned} \tag{25}$$

and for $j = 1,2$

$$\xi_j = \kappa_j n + \omega_j t + \xi_j^0,\ \ \xi_{j,-n}^* = -\kappa_j^* n + \omega_j^* t + \xi_j^{0*}. \tag{26}$$

DOI: 10.54254/2753-8818/106/2025.22571

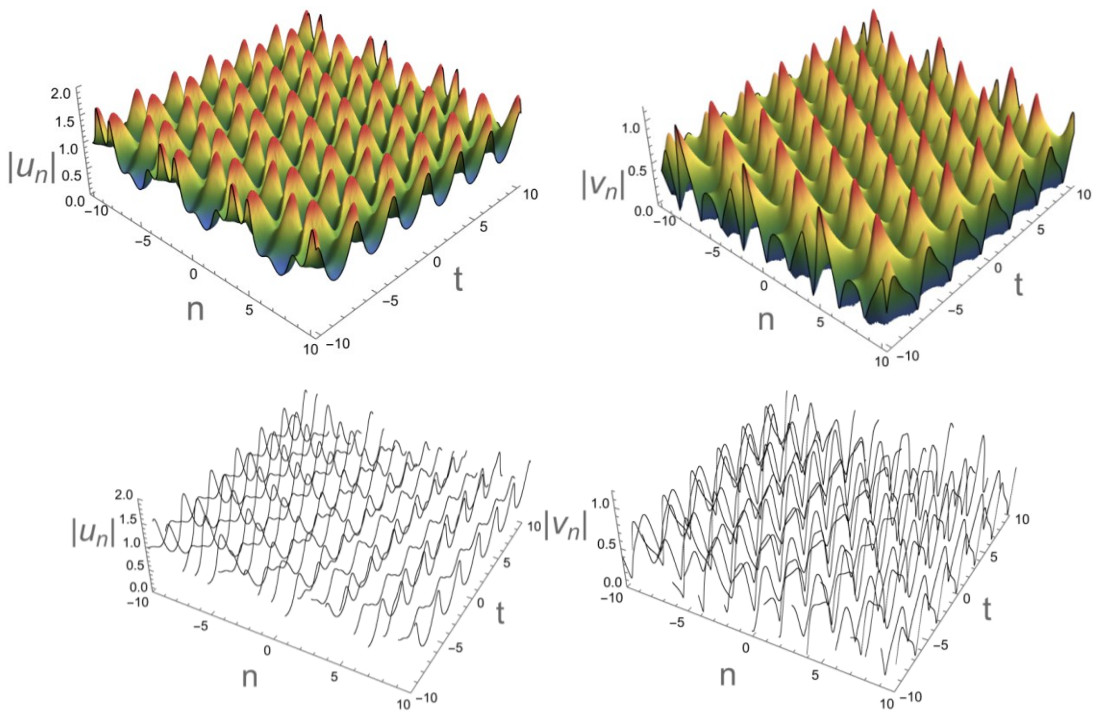

Figure 3: Double spatial period solutions with $\kappa_1 = -2i, \kappa_2 = i$.

we can obtain the dispersion relation $\omega_j = 2i(\cosh\kappa_j - 1)$. Some important coefficients of the soliton solutions are obtained through complex calculations as following

