Proceedings of the 4th International Conference on Computing Innovation and Applied Physics
DOI: 10.54254/2753-8818/107/2025.22644

Understanding and Measuring UHI Using Numerical Methods for Solving the Heat Equation

Ziru Liao[1*], Songrui Liu[2], Junxu Tang[3], Lindsay Lingyuan Guo[4]

[1]*School of Mathematical Sciences, University of Science and Technology of China, Hefei, China*
[2]*College of Science, Northeastern University, Boston, USA*
[3]*Guanghua Cambridge International School, Shanghai, China*
[4]*Chinese International School, Hong Kong, China*
**Corresponding Author. Email: nickiel@mail.ustc.edu.cn*

Abstract: This paper explores the application of numerical methods to solve the two-dimensional heat equation, focusing on modelling the urban heat island (UHI) effect in Beijing, China. By utilizing the finite difference method (FDM) and solving finite difference equations iteratively using Python, our team simulated temperature variations between urban and rural zones and the heat transfer across different areas. Our results confirmed the presence of heat flow from urban to rural areas and the intensifying UHI effect over time.

Keywords: Urban Heat Island, 2D Heat Equation, Numerical Methods

1. Introduction

UHI refers to a metropolitan area that is considerably warmer than the surrounding rural areas. Urban areas take up about 0.5% of the Earth's land surface, yet they are home to more than 50% of the world's population [1]. According to research conducted by the Environmental Protection Agency, the annual air temperature of a city with 1 million people can be 1–3 degrees Celsius warmer than its surroundings [2]. UHI is now a common phenomenon in almost all urban areas in the world.

As indicated in Appendix 1, UHI is caused by multiple factors. First, a lot of energy is created when human beings concentrate on a certain area. Buildings are constructed with materials such as wood, cement, steel, and glass, which capture heat. Such insulation results in areas surrounding these structures being considerably warmer.

Secondly, there is “waste energy,” which comes from cars, buses, and trains that release heat into the environment. As people engage in their daily lives, the heat energy human activities give off makes city areas much warmer than less populated rural areas.

Thirdly, the density in cities also contribute to urban heat – with more people packed into smaller spaces, and buildings constructed closely together, heat generated linger in insulated buildings, leading to even higher temperatures.

Furthermore, with increasing urbanization, trees and greenery are replaced by buildings. The local weather and geographic location of the city are also contributing factors to UHI. Factors such as wind velocity, solar radiation, rainfall volume, climate, and topography all impact urban heat levels.

With such a backdrop, we wanted to research the case of Beijing, which is China’s capital and, with a population of over 22 million, the world’s eighth most populous city. Our hypothesis is that Beijing has an undeniable UHI problem. As indicated in Appendix 2, the red area is in the city center,

DOI: 10.54254/2753-8818/107/2025.22644

and not surprisingly, it is almost mostly developed with buildings. The green shaded areas are grassland and forest land, which are all outside the city. It is apparent that the city's central area lacks vegetation and water coverage, which makes evaporative cooling almost non-existent for city dwellers. As shown in Appendix 3 [3], the city's central areas have temperatures that are considerably higher than the surrounding rural areas, which have more vegetation and fewer people.

2. Research process

Our research process consisted of six steps:

1. Background research: we researched about urban heat island phenomenon and incorporated the UHI effect into a two-dimensional heat equation.
2. Problem definition: we designated a model using a rectangular area with urban and rural zones. We defined an area of 1000×1000 square meters as the study region. Within this area, we further designated a central 400×400 square meters square as the urban area, while the surrounding area represents the rural area.
3. Mathematical modelling: we applied the heat equation to model the temperature distributions.

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\big((t, x, y)\big) \in [0,2] \times [0,10] \times [0,10]$$

$$\alpha \textit{ is the thermal diffusivity } (\textit{for air, this constant is approximately } 2.1 \times 10^{-5} m^2/s).$$

where T is the temperature, t is time, x and y represent spatial coordinates, and α is the thermal diffusivity. The goal is to analyze how temperature evolves over time, especially across the urban-rural boundary.

4. Numerical method: we utilized the method of discretization -- Finite Difference Method (FDM).
5. Simulation setup: we implemented the initial conditions in the programming environment.
6. Results and interpretation: we presented the results in terms of temperature maps, graphs of temperature variation over time, and heat island intensity.

