Proceedings of the 3rd International Conference on Mathematical Physics and Computational Simulation
DOI: 10.54254/2753-8818/2025.22744

Calculation of Two Classes of Determinants by Reduction Method and Order-Increase Method

Yueting Jin

United World College, Changshu, China
ytjin24@uwcchina.org

Abstract: This paper explores two systematic approaches for computing determinants of structured matrices using Laplace Expansion. The reduction (recursion) method leverages recursive expansion to decompose high-order determinants into lower-order counterparts, exploiting structural repetition. This method simplifies complex calculations by iteratively applying Laplace Expansion. The order-increase (edge) method strategically augments matrices with auxiliary rows and columns to transform them into solvable forms. Examples include converting a 3-order determinant into an upper triangular matrix and extending a 4-order matrix into a Vandermonde determinant, enabling direct evaluation via established formulas. Both methods highlight how Laplace Expansion, combined with matrix structure insights, streamlines determinant computation. The reduction method is ideal for matrices with recursive patterns, while the edge method benefits determinants missing key rows/columns but amenable to structural augmentation. Practical applications span linear algebra, physics, and cryptography, where efficient determinant evaluation is critical. The paper underscores the pedagogical and computational value of these techniques, offering educators and researchers accessible strategies for tackling high-order determinants. Future directions include integrating these methods with computational tools and exploring broader interdisciplinary applications.

Keywords: Determinant, Laplace Expansion, Recursion Method, Order-increase Method, Vandermonde Determinant.

1. Introduction

Determinant is a basic and important concept in mathematics. Determinant is defined as function, or a scalar value that is associated with a matrix shaped like

$$D_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}. \tag{1}$$

The calculation of determinants is a critical problem in linear algebra. Afterwards, the function of the introduction of the determinant concept is not only to solve systems of linear equations but also to apply determinant theory to other mathematics fields, including matrices discussion, coordinate transformation, variable substitution for multiple integrals, differential equation systems, and quadratic models. In addition, determinants are also applied in variable crossing fields, like the

DOI: 10.54254/2753-8818/2025.22744

calculation in quantum physics, materials science, and cryptography. Determinant theory and calculation are very useful mathematical tools nowadays. In the future, with the innovation of computing technology and the growth of interdisciplinary demand, determinants will continue to play a key role in both theory and application.

Nowadays, researchers have developed many methods to calculate the value of determinants. Most of the latest research to calculate determinants combined some important mathematical idea with computer science and technology. According to Philip D. Powell, a method for expressing the determinant of block matrices and providing a systematic method for evaluating determinants that might otherwise be analytically intractable in 2011 [1]. There is another method to calculate determinants, with computer algorithm interpretation, by Armend Salihu in 2018 [2]. In addition, Lugen M. Zake Sheet created a method to calculate the determinant of a matrix by a permutation algorithm by fixing two components by computer in 2020 [3]. Furthermore, there are also some papers that utilized some special structures to calculate determinants, including the derivative method by Shi & Jiang and some comparison of mathematics thoughts made by Fang [4,5]. Another article by Bernard shows a method to calculate a special case of Vandermonde determinant in 2018 [6]. These researchers all made great progress in this field.

This paper aims to explain two methods to calculate two classes of special determinants with similar mathematics thought based on Laplace expansion. One is the reduction method, or the recursion method, and the other is the order-increase method, or edge method. This paper also presents some examples that are suitable to calculate with these two methods.

2. Reduction method

2.1. Theory of reduction method

In algebra, researchers always need to calculate high-order determinants. There are many basic rules to calculate some low-order determinants. For a 2-order determinant, shown as

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{2}$$

It can be calculated following the diagonal rule, or Sarrus rule, shown as $A_2 = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$. For high-order determinants, among which a prominent example is

$$D_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}. \tag{3}$$

For this determinant, one can solve by using the basic formula shown as

$$D_n = \sum_{j_1 j_2 \cdots j_n} (-1)^{N\left(j_1 j_2 \cdots j_n\right)} a_{1j_1} a_{2j_2} \cdots a_{nj_n}, \tag{4}$$

where $N(j_1 j_2 \cdots j_n)$ is the number of inversions of the number serie $j_1 j_2 \cdots j_n$, and a_{nj_n} is the element on the nth row and the j_n-th column. Obviously, when n is very large, the procedures of calculation will be complicated. To calculate the value of a determinant in an easier way, researchers have developed a lot of methods and rules to simplify it. To solve a class of high-order determinants that contain some special structures in their expressions, the reduction method, or recursion method, can be used to simplify this kind of determinants during calculation.