$$
\begin{aligned}
& a_{l,m*} = \frac{\alpha_l\alpha_m^* + \beta_l\beta_m^*}{2} sinh^{-2}\frac{\kappa_l - \kappa_m^*}{2}, \\
& a_{1,2,1*} = \frac{1}{p_{12}}(\alpha_1 a_{2,1*}p_{1,1*} - \alpha_2 a_{1,1*}p_{2,1*}), a_{1,2,2*} = \frac{1}{p_{12}}(\alpha_1 a_{2,2*}p_{1,2*} - \alpha_2 a_{1,2*}p_{2,2*}), \\
& b_{1,2,1*} = \frac{1}{p_{12}}(\beta_1 a_{2,1*}p_{1,1*} - \beta_2 a_{1,1*}p_{2,1*}), b_{1,2,2*} = \frac{1}{p_{12}}(\beta_1 a_{2,2*}p_{1,2*} - \beta_2 a_{1,2*}p_{2,2*}), \\
& a_{1,2,1*,2*} = -4(|\alpha_1|^2 + |\beta_1|^2)(|\alpha_2|^2 + |\beta_2|^2)\frac{p_{12*}p_{21*}}{p_{12}p_{1*2*}}\frac{e^{2\kappa_{1R}+2\kappa_{2R}}}{(e^{\kappa_1} - e^{\kappa_1^*})^2(e^{\kappa_2} - e^{\kappa_2^*})^2} \\
& +4(\alpha_1\alpha_2^* + \beta_1\beta_2^*)(\alpha_2\alpha_1^* + \beta_2\beta_1^*)\frac{p_{11*}p_{22*}}{p_{12}p_{1*2*}}\frac{e^{2\kappa_{1R}+2\kappa_{2R}}}{\left(e^{\kappa_2} - e^{\kappa_1^*}\right)^2\left(e^{\kappa_1} - e^{\kappa_2^*}\right)^2}.
\end{aligned} \tag{27}
$$

where

$$
p_{12} = \frac{1+e^{\kappa_1+\kappa_2}}{e^{\kappa_1}-e^{\kappa_2}}, p_{1*2*} = \frac{1+e^{\kappa_1^*+\kappa_2^*}}{e^{\kappa_1^*}-e^{\kappa_2^*}}, p_{ij*} = \frac{1+e^{\kappa_i+\kappa_j^*}}{e^{\kappa_i}-e^{\kappa_j^*}}, \tag{28}
$$

Therefore, for $\varepsilon = 1$ the two-soliton solution is

$$
\begin{cases}
u_n = \dfrac{\alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2} + a_{1,2,1*}e^{\xi_1+\xi_2+\xi_{1,-n}^*} + a_{1,2,2*}e^{\xi_1+\xi_2+\xi_{2,-n}^*}}{1 + a_{1,1*}e^{\xi_1+\xi_{1,-n}^*} + a_{1,2*}e^{\xi_1+\xi_{2,-n}^*} + a_{2,1*}e^{\xi_2+\xi_{1,-n}^*} + a_{2,2*}e^{\xi_2+\xi_{2,-n}^*} + a_{1,2,1*,2*}e^{\xi_1+\xi_2+\xi_{1,-n}^*+\xi_{2,-n}^*}} \\
v_n = \dfrac{\beta_1 e^{\xi_1} + \beta_2 e^{\xi_2} + b_{1,2,1*}e^{\xi_1+\xi_2+\xi_{1,-n}^*} + b_{1,2,2*}e^{\xi_1+\xi_2+\xi_{2,-n}^*}}{1 + a_{1,1*}e^{\xi_1+\xi_{1,-n}^*} + a_{1,2*}e^{\xi_1+\xi_{2,-n}^*} + a_{2,1*}e^{\xi_2+\xi_{1,-n}^*} + a_{2,2*}e^{\xi_2+\xi_{2,-n}^*} + a_{1,2,1*,2*}e^{\xi_1+\xi_2+\xi_{1,-n}^*+\xi_{2,-n}^*}}
\end{cases} \tag{29}
$$

the solution u_n, v_n are double spatial period solutions. In Figure 3, discrete and corresponding continuous double spatial period soliton diagrams are plotted under specific parameters.