3. Methodology

3.1. Numerical methods

In terms of methodology for this research project, we used the Finite Difference Method ("FDM"). We lay out the detailed steps [4] below:

$$u_t = u_x$$

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x+\Delta x,t) - 2u(x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \tag{1}$$

$$u(x, t+\Delta t) = u(x,t) + \frac{\Delta t}{\Delta x^2}\big(u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)\big) \tag{2}$$

$$u_{i,j+1} = u_{i,j} + r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \tag{3}$$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \\ u_{5,j} \end{bmatrix} = \begin{bmatrix} -2u_{1,j} + u_{2,j} \\ u_{1,j} - 2u_{2,j} + u_{3,j} \\ u_{2,j} - 2u_{3,j} + u_{4,j} \\ u_{3,j} - 2u_{4,j} + u_{5,j} \\ u_{4,j} - 2u_{5,j} \end{bmatrix}$$

DOI: 10.54254/2753-8818/107/2025.22644

For 2D heat equation:

$$\frac{\partial T}{\partial t} = \alpha(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) \tag{4}$$

When T_0=0

$$T(x, y, 0) = \eta(x, y) \tag{5}$$

We apply a set of finite difference equations on a discrete grid with grid points $(x_i, y_j\, , t_n)$, where

$$x_i = i\ \Delta x\ , \qquad y_j = \mathrm{j}\ \Delta y, \qquad t_n = \mathrm{n}\ \Delta t$$

Let $T^n_{i,j} \approx \mathrm{T}(x_i, y_j\, , t_n)$ represent the numerical approximation at grid point $(x_i, y_j\, , t_n)$
One natural discretization of (3.4) would be:

$$\frac{T^{n+1}_{i,j} - T^n_{i,j}}{\Delta t} = \alpha\ (\frac{T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j}}{\Delta x^2} + \frac{T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j+1}}{\Delta y^2}) \tag{6}$$

$$T^{n+1}_{i,j} = \alpha\ \Delta t\ (\frac{T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j}}{\Delta x^2} + \frac{T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j+1}}{\Delta y^2}) + T^n_{i,j} \tag{7}$$

*In our research, we have let $\Delta x = \Delta y$

$$T^{n+1}_{i,j} = \alpha\ \frac{\Delta t}{\Delta x^2}\, (T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j} + T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j+1}) + T^n_{i,j} \tag{8}$$

So we let $\frac{\Delta t}{\Delta x^2}$ = r

$$T^{n+1}_{i,j} = \alpha\ \mathrm{r}\ (T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j} + T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j+1}) + T^n_{i,j} \tag{9}$$

3.2. Code explanation

By implementing the Finite Difference Method (FDM) in a 2D grid, we solved the heat equation using Python. Below is the code used to simulate heat transfer between urban and rural areas, along with a detailed explanation of each component.

3.2.1. Importing required libraries

```
import numpy as np
import matplotlib.pyplot as plt
```

3.2.2. Initializing grid and parameter

```
Lx, Ly = 10, 10  # Domain size
Nx, Ny = 100, 100  # Number of grid points
dx = Lx / (Nx - 1)  # Grid spacing in x
dy = Ly / (Ny - 1)  # Grid spacing in y
a = 0.1  # Thermal diffusivity constant
dt = 0.0001  # Time step
Nt = 20000  # Number of time iterations
alpha = a * dt / dx**2  # Stability criterion
```

DOI: 10.54254/2753-8818/107/2025.22644

The grid is defined with 100×100 points, representing the 1000×1000 square meters domain. **Lx** and **Ly** represent the length of the domain in the x and y directions. The thermal diffusivity **a**, time step **dt**, and the stability criterion **alpha** are set based on the heat equation's requirements.

3.2.3. Initial temperature

```
u = np.zeros((Nx, Ny), dtype=np.float64)# Initialize temperature grid

urban_size = 20  # Urban area size
urban_center_x = Nx // 2
urban_center_y = Ny // 2

# Set initial temperature in the urban area to 32.2°C
u[urban_center_x - urban_size//2:urban_center_x + urban_size//2,
  urban_center_y - urban_size//2:urban_center_y + urban_size//2] = 32.2

# Set initial temperature in the rural area to 30.6°C
u[u == 0] = 30.6
```

The temperature field **u** is initialized as a grid of zeros, representing the entire simulation domain. The urban region in the middle of the domain is assigned to an initial temperature of 32.2°C and the surrounding rural areas are set to 30.6°C, which represents the temperature disparity between urban and rural regions at the start of the simulation.

3.2.4. Temperature evolution function

```
def evolve_temperature(u, Nt, alpha, dx, dy):
    u_list = [u.copy()]  # Store initial state
    for n in range(Nt):
        u_new = u.copy()
        for i in range(1, Nx - 1):
            for j in range(1, Ny - 1):
                u_new[i, j] = u[i, j] + alpha * (
                    (u[i+1, j] - 2*u[i, j] + u[i-1, j]) / dx**2 +
                    (u[i, j+1] - 2*u[i, j] + u[i, j-1]) / dy**2
                )
        u = u_new.copy()

        if n % 5000 == 0:  # Save every 5000 iterations
            u_list.append(u.copy())

    return u_list
```

DOI: 10.54254/2753-8818/107/2025.22644

The **evolve_temperature** function updates the temperature values for each grid point over time. The finite difference method is applied to update the temperature based on neighboring points using the formula:

$$\boldsymbol{T_{i,j}^{n+1}} = \alpha\ \mathbf{r}\ (\boldsymbol{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}} + \boldsymbol{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j+1}^{n}}) + \boldsymbol{T_{i,j}^{n}}$$

The code iterates through all grid points (excluding the boundaries) and updates the temperature based on this discretization. Every 5000 iterations, the current temperature grid is stored for later analysis or visualization.

3.2.5. Visualization

```
u_list = evolve_temperature(u, Nt, alpha, dx, dy)

# Plot the final temperature distribution
plt.imshow(u_list[-1], cmap='hot', origin='lower')
plt.colorbar(label='Temperature (°C)')
plt.title('Temperature Distribution at Final Time Step')
plt.show()
```

After evolving the temperature field, the final state is plotted using **matplotlib** and is displayed as a heatmap.

4. Results and interpretation

The simulation of the UHI effect using the 2D heat equation successfully modeled the heat transfer between urban and rural zones over time. By implementing the FDM to solve the heat equation, the evolution of temperature distribution was observed (please refer to Appendix 4 [3]).

Initially, at time step zero, as shown in Fig 1 below, the urban area was set to 32.2°C, while the rural zone began at 30.6°C, representing the typical thermal disparity seen between urban centers and surrounding rural areas. After 5,000-time steps, the temperature within the urban zone remained significantly higher, although heat transfer to the rural areas had started. The urban region retained most of its heat, and diffusion outward was limited.

As the simulation progressed to 10,000-time steps, as shown in Fig 2 below, heat diffusion became more pronounced. The temperature in the urban center began to decrease as heat spread outward, causing a rise in the temperature of the rural areas. By 15,000- and 20,000-time steps, the temperature difference between the urban and rural areas further diminished as the system approached thermal equilibrium. The temperature of urban areas continued to decrease, and more heat transferred to the rural areas.

Between time steps 25,000 and 40,000, the temperature of both urban and rural areas continued to be the same. The previous significant temperature gradient has gradually decreased. It means that when the system moves toward a thermal equilibrium state, the UHI effect is enfeeble.

These findings show how heat transport behaves classically in a two-dimensional system. At first, the surrounding rural regions stayed cold while the metropolitan area continued to keep high temperatures. The temperature gradient gradually decreased as heat from the metropolitan region permeated into the rural areas. Early in the simulation, the UHI impact was more noticeable. However, as heat spread outward, the difference in temperature between the urban and rural areas decreased, and the system became closer to thermal equilibrium.

DOI: 10.54254/2753-8818/107/2025.22644

The simulation highlights the natural heat transfer mechanisms between urban and rural zones and sheds light on the dynamics of the UHI impact. Additional factors like wind, humidity, or various surface materials might be investigated in future research to improve the model's accuracy and suitability for real-world situations.

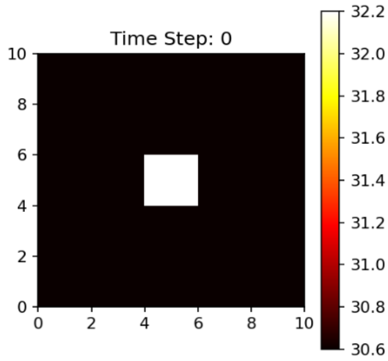


Figure 1: Initial temperature distribution at time step 0

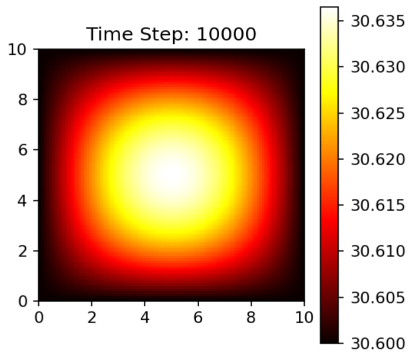


Figure 2: Temperature evolution at time step 10,000

5. Conclusion

In this study, we used numerical methods to solve the heat equation and applied it to the analysis of the urban heat island effect. Using the finite difference method, we simulated the temperature distribution in rectangular urban and rural areas. Our results demonstrated the differences in temperature patterns between urban and rural areas, reflecting the actual impact of urbanization on local climate.