The reduction method, or recursion method, is a strategy to reduce the number of orders of the determinants by using Laplace Expansion combined with the recursion idea. According to Pak, K. &

DOI: 10.54254/2753-8818/2025.22744

Trybulec, Laplace Expansion, a high-order determinant shown in Eq. (2) can be expanded according to row i, which is

$$D_n = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}(i = 1,2,\cdots,n), \tag{5}$$

or expanded according to column j, which is

$$D_n = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}(j = 1,2,\cdots,n), \tag{6}$$

where a_{ij} is the element on the ith row and the jth column, and A_{ij} is the cofactor of the element a_{ij} [7]. The cofactor of an element a_{ij} is $A_{ij} = (-1)^{i+j}M$, where M is the minor of a_{ij}, which is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies.

For some high-order determinants, several-time edge reduction can be used directly to calculate the final value. However, for some complicated cases, the direct reduction method is not enough. Based on Laplace Expansion, recursion idea can be combined and used to calculate the value of a high-order determinant that has the similar structure in both of its original expression and a cofactor of it. Combined with the idea of recursion, researchers can apply Laplace Expansion to this class of determinants mechanically until the original determinant is simplified into the calculation of several 2 or *3* order determinants or some other special determinants. This method is extremely suitable to use in order to solve a determinant that contains a large number of zeros in the same row or column because the expanded expression according to a row or column that contains only a few non-zero elements will be very simple.

2.2. Application instances of reduction method

To explain this method specifically, there are two examples to show the basic steps of this method. The direct reduction method can be used to solve a determinant such as

$$D_4 = \begin{vmatrix} -1 & 2 & -2 & 1 \\ 2 & 3 & 1 & -1 \\ 2 & 0 & 0 & 3 \\ 4 & 1 & 0 & 1 \end{vmatrix}. \tag{7}$$

This determinant can be expanded according to the 3rd row as $D_4 = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34}$, where a_{ij} is the element on the ith row and the jth column, A_{ij} is the cofactor of the element a_{ij}. According to the definition of cofactor, in this case,

$$D_4 = 2(-1)^{3+1}\begin{vmatrix} 2 & -2 & 1 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} + 0 + 0 + 3(-1)^{3+4}\begin{vmatrix} -1 & 2 & -2 \\ 2 & 3 & 1 \\ 4 & 1 & 0 \end{vmatrix}. \tag{8}$$

By this step, a 4-order determinant can be rewritten into the calculation of several 3-order determinants. However, the typical calculation method of 3-order determinants is still very inconvenient in this case. The direct reduction method can be used continuously to simplify this determinant, such as

DOI: 10.54254/2753-8818/2025.22744

$$D_4 = 2\left(1\cdot(-1)^{3+1}\begin{vmatrix}-2 & 1\\ 1 & -1\end{vmatrix} + 0 + 1\cdot(-1)^{3+3}\begin{vmatrix}2 & -2\\ 3 & 1\end{vmatrix}\right) \\ -3\left(4\cdot(-1)^{3+1}\begin{vmatrix}2 & -2\\ 3 & 1\end{vmatrix} + 1\cdot(-1)^{3+2}\begin{vmatrix}-1 & -2\\ 2 & 1\end{vmatrix}\right). \tag{9}$$

After these steps, a 4-order determinant is simplified into the calculation of several 2-order determinants so that the final value which is -69 can be calculated out easily.