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3. The interaction of dark-dark soliton solutions

This section focuses on the exploration of coupled dark-dark soliton solutions of the Equation (5). By using the dependent variable transformations

$$u_n = \rho_1 (ik_1)^n \frac{g_n}{f_n} e^{(\omega_1 - 2i)t}, v_n = \rho_2 (ik_2)^n \frac{h_n}{f_n} e^{(\omega_2 - 2i)t}. \tag{30}$$

where f_n is a real function and g_n, h_n are complex functions, $\omega_j = m(k_j^{-1} - k_j)$, k_j are complex constants with $k_j = -k_j^*, j = 1,2$, Equation (5) can be expressed in the following bilinear form:

$$\begin{aligned} &(\omega_1 + D_t) g_n \cdot f_n + m(k_1 g_{n+1} f_{n-1} - k_1^{-1} g_{n-1} f_{n+1}) = 0, \\ &(\omega_2 + D_t) h_n \cdot f_n + m(k_2 h_{n+1} f_{n-1} - k_2^{-1} h_{n-1} f_{n+1}) = 0, \\ &f_{-n}^* (m f_{n+1} f_{n-1} - f_n^2) = 2\delta_1 |\rho_1|^2 g_n g_{-n}^* f_n + 2\delta_2 |\rho_2|^2 h_n h_{-n}^* f_n. \end{aligned} \tag{31}$$

Expand f_n, g_n and h_n in Equation (31) as follows:

$$\begin{aligned} f_n &= 1 + f_n^{(2)} \varepsilon^2 + f_n^{(4)} \varepsilon^4 + f_n^{(6)} \varepsilon^6 + \cdots, \\ g_n &= 1 + g_n^{(2)} \varepsilon^2 + g_n^{(4)} \varepsilon^4 + g_n^{(6)} \varepsilon^6 + \cdots, \\ h_n &= 1 + h_n^{(2)} \varepsilon^2 + h_n^{(4)} \varepsilon^4 + h_n^{(6)} \varepsilon^6 + \cdots. \end{aligned} \tag{32}$$

for obtaining the two dark-dark solution, functions are assumed to be

$$f_n = 1 + f_n^{(2)} \varepsilon^2 + f_n^{(4)} \varepsilon^4, g_n = 1 + g_n^{(2)} \varepsilon^2 + g_n^{(4)} \varepsilon^4, h_n = 1 + h_n^{(2)} \varepsilon^2 + h_n^{(4)} \varepsilon^4. \tag{33}$$

Substituting Equation (32) into Equation (31), the forms of g_n, h_n, f_n are assumed to be

$$\begin{aligned} g_n^{(2)} &= a_1 e^{\xi_1} + a_2 e^{\xi_{1,-n}}, h_n^{(2)} = b_1 e^{\xi_1} + b_2 e^{\xi_{1,-n}}, f_n^{(2)} = e^{\xi_1} + e^{\xi_{1,-n}}, \\ g_n^{(4)} &= a_1 a_2 M e^{\xi_1 + \xi_{1,-n}}, h_n^{(4)} = b_1 b_2 M e^{\xi_1 + \xi_{1,-n}}, f_n^{(4)} = M e^{\xi_1 + \xi_{1,-n}}. \end{aligned} \tag{34}$$

where $\xi_1 = Pn + \Omega t + \xi_1^0$, $\xi_{1,-n} = -Pn + \Omega t + \xi_1^0$, $m = 1 + 2\delta_1 |\rho_1|^2 + 2\delta_2 |\rho_2|^2$, $\omega_j + m(k_j - k_j^{-1}) = 0$, and P, Ω, ξ_1^0 are real constants, a_1, a_2, b_1, b_2 are arbitrary complex constants, when $k_1 = k_2 = -i$, then $\omega_1 = \omega_2 = 2mi$, Ω and P satisfy the following equation

$$m e^{-3P} (e^P - 1)^2 (\Omega^2 e^{2P} + m (e^P - 1)^2 (2e^P (m - 2) + m + m e^{2P})) = 0. \tag{35}$$

