DOI: 10.54254/2753-8818/108/2025.22623

An Approach to the Recursive Formula of Riemann Zeta Function at Even Natural Numbers

Yifan Chen

Woodland Christian High School, Breslau, Kitchener, Canada
justinchenyifan2007@gmail.com

Abstract: This paper aims to derive a recursive relationship of the values of Riemann zeta function at even natural numbers by using the principles of elementary symmetric polynomials, and associations between them and zeta functions, thereby expressing $\zeta(2k)$ in only terms of previous zeta function's values. Moreover, this recursive formula is going to be proven equivalently to the explicit formula of $\zeta(2k)$.

Keywords: Bernoulli numbers, Riemann zeta function, recursive formula

1. Introduction

The explicit formula of Riemann zeta function over $2n$ $(n \in \mathbb{N})$ is originally given by:

$$\zeta(2n) = \frac{|B_{2n}|(2\pi)^{2n}}{2(2n)!}$$

Where B_n denotes as the nth Bernoulli number.

Also, the values of B_n can be evaluated recursively by this following relationship, and it is the only simplest way to calculate their exact values:

$$B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k$$

Which is just being rewritten from the identity that $\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0$.

However, when we are evaluating $\zeta(2n)$ based on Bernoulli numbers, then we have to find the Bernoulli numbers at corresponding term first, then plug them in into the explicit formula. It will be very complex especially when n becomes larger.

However, there is a way to eliminate the Bernoulli numbers in the formula when we are calculating $\zeta(2n)$, which can simplify the process at some extend.

That is, the recursive formula of Riemann zeta function $\zeta(n)$ at even natural numbers can be expressed by the following equation:

$$\zeta(2n) = (-1)^{n-1}\left(\frac{n\pi^{2n}}{(2n+1)!} + \sum_{k=1}^{n-1}(-1)^k \frac{\pi^{2(n-k)}\zeta(2k)}{(2(n-k)+1)!}\right) \text{ for } n \geq 2$$

From using this recursive formula:

$$\zeta(2) = \frac{\pi^2}{3!} = \frac{\pi^2}{6}$$

$$\zeta(4) = -\frac{2\pi^4}{5!} + \frac{\pi^2}{3!}\zeta(2) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{3\pi^6}{7!} - \frac{\pi^4}{5!}\zeta(2) + \frac{\pi^2}{3!}\zeta(4) = \frac{\pi^6}{945}$$

$$\zeta(8) = -\frac{4\pi^8}{9!} - \frac{\pi^6}{7!}\zeta(2) + \frac{\pi^4}{5!}\zeta(4) - \frac{\pi^2}{3!}\zeta(6) = \frac{\pi^8}{9450}$$

$$\cdots$$

This formula is derived from the principles of elementary symmetric polynomials and the associations between them and the values of zeta function. Also, it is equivalent to the original explicit formula of zeta function, that is, $\zeta(2n) = \frac{|B_{2n}|(2\pi)^{2n}}{2(2n)!}$.

2. Elementary symmetric polynomials & Newton's identities

For variables $x_1, \dots, x_k$ $(k > 1)$, let $p_k(x_1, \dots, x_n)$ be the power sum $\sum_{i=1}^{n} x_i^k$ and $e_k(x_1, \dots, x_n)$ be the elementary symmetric polynomial such that $e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1} \cdot x_{i_2} \dots x_{i_k}$.

For example, we know that $(a+b)^2 = a^2 + 2ab + b^2$, replace $a+b$ by p_1; a^2+b^2 by p_2; ab by e_2 and we obtain $p_1^2 = 2e_2 + p_2 \Rightarrow e_2 = \frac{1}{2}(p_1^2 - p_2)$. This is important because it generalizes the following summation identity:

$$\sum_{1 \leq i < j} ij = \frac{1}{2}\left[\left(\sum_{n=1}^{\infty} i\right)^2 - \sum_{n=1}^{\infty} j^2\right]$$

Similarity for e_3, expanding $(a+b+c)^3$ and do some elementary algebra:

$$\Rightarrow (a+b+c)^3 = a^3 + b^3 + c^3 + 6abc + 3a^2b + 3a^2c + 3b^2a + 3b^2c + 3c^2a + 3c^2b$$

$$= \left(a^3 + b^3 + c^3\right) + 6abc + 3a^2(b + c + a - a) + 3b^2(a + c + b - b) + 3c^2(a + b + c - c)$$

$$= \left(a^3 + b^3 + c^3\right) + 6abc + \left(3a^2 + 3b^2 + 3c^2\right)(a + b + c) - \left(3a^3 + 3b^3 + 3c^3\right)$$

$$= -2\left(a^3 + b^3 + c^3\right) + 6abc + 3\left(a^2 + b^2 + c^2\right)(a + b + c)$$

$$\Rightarrow p_1^3 = -2p_3 + 6e_3 + 3p_2p_1 \Rightarrow e_3 = \frac{1}{6}\left(p_1^3 - 3p_1p_2 + 2p_3\right)$$

We can keep deriving, but it becomes complicated when k goes larger since it involves a lot of substitution and the coefficients are irregular.