There is also an example which is suitable to use the reduction or recursion method, combined with recursion idea during the calculation process. There is a $2n$-order determinant shown as

$$D_{2n} = \begin{vmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & \ddots & 0 & 0 & 0 & 0 & ⋰ & 0 \\ 0 & 0 & a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & c & d & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d & 0 & 0 \\ 0 & ⋰ & 0 & 0 & 0 & 0 & \ddots & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & d \end{vmatrix}. \tag{10}$$

According to Laplace Expansion, D_{2n} can be expanded according to the 1st row as

$$D_{2n} = a\begin{vmatrix} a & 0 & 0 & 0 & 0 & b & 0 \\ 0 & \ddots & 0 & 0 & ⋰ & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 \\ 0 & ⋰ & 0 & 0 & \ddots & 0 & 0 \\ c & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d \end{vmatrix} - b\begin{vmatrix} 0 & a & 0 & 0 & 0 & 0 & b \\ 0 & 0 & \ddots & 0 & 0 & ⋰ & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & ⋰ & 0 & 0 & \ddots & 0 \\ 0 & c & 0 & 0 & 0 & 0 & d \\ c & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \tag{11}$$

These two $2n-1$-order determinants are still difficult to calculate directly. However, obviously, there is the same structure in these two determinants as that in the original determinant D_{2n}. In another point of view, there is only one element in the redundant row and column and this element lies on the intersection of these two lines in each determinant. Hence, the first $2n-1$-order determinant can be expanded according to the $2n-1$th row or column; at the same time, the second $2n-1$-order determinant can be expanded according to the element on the $2n-1$th row or column as

$$D_{2n} = ad\begin{vmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & \ddots & 0 & 0 & 0 & 0 & ⋰ & 0 \\ 0 & 0 & a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & c & d & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d & 0 & 0 \\ 0 & ⋰ & 0 & 0 & 0 & 0 & \ddots & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & d \end{vmatrix} - bc\begin{vmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & \ddots & 0 & 0 & 0 & 0 & ⋰ & 0 \\ 0 & 0 & a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & c & d & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d & 0 & 0 \\ 0 & ⋰ & 0 & 0 & 0 & 0 & \ddots & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & d \end{vmatrix}. \tag{12}$$

Notice that the structure of this $2n-2$-order determinant is similar to the $2n$-order determinant D_{2n} after this re-expansion. Based on the recursion idea, determinant D_{2n} can be expanded continuously as $D_{2n} = (ad-bc)D_{2n-2} = (ad-bc)^2 D_{2n-4} = \cdots$. In this process, determinant D_{2n-2k}, where k is natural number and $k < n$, has the similar structure to D_{2n}. After (n − 1) times expansion, the original determinant is rewritten into $D_{2n} = (ad-bc)^{n-1}D_2$ where $D_2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Finally, this determinant can be calculated by the diagonal rule, or Sarrus rule, shown as

$$D_{2n} = (ad-bc)^n. \tag{13}$$

In these cases, the reduction method, or recursion method, uses Laplace expansion to expand the original high-order determinants into lower-order determinants. The recursion idea is also combined to solve the class of determinants that have similar structures during the expansion process. Finally, all the high-order determinants can be simplified into 2 or 3-order determinants, or some other special determinants. In this way, the calculation of high-order determinants is simplified.

3. Order-increase method

3.1. Theory of order-increase method

The order-increase method, or edge method, is another strategy to calculate a class of high-order determinants. This strategy is also based on Laplace Expansion. However, compared to the reduction method, or recursion method above, the order-increase method applies Laplace Expansion inversely.

For some special m-order determinants, after adding one suitable row and one suitable column as the first row and first column, these $m+1$-order determinants can be written into determinants with special structures that can be calculated directly or can be simplified. In addition, the value of the $m+1$-order determinant after adding one row and one column is still equal to the value of original m-order determinant. The order-increase method is used to calculate this class of determinants. To guarantee the value of the $m+1$-order determinant, which is D_{m+1}, unchanged after adding one row and one column as the first row and first column, the elements on one of these two added lines, either row or column, should be $(1, 0, 0 \cdots 0)$.