We get the two dark-dark solution for Equation (5) as

$$\begin{cases} u_n = \rho_1 (ik_1)^n e^{(\omega_1 - 2i)t} \frac{1 + a_1 e^{\xi_1} + a_2 e^{\xi_{1,-n}} + a_1 a_2 M e^{\xi_1 + \xi_{1,-n}}}{1 + e^{\xi_1} + e^{\xi_{1,-n}} + M e^{\xi_1 + \xi_{1,-n}}}, \\ v_n = \rho_2 (ik_2)^n e^{(\omega_2 - 2i)t} \frac{1 + b_1 e^{\xi_1} + b_2 e^{\xi_{1,-n}} + b_1 b_2 M e^{\xi_1 + \xi_{1,-n}}}{1 + e^{\xi_1} + e^{\xi_{1,-n}} + M e^{\xi_1 + \xi_{1,-n}}}. \end{cases} \tag{36}$$

where

$$\begin{aligned} a_1 = a_2 = b_1 = b_2 &= \frac{\Omega + im(e^P + e^{-P} - 2)}{\Omega - im(e^P + e^{-P} - 2)}, \\ M &= \frac{e^{-3P} (1 + e^P)^2 (m^2 (e^P - 1)^4 + e^{2P} \Omega^2)}{4\Omega^2}. \end{aligned} \tag{37}$$

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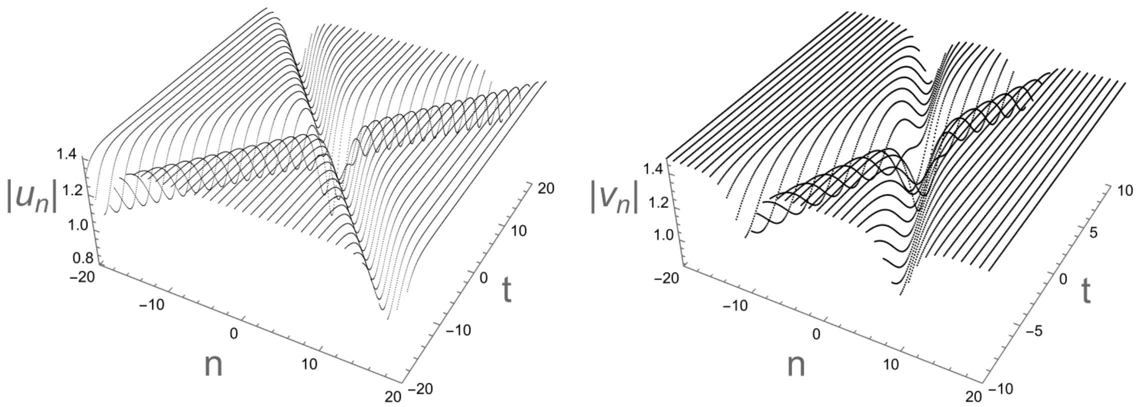

Figure 4: dark-dark soliton solutions with parameters as:
$k_1 = k_2 = -i, P = -1, \delta_1 = 1, \delta_2 = -1, \rho_1 = 1 - i, \rho_2 = 1 + 1.1i.$

Formally speaking, the expansion of f_n, g_n and h_n in this section and the form of solutions are derived from continuous equations, However, in continuous systems, they correspond to breathers, while in non-local systems, they exhibit dark-dark soliton waves, dark-antidark soliton waves or antidark-antidark soliton waves. In Figure 4, we obtain the dark-dark soliton waves with the parameters: $k_1 = k_2 = -i, P = -1, \delta_1 = 1, \delta_2 = -1, \rho_1 = 1 - i, \rho_2 = 1 + 1.1i.$

4. Conclusion

The coupled discrete non-local nonlinear Schrödinger Equation (5) represents a novel integrable system in discrete mathematics. This study employs the Hirota bilinear approach to derive the bilinear representation of this coupled nonlocal discrete system. Through systematic analysis, we establish both bright-bright single and two-soliton solutions, followed by the development of bilinear expansions for dark-dark soliton configurations. Furthermore, computational methods are implemented to generate graphical representations of these soliton solutions. We found that compared to non-local NLS equations, integrable coupled discrete non-local NLS equations have a richer variety of soliton solution types, and the interactions between soliton solutions and the evolution properties of solutions over time are also significantly different.

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