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Mathematical Properties and Interactions of Catalan Numbers and Symmetry Groups

Xiangying Li

Rosedale Global High School, Markham, Canada
LXY58798437@163.com

Abstract: This study focuses on the history of Catalan numbers, including definitions, formulas, combinatorial meanings, and geometric interpretations. Catalan numbers are a sequence of natural numbers widely used in combinatorial mathematics, often used to represent arrangements of structures such as balanced brackets, binary trees and polygonal triangular dissections. In terms of the definition and properties of symmetry groups, the study includes the structure of orders and subgroups as well as representation theory. Symmetry groups describe the symmetry of geometric objects, including rotations, reflections, and axes of rotational reflections. The connection between Catalan numbers and symmetry groups, including influences and interactions, is further analysed.

Keywords: Catalan numbers, Symmetric groups, Mathematical characteristics, Connection

1. Catalan numbers and symmetry groups: history, definitions, and properties

1.1. The history of Catalan numbers

Catalan number originated in the 18th century when mathematicians gradually discovered this unique series while solving various combinatorial mathematical problems. [1] Initially, it only appeared in the solution of specific problems, but in the 19th century, as the study of mathematics continued, Catalan numbers began to receive more attention. Mathematicians began to systematically study their properties and patterns, and this period of research laid the foundation for the widespread use of Catalan numbers. [10]

1.2. The definition and characteristic of symmetry group

A symmetric group is a group consisting of all possible substitution operations on a given set. It follows that for a set containing an infinite number of elements, the elements of a symmetric group are the subject their possible permutations. For example, a set {1, 2, 3, 4, 5} has a symmetry group consisting of 10 elements corresponding to 10 different permutations of these five elements. And the symmetry group is closed in the sense that the result of a composite operation on any two elements of the symmetry group remains in the group. And every element in it has inverse element, through which it can be restored to the arrangement state at the beginning. Meanwhile, the order of the elements of the symmetric group can return to the smallest positive integer number of times after countless substitutions, but in the symmetric group, the order of different elements is also different. [2]

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2. Catalan numbers: definitions, formulas, combinatorial meaning and geometric interpretation

2.1. Definition and formula of Catalan numbers

Catalan numbers are a sequence of natural numbers. And the definition and derivation of formulas for Catalan numbers can be applied in combinatorial mathematics and practical applications. It can be found that the first few terms of Catalan numbers show certain patterns and trends. [2] When the number of terms is 0, the value of Catalan numbers is 1, which is the number of division methods for the empty set; when the number of terms is 1, the value is still 1, which corresponds to the number of division methods for a set of elements. [11] As the number of terms increases, the value of the Catalan number is increased, and the combinatorial meanings become more varied and complicated.

Where the number of terms is 2, the value of the Catalan number changes to 2. It reflects how many ways there may be to partition the set with two elements. When the number of terms is 3, the value is 5, denoting the number of ways of partitioning for a three-element set. [3] Starting from the numerical and combinatorial meaning of this change reflects different properties of Catalan numbers in different scale problems.

The Catalan numbers are a sequence of natural numbers arising in various counting problems, often involving recursively defined objects. The nth Catalan number is given by the formula.[4]

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

When $\binom{2n}{n}$ is a binomial coefficient, the formula goes as:

$$\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!}$$

Therefore, Catalan numbers can also be defined as:

$$C_n = \frac{(2n)!}{(n!)(n+1)!}$$

In accordance with the formula, we present the following table 1,

Table 1: Definition of Catalan numbers and list of values

Item count	Catalan number value
0	1
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1430
9	4862
10	16796

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2.2. Combinatorial meaning of the Catalan numbers

In combinatorial mathematics, Catalan numbers have many combinatorial meanings that not only provide a deeper understanding of mathematical structures and patterns, but also help us to solve problems in practical applications.

Catalan numbers play a key role in polygon-splitting problems. For an n-sided convex polygon, the Catalan number C_{n-2} represent the total number of different ways to partition it into triangles. [1]

$$C_{n-2} = \frac{1}{(n-1)}\binom{2(n-2)}{(n-2)} = \frac{(2n-4)!}{(n-1)!\,(n-2)!}$$

So, for a quadrilateral (n = 4), it can be divided into two triangles. There are two divisions, so the corresponding Catalan number is $C_2 = 2$.

For a pentagon (n = 5), it can be divided into three triangles. There are five ways to divide it, so the corresponding Catalan number is $C_3 = 5$.