For example, for this m-order determinant

$$D_{2m} = \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{vmatrix}, \tag{14}$$

if the added row is $(1, 0, 0 \cdots 0)$ and the added column is $(1, b_2, \cdots b_{n+1})$, D_n can be rewritten as

DOI: 10.54254/2753-8818/2025.22744

$$D_{m+1} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ b_2 & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m+1} & a_{m1} & \cdots & a_{mm} \end{vmatrix}. \tag{15}$$

According to Laplace expansion, if $D_{m+1}$is expanded according to the first line, D_{m+1}, can be simplified into

$$D_{m+1} = 1 \cdot \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{vmatrix} + 0 \cdot \begin{vmatrix} b_2 & a_{12} & \cdots & a_{1m} \\ b_3 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m+1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} + \cdots + 0 \cdot \begin{vmatrix} b_2 & a_{11} & \cdots & a_{1(m-1)} \\ b_3 & a_{21} & \cdots & a_{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m+1} & a_{m1} & \cdots & a_{m(m-1)} \end{vmatrix} \tag{16}$$

which is equal to the original determinant

$$D_m = \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{vmatrix}. \tag{17}$$

After adding edge, the new higher-order determinants can be observed to have some special structures. Hence, these determinants can be simplified easily.

3.2. Application instances of order-increase method

To explain this method specifically, there are two examples to show the basic steps of this method. There is a 4-order determinant, shown as

$$D_3 = \begin{vmatrix} b_1 & a_2 & a_3 \\ a_1 & b_2 & a_3 \\ a_1 & a_2 & b_3 \end{vmatrix}, \tag{18}$$

where $a_i \neq b_i$, $i = 1, 2, 3, 4$. Add one row and one column as its first row and first column, where the added row is $(1, 0, 0, 0, 0)$ and the added column is $(1, 1, 1, 1, 1)$, the value of this determinant is unchanged. Hence, A_3 can be expressed as

$$D_3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & b_1 & a_2 & a_3 \\ 1 & a_1 & b_2 & a_3 \\ 1 & a_1 & a_2 & b_3 \end{vmatrix}. \tag{19}$$

According to MIT OpenCourseWare,adding t times row i or column i to another row or column does not change the value of the determinant [8]. Hence, add $-a_1$ times of corresponding elements in the first column to the second column, add $-a_2$ times of corresponding elements in the first column

DOI: 10.54254/2753-8818/2025.22744

to the third column and add $-a_3$ times of corresponding elements in the first column to the corresponding elements in the fourth column, shown as

$$D_3=\begin{vmatrix}1 & 0-a_1 & 0-a_2 & 0-a_3\\ 1 & b_1-a_1 & a_2-a_2 & a_3-a_3\\ 1 & a_1-a_1 & b_2-a_2 & a_3-a_3\\ 1 & a_1-a_1 & a_2-a_2 & b_3-a_3\end{vmatrix}=\begin{vmatrix}1 & -a_1 & -a_2 & -a_3\\ 1 & b_1-a_1 & 0 & 0\\ 1 & 0 & b_2-a_2 & 0\\ 1 & 0 & 0 & b_3-a_3\end{vmatrix}. \tag{20}$$

This determinant is still complicated to calculate directly. However, obviously, the structure of this determinant is similar to an upper triangular determinant. The basic property of the determinant above can still be used to simplify this determinant. Add $-1/(b_1-a_1)$ times of corresponding elements in the second column, $-1/(b_2-a_2)$ times of corresponding elements in the third column and $-1/(b_3-a_3)$ times of corresponding elements in the third column to the corresponding elements in the first column. Hence, this determinant can be rewritten as

$$D_3=\begin{vmatrix}1+\dfrac{a_1}{b_1-a_1}+\dfrac{a_2}{b_2-a_2}+\dfrac{a_3}{b_3-a_3} & -a_1 & -a_2 & -a_3\\ 0 & b_1-a_1 & 0 & 0\\ 0 & 0 & b_2-a_2 & 0\\ 0 & 0 & 0 & b_3-a_3\end{vmatrix}. \tag{21}$$