However, there is a way to make it simpler:

Define $E(t)$ to be the generating function of e_k, that is,

$$E(t) = \sum_{k=1}^{\infty} e_k t^k$$

It can be observed that by the definition of e_k,

$$E(t) = \prod_{i=1}^{n} (1 + x_i t)$$

Taking logarithms from both sides:
(Note that $\ln X$ is an alternative notation for $\log_e X$)

$$\ln E(t) = \ln \prod_{i=1}^{n} (1 + x_i t) = \sum_{i=1}^{n} \ln(1 + x_i t)$$

$$\because \ln(1 + x_i t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (x_i t)^k}{k} \quad \text{(Taylor's expansion for } \ln(1 + X))$$

$$\therefore \ln E(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} p_k t^k}{k}$$

And taking derivatives respect to t from both sides of the equation:

$$\frac{E'(t)}{E(t)} = \sum_{k=1}^{\infty} (-1)^{k-1} p_k t^{k-1}$$

$$\Rightarrow E'(t) = E(t) \sum_{k=1}^{\infty} (-1)^{k-1} p_k t^{k-1}$$

$$= \left(1 + e_1 t + e_2 t^2 + \cdots\right)\left(p_1 - p_2 t + p_3 t^2 - \cdots\right)$$

By comparing and equating the coefficient of t^{k-1} and therefore obtain:

$$k e_k = \sum_{i=1}^{k} (-1)^{k-1} e_{k-i} p_i$$

Which is so called *Newton's Identities* for elementary symmetric polynomials. From using this principle, we may express e_k in terms of $p_1, p_2, \dots, p_k$ (Note that $e_1 = p_1$):

$$e_1 = p_1$$

$$e_2 = \tfrac{1}{2}\left(p_1^2 - p_2\right)$$

DOI: 10.54254/2753-8818/108/2025.22623

$$e_3 = \frac{1}{6}\left(p_1^3 - 3p_1p_2 + 2p_3\right)$$

$$e_4 = \frac{1}{24}\left(p_1^4 - 6p_1^2p_2 + 8p_1p_3 + 3p_2^2 - 6p_4\right)$$

$$\dots$$

This property will further be using to derive the recursive formula of $\zeta(2k)$ essentially.

3. Associations with Riemann zeta functions

According to Euler's product formula, for any polynomial function $P(x)$, it can be written as

$$P(x) = a_n \prod_{i=1}^{n} (x - x_i)$$

where $x_1, x_2, \dots, x_i$ are the roots of $P(x)$ and a_n is the leading coefficient of $P(x)$.

Let $P(x) = \sin x$, then

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2}\right)$$

$$\Rightarrow \frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2}\right)$$

And associate it with Taylor's expansion of $\frac{\sin x}{x}$:

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-2}}{(2n-1)!}$$

Equating the coefficients of $x^2, x^4, \dots, x^{2n}$:

$$x^2 : -\frac{1}{3!} = -\frac{1}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2}$$

$$x^4 : \frac{1}{5!} = \frac{1}{\pi^4} \sum_{1 \le i < j} \frac{1}{i^2 j^2}$$

$$\vdots$$

$$x^{2n} : \frac{(-1)^n}{(2n+1)!} = \frac{(-1)^n}{\pi^{2n}} \sum_{1 \le i_1 < i_2 < \dots < i_n} \frac{1}{i_1^2 i_2^2 \dots i_n^2} (n > 2)$$

$$\Rightarrow x^{2n} : \frac{1}{(2n+1)!} = \frac{1}{\pi^{2n}} \sum_{1 \le i_1 < i_2 < \dots < i_n} \frac{1}{i_1^2 i_2^2 \dots i_n^2} (n > 2)$$

DOI: 10.54254/2753-8818/108/2025.22623

The summation part

$$\sum_{1 \le i_1 < i_2 < \cdots < i_n} \frac{1}{i_1^2 i_2^2 \dots i_n^2} (n > 2)$$

can be considered as an elementary symmetric polynomial $e_n(x_1, \dots, x_k)$ where $x_1, x_2, \dots, x_k = \frac{1}{i_1^2}, \frac{1}{i_2^2}, \dots, \frac{1}{i_n^2} (n = k)$

$$\therefore e_n = \sum_{1 \le i_1 < i_2 < \cdots < i_n} \frac{1}{i_1^2 i_2^2 \dots i_n^2} = \frac{\pi^{2n}}{(2n+1)!}$$

Therefore, the power sum $p_n(x_1, \dots, x_k)$ has

$$p_n = \sum_{i=1}^{\infty} \frac{1}{i^{2n}} = \zeta(2n)$$

Associate with Newton's identities which has been proven in 1.1:

$$ne_n = \sum_{k=1}^{n} (-1)^{k-1} e_{n-k} p_n$$

$$\Rightarrow \frac{n\pi^{2n}}{(2n+1)!} = \sum_{k=1}^{n} (-1)^{k-1} \frac{\pi^{2(n-k)}}{(2(n-k)+1)!} \zeta(2k)$$

$$\Rightarrow \frac{n\pi^{2n}}{(2n+1)!} = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{\pi^{2(n-k)}}{(2(n-k)+1)!} \zeta(2k) + (-1)^{n-1} \zeta(2n)$$

$$\Rightarrow (-1)^{n-1} \zeta(2n) = \frac{n\pi^{2n}}{(2n+1)!} + \sum_{k=1}^{n-1} (-1)^{k} \frac{\pi^{2(n-k)}}{(2(n-k)+1)!} \zeta(2k)$$

$$\Rightarrow \zeta(2n) = (-1)^{n-1} \left(\frac{n\pi^{2n}}{(2n+1)!} + \sum_{k=1}^{n-1} (-1)^{k} \frac{\pi^{2(n-k)}}{(2(n-k)+1)!} \zeta(2k) \right), \text{for } n \ge 2$$

Which obtain the recursive formula for $\zeta(2n)$ where $n \ge 2$.

4. Associations with the explicit formula of Riemann zeta functions

For $n \in \mathbb{N}^*$, the explicit formula of Riemann zeta function is given by:

$$\zeta(2n) = \frac{|B_{2n}| (2\pi)^{2n}}{2(2n)!}$$

where B_n is the nth Bernoulli number.

Then according to the recursive formula derived before, the equation

$$\frac{\left|B_{2n}\right|(2\pi)^{2n}}{2(2n)!} = (-1)^{n-1}\left(\frac{n\pi^{2n}}{(2n+1)!} + \sum_{k=1}^{n-1}(-1)^k \frac{\pi^{2(n-k)}}{(2(n-k)+1)!}\zeta(2k)\right) \text{ for } n > 1$$

must be satisfied.

Lemma:

$$x\cot x = 1 - \sum_{n=1}^{\infty}\frac{2^{2n}\left|B_{2n}\right|}{(2n)!}x^{2n}$$

Proof:

Recall that the generating function of Bernoulli numbers B_n is

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \text{ for } |x| < 2\pi$$

$$\Longrightarrow \frac{x}{2}\cdot\frac{e^x+1}{e^x-1} = 1 + \sum_{n=2}^{\infty}\frac{B_n}{n!}x^n$$

Since LHS is an even function and $B_{2n+1} = 0$ for $n = 1,2,3 \ldots$

$$\Longrightarrow \frac{x}{2}\cdot\frac{e^x+1}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n} \text{ for } |x| < 2\pi$$

$$\Longrightarrow \frac{x}{2}\cdot\frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n}$$

$$\Longrightarrow \frac{x}{2}\coth\frac{x}{2} = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n}$$

$$\Longrightarrow x\coth x = \sum_{n=0}^{\infty}\frac{B_{2n}2^{2n}}{(2n)!}x^{2n},\ \ |x| < \pi$$

Since $\coth ix = -i\cot x$, substitute $ix \to x$ to obtain:

$$ix\coth ix = \sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}2^{2n}}{(2n)!}x^{2n}$$

$$\Longrightarrow x\cot x = \sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}2^{2n}}{(2n)!}x^{2n},\ \ |x| < \pi$$

DOI: 10.54254/2753-8818/108/2025.22623

$$\because (-1)^{n+1} B_{2n} = \left|B_{2n}\right|$$

$$\therefore x\cot x = -\sum_{n=0}^{\infty} \frac{\left|B_{2n}\right| 2^{2n}}{(2n)!} x^{2n}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{\left|B_{2n}\right| 2^{2n}}{(2n)!} x^{2n}, \quad |x| < \pi$$

As we successfully obtain the generating function of B_{2k}, we may use it to deduce the generating function of $\zeta(2k)$:

Since

$$\zeta(2k) = \frac{\left|B_{2k}\right|(2\pi)^{2k}}{2(2k)!} \tag{1}$$

and

$$x\cot x = 1 - \sum_{k=1}^{\infty} \frac{2^{2k}\left|B_{2k}\right|}{(2k)!} x^{2k} \tag{2}$$

By substituting $x = \pi x$ into (2):

$$\Rightarrow \pi x \cot \pi x = 1 - \sum_{k=1}^{\infty} \frac{2^{2k}\left|B_{2k}\right|}{(2k)!} (\pi x)^{2k}$$

$$\Rightarrow 1 - \pi x \cot \pi x = \sum_{k=1}^{\infty} \frac{(2\pi)^{2k}\left|B_{2k}\right|}{(2k)!} x^{2k}$$

$$\Rightarrow \frac{1}{2}(1 - \pi x \cot \pi x) = \sum_{k=1}^{\infty} \frac{(2\pi)^{2k}\left|B_{2k}\right|}{2(2k)!} x^{2k}$$

And associate with (1):

$$\Rightarrow \frac{1}{2}(1 - \pi x \cot \pi x) = \sum_{k=1}^{\infty} \zeta(2k) x^{2k}$$

Now consider another series which

$$x\cot x = \sum_{k=1}^{\infty} a_k x^{2k}$$

Where a_k is any real valued function

$$\Rightarrow \cos x = \frac{\sin x}{x} \sum_{k=0}^{\infty} a_k x^{2k}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n+1)!} \sum_{k=0}^{\infty} a_k x^{2k}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k \frac{a_{n-k}}{(2k+1)!} \right) x^{2n}$$

By equating the coefficients of x^{2n} therefore obtain:

$$\frac{(-1)^n}{(2n)!} = \sum_{k=0}^{n} (-1)^k \frac{a_{n-k}}{(2k+1)!}$$

$$\Rightarrow a_n = \frac{(-1)^n}{(2n)!} - \sum_{k=1}^{n} (-1)^k \frac{a_{n-k}}{(2k+1)!}$$

$$= \frac{2n(-1)^n}{(2n+1)!} - \sum_{k=1}^{n-1} (-1)^k \frac{a_{n-k}}{(2k+1)!}$$

From (3), it can be deduced that

$$a_n = -\frac{2\zeta(2n)}{\pi^{2n}}$$

Therefore,

$$-\frac{2\zeta(2n)}{\pi^{2n}} = \frac{2n(-1)^n}{(2n+1)!} - \sum_{k=1}^{n-1} (-1)^k \frac{2\zeta\left(2(n-k)\right)}{(2k+1)!\,\pi^{2(n-k)}}$$

$$\Rightarrow \zeta(2n) = \frac{n\pi^{2n}(-1)^{n-1}}{(2n+1)!} - \sum_{k=1}^{n-1} (-1)^k \frac{\zeta\left(2(n-k)\right)}{(2k+1)!}$$

$$= (-1)^{n-1} \left(\frac{n\pi^{2n}}{(2n+1)!} + \sum_{k=1}^{n-1} (-1)^k \frac{\pi^{2k}}{(2(n-k)+1)!} \zeta(2k) \right)$$

Which corresponds to the recursive formula of $\zeta(2n)$

Therefore, it has been successfully proven the equivalent relationship between the recursive formula and the explicit formula for Riemann zeta function at even positive integers.

DOI: 10.54254/2753-8818/108/2025.22623

5. Conclusion

By associating the coefficients of the terms of Taylor's expansion and Euler's product of $\frac{\sin x}{x}$ with the essential properties of elementary symmetric polynomials (Newton's identities), the recursive formula of Riemann zeta function at natural even numbers can be expressed by the previous values of it. This method of evaluating the values of zeta function at large even numbers, can be helpful in particular occasion than applying the explicit formula since it is difficult to calculate the values of large Bernoulli numbers.

References

[1] *https://en.wikipedia.org/wiki/Riemann_zeta_function*

[2] *Titchmarsh, E.C., "The Theory of the Riemann zeta-function." Oxford Science Publications (1986)*

[3] *Segarra, Elan, "An Exploration of Riemann Zeta Function and its application to the Theory of Prime Distribution" (2006). HMC Senior Theses. 189.*

[4] *H.M. Edwards., Riemann's zeta function. Dover Publications, Inc., Mineola, N.Y. (2001)*

[5] *https://brilliant.org/wiki/riemann-zeta-function/*

[6] *Pascal Sebah and Xavier Gourdon, "Introduction on Bernoulli numbers" (2002)*

[7] *https://en.wikipedia.org/wiki/Elementary_symmetric_polynomial*