The maps and graphs we cited clearly depict the increasing heat island effect over time while demonstrating the accuracy of the model. These provided valuable insights into understanding and addressing the environmental challenges posed by rapid urban growth.

Looking forward, further refinements to the model could enhance the productivity of our simulations, such as improving stability, using the Crank-Nicolson method, and incorporating more complex factors such as humidity, wind speed, and changing urban infrastructure. Ultimately, this work will provide a foundation for addressing urban heat challenges in future urban planning endeavors.

6. Author contributions

Ziru Liao and Songrui Liu contributed equally to this work and should be considered co-first authors.

DOI: 10.54254/2753-8818/107/2025.22644

Songrui Liu - coding and computation
Ziru Liao - background research and presentation
Junxu Tang - numerical methods research and report compilation
Lindsay Guo - background research and report compilation

References

[1] *Wang, K. (2017) Comparing the diurnal and seasonal variabilities of atmospheric, and surface urban heat islands based on the Beijing Urban Meteorological Network. Journal of Geophysical Research: Atmospheres. Volume 122 , Issue 4 . 2017. PP 2131-2154*

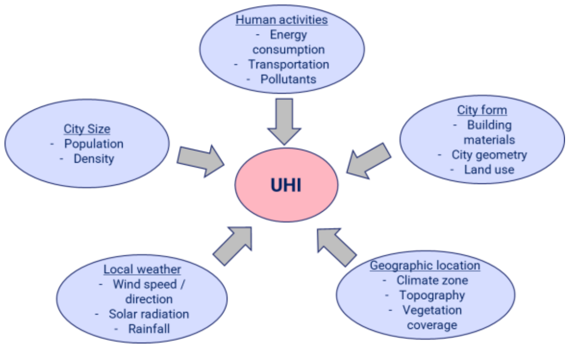
[2] *National Geographic Society. (2024) Urban Heat Island. http://www.education.nationalgeographic.org/resource/urban-heat-island/*

[3] *Zhang J Z, Liang Y N, Wang J. (2024) Temporal and spatial characteristics of urban heat island effect and its influencing factors in Beijing from 1981 to 2020[J]. Trans Atmos Sci, 47(4):581-591*

[4] *Randall J. LeVeque. (2007) Finite Difference Methods for Ordinary and Partial Differential Equation. Society for Industrial and Applied Mathematics, Philadelphia.*

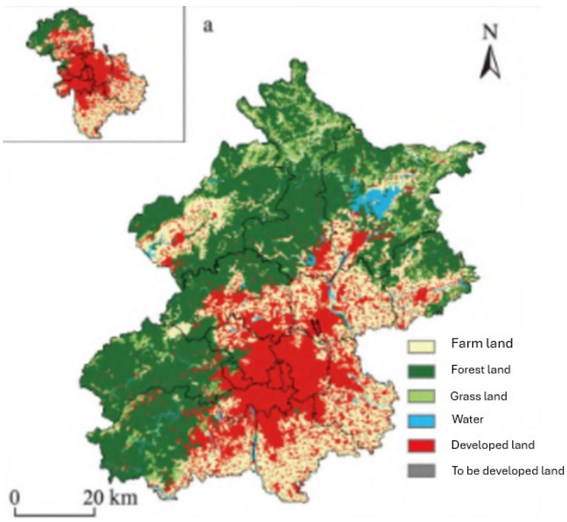
Appendix 1

What causes UHI?



Appendix 2

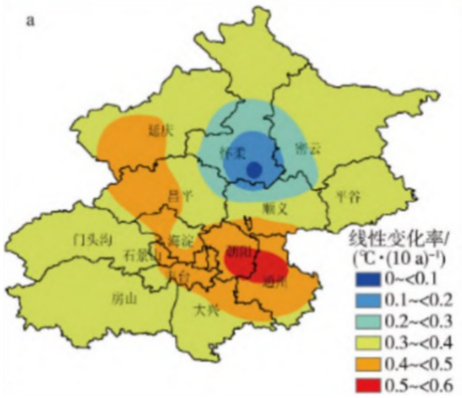
Land use of Beijing (2020)



Appendix 3

Spatial distribution of annual mean temperature

DOI: 10.54254/2753-8818/107/2025.22644



Appendix 4

Coding data output