According to Krista King Math, the value of an upper triangular determinant is the product of all elements on the main diagonal of this determinant [9]. Hence, the value of A_3 is calculated out:

$$D_3=\left(1+\frac{a_1}{b_1-a_1}+\frac{a_2}{b_2-a_2}+\frac{a_3}{b_3-a_3}\right)\left(b_1-a_1\right)\left(b_2-a_2\right)\left(b_3-a_3\right). \tag{22}$$

Another example for the order-increase method is shown as

$$A_4=\begin{vmatrix}1 & 2 & 3 & 4\\ 1 & 2^2 & 3^2 & 4^2\\ 1 & 2^3 & 3^3 & 4^3\\ 1 & 2^4 & 3^4 & 4^4\end{vmatrix}. \tag{23}$$

Notice that the structure of this determinant is similar to a Vandermonde determinant but misses some rows and columns. According to Li & Ding, Vandermonde determinant was named after Alexandre-Theophile Vandermonde, who is believed to be the founder of determinant theory [10]. For a positive integer $n\geq 2$, the Vandermonde determinant of order n is defined as follows

$$V_n=\begin{vmatrix}1 & 1 & 1 & \cdots & 1\\ x_1 & x_2 & x_3 & \cdots & x_n\\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1}\end{vmatrix}. \tag{24}$$

DOI: 10.54254/2753-8818/2025.22744

The value of a Vandermonde determinant can be calculated by this formula $V_n = \prod_{1 \le j < i \le n} (x_{i} - x_j)$. Hence, to calculate determinant A_4, researchers can try to add one row, which is $(1, 1, 1, 1, 1)$, as the first row, and one column, which is $(1, 0, 0, 0, 0)$ as the first column of A_4, shown as

$$V_n = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2^2 & 3^2 & 4^2 \\ 0 & 1 & 2^3 & 3^3 & 4^3 \\ 0 & 1 & 2^4 & 3^4 & 4^4 \end{vmatrix}. \tag{25}$$

By adding one row and one column, A_4 is transferred into a typical Vandermonde determinant. A_4 can be calculated according to the formula, shown as

$$\begin{aligned} V_n &= \prod_{1 \le j < i \le 4} (x_{i} - x_j) \\ &= (4-3)(4-2)(4-1)(4-0)(3-2)(3-1)(3-0)(2-1)(2-0) = 288 \end{aligned} \tag{26}$$

where $i, j \in \{1, 2, 3, 4\}$, x_i is the element on the second row and the ith column, x_j is the element on the second row and the jth column. Through the order-increase method, this class of determinants with special structures but missing several rows and columns can be simplified and calculated.

4. Conclusion

This paper explains two methods to calculate determinants based on Laplace Expansion with several cases respectively. One method is the reduction method, or recursion method, which is to reduce the order of determinants combined with the recursion idea. Another method is oder-increase method, or edge method, which is to add one row and one column to a special class of determinants and keep the value of the determinants unchanged. This method is suitable to calculate special m-order determinants, after adding one suitable row and one suitable column as the first row and first column, these $m + 1$-order determinants can be written into determinants with special structures that can be calculated directly or can be simplified. This paper shows some examples of this method step by step and points out properties of these two classes of determinants separately. This paper also concludes some basic computing skills in these two methods. However, there are still many questions that remain to be answered. For example, is there an easier and more generalized method to calculate determinants? Is there a better way to combine the calculation of determinants with computer science and technology? Is there a better strategy to apply determinant theory to other crossing fields? They are all valuable questions to research in the future.

References

[1] *Powell, P. D. (2011). Calculating determinants of block matrices. arXiv:1112.4379.*

[2] *Salihu, A., & Salihu, F. (2018). A method to calculate determinants, with computer algorithm interpretation. 2017 UBT International Conference, 79.*

[3] *Sheet, L. M. Z. (2020). Calculation the Determinants of matrix by Permutation Algorithm by fixing two components by Computer. IOP Conference Series Materials Science and Engineering, 928(4), 042023.*

[4] *Shi, D. Q., & Jiang, X. H. (2022). A method for calculating the derivative of a class of determinants. Science & Technology Information, 20(5), 185-188.*

[5] *Fang, X. M. (2018). Some comparisons related to determinants. