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Volume of N-Dimensional Spheres

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Abstract: In this paper, we present two methods to derive the formula for the volume of n-dimensional spheres in Euclidean space and analyze its asymptotic behavior as n approaches infinity using Stirling's formula. We then establish a connection between the infinite sum of all even-dimensional spheres of radius r and the exponential function through a differential equation. Our findings highlight the decay characteristics of high-dimensional volumes and reveal a novel link between geometry and analysis. This work not only refines classical volume estimation techniques but also offers valuable insights into applications in higher-dimensional mathematics, contributing to fields such as statistical mechanics and theoretical physics.

Keywords: n-dimensional spheres, Euclidean space, Stirling's formula, asymptotic behavior, high-dimensional volumes

1. Deriving the formula for the volume of n-dimensional sphere

1.1. method1

reference: [1][7][9][10]

Lemma:

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

Proof: We prove this by first squaring the expression to obtain a double integral

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x_1^2-x_2^2}\, dx_1 dx_2$$

Using polar coordinates, we obtain

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$$\begin{aligned}&\int_{\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x_1^2-x_2^2}\,dx_1dx_2\\&=\int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,rdrd\theta\\&=\left(\int_0^{\infty} e^{-r^2}\,rdr\right)\left(\int_0^{2\pi} d\,\theta\right)\\&=\left(\frac{1}{2}\int_0^{\infty} e^{-r^2}\,dr^2\right)\times 2\pi\\&=\frac{1}{2}\left(-e^{-r^2}\Big|_0^{\infty}\right)\times 2\pi\\&=\pi\end{aligned}$$

which gives us the result:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$$

Proposition: Let $V_n(r)$ be the volume of n-dimensional sphere of radius r, then

$$V_n(r)=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}r^n$$

Proof: Let v(n) and s(n) denote the volume and surface area of a n-dimensional unit sphere respectively. We have the relations

$$V_n(r)=v(n)r^n$$
$$S_n(r)=s(n)r^{n-1}$$

Representing $V_n(r)$ using both Cartesian coordinate and spherical coordinate, we obtain

$$\begin{aligned}V_n(R)&=\int_{x_1^2+x_2^2+\ldots+x_n^2\le R^2} d\,x_1dx_2\cdots dx_n\\&=\int_0^{R}\int_0^{2\pi} r^{\,n-1}\,drd\Omega_{n-1}\\&=\frac{1}{n}R^n\int_0^{2\pi} d\,\Omega_{n-1}\\&=v(n)R^n\end{aligned}$$

which implies

$$nv(n)=\int_0^{2\pi} d\,\Omega_{n-1}$$

Representing the hypersphere to be the union of several concentric spherical shells whose thicknesses are closed to 0, the volume of the hypersphere can be expressed as:

$$V_n(R)=\int_0^{R} S_n\,(r)dr$$

Using the fundamental theorem of calculus, we obtain

$$\frac{d}{dR}\left(v(n)R^n\right)=S_n(R)$$

This can be reputed as the following:

$$v(n)nR^{n-1}=s(n)R^{n-1}$$
$$nv(n)=s(n)$$

Now, a relationship between v(n) and s(n) independent of r is built
Consider the integral:

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-x_1^2-x_2^2-\ldots-x_n^2}\,dx_1dx_2\ldots dx_n$$

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where $(x_1,x_2,...x_n)$ is the cartesian coordinate in the n-th dimension.

As it is a sphere, its radius can be expressed as:

$$r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

After making some substitutions, we have

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-x_1^2-x_2^2-\ldots-x_n^2}\, dx_1 dx_2 \ldots dx_n$$

where V is the volume element in spherical coordinates, and

$$dV = dx_1 dx_2 \ldots dx_n$$

Computing the RHS we have

$$\begin{aligned} \text{RHS} &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r^{n-1} d\Omega_{n-1} dr \\ &= \left(\int_0^{2\pi} d\,\Omega_{n-1}\right)\left(\int_0^{\infty} e^{-r^2}\, r^{n-1} dr\right) \\ &= s(n)\int_0^{\infty} e^{-r^2}\,(r^2)^{\frac{n-1}{2}}\frac{1}{2}r^{-1}dr^2 \\ &= \frac{s(n)}{2}\int_0^{\infty} e^{-r^2}\,(r^2)^{\frac{n}{2}-1}dr^2 \end{aligned}$$

Substituting r^2 with t, we have

$$\begin{aligned} &= \frac{s(n)}{2}\int_0^{\infty} e^{-t}\, t^{\frac{n}{2}-1} \\ &= \frac{s(n)}{2}\Gamma\left(\frac{n}{2}\right) \end{aligned}$$

Notice that the equation below is the definition of gamma function:

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1} dt$$

Since

$$\text{LHS} = \left(\pi^{\frac{1}{2}}\right)^n = \pi^{\frac{n}{2}}$$

combining with RHS, we get

$$\begin{aligned} \pi^{\frac{n}{2}} &= \frac{s(n)}{2}\Gamma\left(\frac{n}{2}\right) \\ s(n) &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \end{aligned}$$

Combining the results, we have

$$nv(n) = s(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

which implies

$$v(n) = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$$

Substituted v(n) into the first equation we had at the beginning, we obtain

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$$\begin{aligned} V_n(r) &= v(n)r^n \\ &= \frac{2\pi^{\frac{n}{2}}r^n}{n\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{\pi^{\frac{n}{2}}r^n}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)} \end{aligned}$$

Since

$$\Gamma(t+1) = t\Gamma(t)$$

we have

$$\frac{n}{2}\Gamma\left(\frac{n}{2}\right) = \Gamma\left(\frac{n}{2}+1\right)$$

The volume of a n-dimensional sphere is then successfully derived:

$$\boxed{V_n(r) = \frac{\pi^{\frac{n}{2}}r^n}{\Gamma\left(\frac{n}{2}+1\right)}}$$

1.2. method2

reference: [2][6]

It is well known that the volume of an n-dimensional object is proportional to the nth power of its one-dimensional element, so we can set

$$V_n = C_n \cdot R^n$$

Consider an (n-1)-dimensional hyperplane intersecting an n-dimensional hypersphere, resulting in an (n-1)-dimensional hypersphere. Let the distance between the two spheres be n, the radius of the n-dimensional hypersphere be R, and the radius of the (n-1)-dimensional hypersphere be r. In the end, we can decompose Cn into:

$$C_nR^n = 2\int_0^R C_{n-1}r^{n-1} \times dh$$

Let

$$h = R\sin\theta$$

$$\begin{aligned} C_nR^n &= 2C_{n-1}\int_0^R\left(\sqrt{R^2-h^2}\right)^{n-1}dh \\ &= 2C_{n-1}\int_0^{\frac{\pi}{2}}R^{n-1}\cos^{n-1}\theta d(R\sin\theta) \\ &= 2C_{n-1}\int_0^{\frac{\pi}{2}}R^n\cos^n\theta d\theta \\ &= 2C_{n-1}\int_0^{\frac{\pi}{2}}\cos^n\theta d\theta \end{aligned}$$

Next, we observe that the expression allows for simplification of the integral. Therefore, we separate the integral and let

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$$I_n \quad = \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$$
$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} \theta d(\sin\theta)$$
$$= n - 1\left(\int_0^{\frac{\pi}{2}} \cos^{n-2} d\theta - \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta \right)$$
$$= n - 1(I_{n-2} - I_n)$$

Through calculation, we can determine the recurrence relation for In. Using this recurrence relation to derive the general term for In, we can find these two special values of In.

$$I_n = \frac{n-1}{n} I_{n-2} \quad I_1 = \int_0^{\frac{\pi}{2}} \cos x dx = 1 \quad I_n = \int_0^{\frac{\pi}{2}} d\, x = \frac{\pi}{2}$$

After expanding the general term of n, we find that when n is odd:

$$I_n = \frac{n-2}{n}\frac{n-3}{n-2}\cdots\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{(n-1)!!}{n!!}$$

And when n is even:

$$I_n = \frac{n-2}{n}\frac{n-3}{n-2}\cdots\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{(n-1)!!}{n!!}\times\frac{\pi}{2}$$

If we represent In and the volume of an n-dimensional sphere solely using double factorials, the calculations can still be quite large. Therefore, we continue to simplify the double factorial using the Gamma function. The Gamma function is related to non-positive integers *z* as follows:

$$\Gamma(z+1) = z\Gamma(z)$$

Next is the process of using the Gamma function to prove the relation involving double factorials.

$$\Gamma\left(n+\frac{1}{2}\right) \quad = \left(n+\frac{1}{2}-1\right)\left(n+\frac{1}{2}-2\right)\left(n+\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$
$$2^n\Gamma\left(n+\frac{1}{2}\right) \quad = (2n+1-2)(2n+(-4)(2n+1-6)\cdots(5)(3)(1)\sqrt{\pi}$$
$$2^n\Gamma\left(n+\frac{1}{2}\right) \quad = (2n-1)!!\sqrt{\pi}$$

$$\Gamma(n) \quad = (n-1)(n-2)\cdots 3\times 2\times 1$$
$$2^{n-1}\Gamma(n) \quad = (2n-2)(2n-4)\cdots 6\times 4\times 2$$
$$2^{n-1}\Gamma(n) \quad = (2n-2)!!$$

Replace n with m:

$$\begin{cases}(2m-1)!! = 2^m\Gamma\left(m+\frac{1}{2}\right)/\sqrt{\pi}\\(2m-2)!! = 2^{m-1}\Gamma(m)\end{cases}$$

When n is even number:

$$n!! = 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)$$

When n is odd number:

$$n!! = 2^{\frac{m}{2}+\frac{1}{2}}\Gamma\left(\frac{n}{2}+1\right)/\sqrt{\pi}$$

By calculating the results for both odd and even values of n , we can obtain the final expression.

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$$I_n = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{\sqrt{\pi}}{2}$$

Substitute In into Cn and then simplify and split it.

$$\begin{aligned} \text{Cn} &= \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \sqrt{\pi} C_{n-1} \\ &= \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} \left(\sqrt{\pi}\right)^2 C_{n-2} \\ &= \frac{\Gamma\left(\frac{\pi}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\sqrt{\pi}\right)^3 C_{n-3} \\ &= \frac{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{\Gamma\left(\frac{1}{2}-1\right)}{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)} \left(\sqrt{\pi}\right)^4 C_{n-4} \end{aligned}$$

By observation, we can summarize the pattern of Cn as follows i:

$$\frac{\Gamma\left(\frac{n}{2}+1-\frac{i}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \left(\sqrt{\pi}\right)^i C_n - i$$

When i equals n-1:

$$C_n = \frac{\Gamma\left(\frac{n}{2}+1-\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \left(\sqrt{\pi}\right)^{n-1} C_1$$

$$C_n = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \left(\sqrt{\pi}\right)^{n-1} C_1$$

$$C_n = \frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+1\right)} \left(\sqrt{\pi}\right)^{n-1} \times 2$$

$$C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

Substitute Cn into Vn, from this, we can derive the volume formula for an n-dimensional sphere.

$$\begin{aligned} V_n &= \text{Cn} R^n \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} R^n \end{aligned}$$

2. Asymptotic behavior

Reference: [3][4][5]

Stirling's formula states that

$$\Gamma(n) \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Substituting it into $V_n(R)$ we obtain

$$V_n(R) \sim \frac{\pi^{n/2} R^n}{\sqrt{2\pi\left(\frac{n}{2}+1\right)} \left(\frac{\frac{n}{2}+1}{e}\right)^{\frac{n}{2}+1}}$$

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From this, it is clear that $V_n(R)$ goes to 0 as n goes to ∞. The intuitive explanantion for this is that as n becomes larger, $V_n(R)$ occupies smaller proportion of space in the n-dimensional cube inscribing the n-dimensional sphere.

The sum of all even dimensional spheres of radius r.

Adding up the volume of all even dimensional unit spheres, we obtain

$$V_0(1)+V_2(1)+V_4(1)+\ldots=\frac{\pi^0}{\Gamma(1)}+\frac{\pi^1}{\Gamma(2)}+\frac{\pi^2}{\Gamma(3)}+\ldots=1+\frac{\pi^1}{1!}+\frac{\pi^2}{2!}+\frac{\pi^3}{3!}+\ldots=e^{\pi}$$

Proposition:

$$\sum_{k=0}^{\infty} V_{2k}(r)=e^{\pi r^2}$$

Proof:

Generally, when trying to connect something to the exponential function, it's probably good to consider the differential equation

$$f'(x)=cf(x)$$

which also is the key idea in this case.

We begin by considering the following facts: let $a_n(r)$ denote the surface area of the n-dimensional sphere with radius r, then we have

(1) . $a_n(r)=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}r^{n-1}=\frac{n\pi^{\frac{n}{2}}r^{n-1}}{\Gamma\left(\frac{n}{2}+1\right)}=\frac{dV_n(r)}{dr}$

(2). $a_{n+1}(r)=2\pi r V_n(r)$

Let

$$S(r)=\sum_{k=0}^{\infty} V_{2k}(r)$$

then we have

$$\begin{aligned} S'(r) &= 0+a_1(r)+a_3(r)+\cdots \\ &= 2\pi r\cdot 1+2\pi r V_2(r)+2\pi r V_4(r)\cdots \\ &= 2\pi r(1+V_2(r)+V_4(r)+\cdots) \\ &= 2\pi r\cdot S(r) \end{aligned}$$

which implies

$$S(r)=e^{\pi r^2}$$

3. Conclusion

In conclusion, this paper explores the methods for deriving the volume of n-dimensional spheres, presenting two distinct approaches. By leveraging both Cartesian and spherical coordinates, we successfully obtained the general formula and analyzed the asymptotic behavior as the number of dimensions approaches infinity. Notably, as n increases, the volume of n-dimensional spheres tends to zero, offering a geometric interpretation of diminishing space occupation. Moreover, the sum of volumes of all even-dimensional spheres of a fixed radius connects intriguingly with the exponential function through a differential equation, highlighting potential applications in higher-dimensional mathematics and physics.

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