Catalan numbers also have applications in stack sorting problems, where the number of schemes in which n elements are arranged in the order of entry and exit is precisely the Catalan number. A stack-out sequence refers to all possible permutations of a sequence obtained by performing in-stack and out-stack operations on the sequence through a stack. Moreover, A legal outgoing sequence is one that satisfies the condition that no prefix k can be preceded by a number greater than k. This is the ‘post-stack’ condition of the stack. This is a manifestation of the ‘last in, first out’ property of the stack. [5]

For a sequence of length n, the total number of legal outgoing stack sequences is the Catalan number C_n, which is calculated by the formula [1]

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

For the sequence [1] with n=1, there is only one possibility for the outgoing sequence [1], hence $C_1 = 1$.

For the sequence [1,2] with n = 2, there are two possibilities for the outgoing stack sequence [1,2] and [2,1], hence $C_2 = 1$.

For the sequence [1,2,3] with n = 3, there are five possibilities for a legal outgoing sequence: [1,2,3], [1,3,2], [2,1,3], [2,3,1] and [3,2,1], hence $C_3 = 5$.

The Catalan number C_n counts exactly these legal outgoing sequences, since each legal outgoing sequence corresponds to a unique binary tree structure, or other combinatorial structures related to the Catalan number, such as bracketed pairs, non-crossed pairs and so on.

Subsequently, the Dyck path is a classical combinatorial mathematical structure closely related to the Catalan number. [6] The Dyck path is defined as a lattice-point path starting from the far point (0,0) to the point (2n,0), consisting of steps of length (1,1) and (1, -1), and the path is always no lower than the x-axis. The total number of such paths is given precisely by the n-th Catalan number C_n. So according to the following equation,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

It can be found, when n = 1, the Dyck path with path length 2 has one and only one kind: one step up followed by one step down, denoted as (1,1) and (1, -1).

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When n=2, there are two possible Dyck paths with path length 4: the first path is to ascend twice and then descend twice; the second path is to ascend once, descend once, ascend once more, and descend once more. Neither path is ever below the x-axis, so $C_2 = 2$.

When n = 3, there are five possible Dyck paths of path length 6, corresponding to $C_3 = 5$.

So we can know that every legal Dyck path must satisfy the following conditions.

1. The path starts from the point (0,0) and reaches the point (2n,0).
2. The path consists of n steps (1,1) and n steps (1, -1), all 2n steps.
3. At each step of the path, the path cannot go below the x-axis, so the height of the path is always non-negative. [7]

These conditions ensure the uniqueness and legitimacy of the Dyck paths, and by constructing these paths recursively it is possible to derive the Catalan number calculation, it can be seen in the following table 2. That is, each Dyck path can be decomposed into a combination of sub-paths, a recursive structure that characterizes the Catalan number.

Table 2: Examples of the combinatorial significance of Catalan numbers

Application Scenarios	Catalan number value
Division of a convex n-edge	C_n
Stack sorting problems	C_n
Bracketing of binary trees	C_n
Dyck paths	C_n
Arrangement Avoidance Patterns	C_n
Nested Brackets	C_n
Triangular Sections	C_n
Diagonalizationn of convex polygons	C_{n-2}
Greve sequences	C_n
Perrin numbers	C_{n-1}
Scherbinski Triangle	C_n
Cartesian trees	C_n
Generalized Catalan numbers	$C_{n,k}$
Hyper-Catalan numbers	$C_{n,k}$
Multiple Catalan numbers	$C_{n,k}$

2.3. Geometric interpretation of Catalan numbers

In the field of mathematics, delving into the geometric interpretation of Catalan numbers allows for a better view of the patterns and properties underlying Catalan numbers.

Starting from triangle splitting, we find that the number of non-intersecting diagonals that divide a convex polygon into triangles is closely related to the Catalan number. [12] For triangle splitting, the value of the Catalan number is 1, which means that there is only one way to split in this simple geometry.

As the number of sides of the polygon increases, as in the case of convex quadrilateral splitting, the Catalan number changes to 2 and the number of splitting methods increases. In the case of convex pentagonal sections, the Catalan number reaches a value of 5 and there are more possibilities for sectioning. For hexagonal dissections, the value of Catalan number is 14, and for convex heptagonal dissections it is 42, which shows that as the complexity of the geometry increases, the value of Catalan number also increases rapidly, which shows that the variety and complexity of geometric dissections increases exponentially.

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The Catalan number is significant not only in polygonal profiles but also in tree structures. For instance, the number of binary trees with n leaves has a Catalan number value of 1, trinomials 2, quadtrees 5, and so on. These different types of geometries together with their respective values of the Catalan number give an indication of the wide applications that Catalan number has in geometry. [1]

3. Symmetry groups: order, subgroup structure, representation theory, and their role in group theory

3.1. Order and subgroup structure of symmetric groups

The symmetric group, being the most basic object of study in one branch of mathematics, has order and subgroup structure that find such a wide field of applications and great mathematical values. In particular, the elements of a theory of groups and the solution of the related mathematical problems stand at the heart of the investigation into symmetric group order and subgroup structure.[2]

The order in the symmetric group would reflect the number of its elements, while subgroup structure reflects the internal structure and regularity of the group. Analyzing symmetric groups of various orders allows us to trace certain tendencies or features.

Taking for example the low-order symmetric group, when the order is 2, the symmetric group order is 2, there is only 1 subgroup, namely the unit group. As the order increases, the number and type of subgroups also increase. To illustrate, when the order is 3, the symmetric group order is 6 and there are two subgroups, namely the cyclic group C3 and the cyclic group C2. [8]

As the order rises to 4, the symmetric group order is 12 and the number of subgroups reaches 5, including cyclic group C4, cyclic group C3, cyclic group C2, Klein's quaternion group, and the unitary group. At higher orders of the symmetric group, e.g., order 5, the type and number of subgroups increase further. As the order increases, the composition of subgroups becomes more complex and varied with certain patterns and trends as shown in table 3 below.

Table 3: Order and subgroup structure of the symmetric group

Order of the Symmetric Group	Degree of the Symmetric Group	Number of subgroups	Type of subgroup
2	2	1	Trivial group
3	6	2	Cyclic group C3 Cyclic group C2
4	12	5	Cyclic group C4 Cyclic group C3 Cyclic group C2 Klein quaternion group Trivial group
5	20	10	Cyclic group C5 Cyclic group C4 Cyclic group C3 Cyclic group C2 Kleinian quaternion group Trivial group
6	30	15	Cyclic group C6 Cyclic group C5 Cyclic group C3 Cyclic group C2 Kleinian quaternion group

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Table 3: (continued)

			Dihedral group D3 Trivial group
7	42	21	Cyclic group C7 Cyclic group C6 Cyclic group C5 Cyclic group C3 Cyclic group C2 Kleinian quaternion group Dihedral group D3 Trivial group
8	60	28	Cyclic group C8 Cyclic group C6 Cyclic group C5 Cyclic group C4 Cyclic group C3 Cyclic group C2 Kleinian quaternion group Dihedral group D4 Trivial group
9	84	36	Cyclic group C9 cyclic group C8 Cyclic group C6 Cyclic group C5 Cyclic group C4 Cyclic group C3 Cyclic group C2 Kleinian quaternion group Dihedral group D3 Dihedral group D4 Trivial group
10	120	45	Cyclic group C10 cyclic group C8 Cyclic group C6 Cyclic group C5 Cyclic group C4 Cyclic group C3 Cyclic group C2 Kleinian quaternion group Dihedral group D3 Dihedral group D4 Dihedral group D5 Trivial group

3.2. Representation theory of symmetric groups

The representation theory of symmetric groups constitutes a significant branch of group theory research. Its fundamental laws and structural features can be elucidated through an analysis of the representation types and eigenvalues of symmetric groups of varying orders.

From a fundamental perspective, the representations of the symmetric groups of lower orders, such as the symmetric groups of order 2 and order 3, are relatively straightforward. They primarily

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comprise one-dimensional prime representations with eigenvalues of 1. For instance, in the case of a symmetric group of order 4, in addition to the aforementioned one-dimensional prime representations, two-dimensional symbolic representations with eigenvalues of 1 and -1 also emerge. [8]

In the case of higher-order symmetric groups, for example those with orders 5, 6 and so forth, there exist not only one-dimensional prime representations and two-dimensional symbolic representations, but also more complex types of representations, such as three-dimensional regular expressions.

By analysing the representation types of symmetric groups of different orders, for example, the distribution and variation rules of eigenvalues reflect the symmetry and invariance of the symmetric groups, while different representation types in table 4 correspond to different mathematical properties.

Table 4: Representation theory of symmetric groups [8]

Order of the Symmetric Group	Representation type	Eigenvalue descriptions
2	1-dimensional trivial representation	1
3	1-dimensional trivial representation	1
4	1-dimensional trivial representation	1
4	2-dimensional symbolic representation	1, -1
5	1-dimensional trivial representation	1
5	2-dimensional symbolic representation	$1, \zeta, \zeta^2, \zeta^3, \zeta^4$
6	1-dimensional trivial representation	1
6	2-dimensional symbolic representation	$1, \omega, \omega^2$
6	3-dimensional regular representation	1, 1, 1
8	1-dimensional trivial representation	1
8	2-dimensional symbolic representation	1, i, -1, -i
8	3-dimensional regular representation	$1, \omega, \omega^2$
8	4-dimensional symbolic representation	$1, \zeta, \zeta^2, \zeta^3, \zeta^4$
10	1-dimensional trivial representation	1
10	2-dimensional symbolic representation	$1, \zeta, \zeta^3, \zeta^5, \zeta^7$
10	3-dimensional regular representation	$1, \omega, \omega^2$
10	4-dimensional symbolic representation	1, i, -1, -i
10	5-dimensional regular representation	1, 1, 1, 1, 1

3.3. The place of symmetry groups in group theory

Symmetric groups provide a rich paradigm for group theory. An in-depth study of symmetric groups enables us to understand the basic concepts and properties of groups more clearly. For example, the elements of symmetric groups, the structure of subgroups, and the rules of arithmetic help us construct a cognitive framework for groups in general. And many properties and characteristics of groups can be explored and understood by comparing and relating them to symmetric groups.

Moreover, symmetric groups are a basis in the classification of groups or proof of structure theorems. For complicated group structures, it can sometimes be useful to use the symmetry group itself as a vehicle for comparison and analysis of properties in attempting to gain information on how to solve the problem.

The study of symmetric groups played an important role in the development of substantial theorems and results in group theory. [13] For example, in discussions of basic notions, such as those of isomorphisms and homomorphisms of groups, symmetric groups frequently illustrate and at times even constitute the very proof of general theory.[2]

Besides, symmetry groups play a very important role in most of the areas of physics, chemistry, computer science, and many others. Examples include the use of symmetry groups in the study of quantum mechanics for defining particle symmetries. Group, likewise, and properties of symmetry groups are used in cryptography to define secure encryption algorithms.[2]

In addition, research on symmetry groups allows for integration and interdisciplinary communication with other branches of mathematics. The subject is closely connected with algebraic topology and combinatorial mathematics and hence provides powerful tools for solving interdisciplinary problems.

4. Catalan numbers and symmetry groups: applications, interactions, and theoretical insights

4.1. The application of Catalan numbers in symmetric groups

The application of Catalan numbers is extensive and diverse within the field of symmetric groups. The Catalan number can be used as a counting tool in combinatorial counting problems related to symmetric groups. To illustrate, when examining specific permutations, the definition and properties of Catalan numbers can be used to produce concise and accurate results as the table 5 respect. [9]

Table 5: Alignments to avoid certain patterns

Application Scenarios	Corresponding symmetry groups S_n	Formulas
Avoiding pattern alignment	There are no incremental subsequences in the permutation $(k+1)$	$C_n = \frac{1}{n+1}\binom{2n}{n}$
Alignment in standard Young's diagram	For Standard Young's graph for n frames	$C_n = \frac{1}{n+1}\binom{2n}{n}$
Convex polygonal division	Divide the $n+2$ side into n triangles.	$C_n = \frac{1}{n+1}\binom{2n}{n}$
Arrangement of nested brackets	Sequence of nested brackets of length 2n	$C_n = \frac{1}{n+1}\binom{2n}{n}$
Non-cross-matching	Number of non-cross pairs for 2n points	$C_n = \frac{1}{n+1}\binom{2n}{n}$
Monotonic stack sort	The number of permutations of length n that can be sorted by single-stack sorting	$C_n = \frac{1}{n+1}\binom{2n}{n}$
Arrangement of binary trees	Number of binary tree structures with n nodes	$C_n = \frac{1}{n+1}\binom{2n}{n}$

Set up a scenario to compute and show the Catalan number in a symmetric group using an arrangement of avoidance patterns. For example, avoiding permutations of incremental subsequences (3)

For the symmetric group S_3, we count the number of permutations of length 3 with no incremental subsequence (3). The Catalan number C3 describes the total number of these permutations.

All permutations of the symmetric group S_3,

(1,2,3) (1,3,2) (2,1,3) (2,3,1) (3,1,2) (3,2,1)

Avoiding permutations of incremental subsequences (3),

$$C_3 = \frac{1}{3+1}\binom{6}{3} = \frac{1}{4} \times 20 = 5$$

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So the number of permutations in S_3 that avoid any incremental subsequence (3) is 5. This corresponds to the value of the Catalan number C_3.

In addition, Catalan numbers have value in the representation theory of symmetry groups. In the study of the representation of symmetric groups, it is often necessary to consider the combination and arrangement of various elements, and the relevant properties of Catalan numbers can help to advance the study.

The irreducible representation of the symmetry group S_4 can be constructed by means of a standard Young's graph, where the Catalan number $C_4 = 14$ describes the number of specific combinatorial structures for n = 4.

The symmetric group S4 has 5 irreducible representations, which correspond to Young's graphs of:

(3,1) (2, 2) (2, 1,1) (1,1,1,1)

We list the irreducible representations of the symmetric group S_4 and their corresponding Young graphs in table 6:

Table 6: Irreducible representations and their corresponding Young's graphs

Young's diagram	Corresponding irreducible representation	Catalan number applications
(4)	All Arrangements	Describe a trivial representation
(3,1)	Incremental sequences are avoided in one arrangement (4)	Avoiding patterns with Catalan numbers
(2,2)	Arrangement of two identical elements	Counts associated with Dyck paths
(2,1,1)	Two nested subarrangements	Expression nested structure
(1,1,1,1)	Totally disjointed alignment	Describe the unit element of the symmetry group

4.2. Effect of symmetry groups on Catalan number

In the field of mathematics, it is important to investigate in depth the relationship between Catalan numbers and symmetry groups. From the point of view of combinatorial mathematics, the structure and properties of symmetry groups provide new perspectives for understanding the combinatorial significance of Catalan numbers. [9] The way the elements in the symmetry group are arranged and transformed is intrinsically related to the particular combinatorial problems described by Catalan numbers. For example, in some combinatorial problems concerned with permutations and combinations, their hidden pattern of Catalan number may even be reflected from the operating rule of a symmetry group.[1]

Based on these findings, checking into the order and subgroup structure of the symmetry group in terms of algebraic structure presents promising ways of computation and derivation of Catalan numbers. Perhaps the order character of the symmetry group and the composition of its subgroups can present a basis for derivation.

Besides, the representation theory about symmetric groups gave some impetus to the research of Catalan number. In this regard, the symmetry group representation can make some difficult problems transform into the operation of the matrix in the linear algebra, so it provides a very powerful tool for analyzing the properties of Catalan numbers. [8]

4.3. Interaction of Catalan numbers with symmetry groups

There indeed does exist a deep and intricate relationship between Catalan numbers and symmetric groups. In combinatorial mathematics, the combinatorial meaning of Catalan numbers is very often related to some permutations and combinations of elements of the symmetric group. To illustrate, in some special problems of permutation analysis, the Catalan number can be further used to determine the number satisfying some conditions. Many of these permutations are inherently interrelated with the transformations of elements in the symmetry group.[1]

Considering the algebraic structure, the nature of the symmetry group and the rules of arithmetic are important in yielding significant consequences for the theory of Catalan numbers. The order and subgroup structure of the symmetry group form the basic pillars upon which an understanding of Catalan numbers is constructed. The subgroup structure allows for discovery of the hidden patterns in the symmetry group, while discoveries, in turn, allow proof of how the Catalan numbers change in various contexts. [2]

Geometrically speaking, the geometric interpretation of Catalan numbers reflects the symmetry of its respective symmetry group. While the geometric figures are allowed to transform in the symmetry contained within the symmetry groups, the Catalan numbers hold a privileged value in terms of characterization and quantification of this geometric structure.

Also, the relationship between Catalan numbers and symmetry groups will have exemplary implications for the practical use of mathematics. For example, the designing and optimization of algorithms in computer science will be able to consider such a relationship to develop more advanced algorithms with high efficiency and accuracy.

Theoretically, a comprehensive discussion of the mutual relationship between Catalan numbers and symmetric groups might allow giant leaps in combinatorial mathematics and group theory, for the new theoretical results could possibly offer insights and methods in the solution of other mathematical problems.[9]

5. Conclusion

In conclusion, the interaction between Catalan numbers and symmetry groups is a many-dimensional and multilevel problem. Detailed analysis of the interplay between these two concepts has the propensity not only to develop an understanding of each separate mathematical entity but also to provide an explanation for the intrinsic relationships shared between both. Further research in this area may help explain the value of such concepts in the field of mathematics they pertain to and their practical applications.

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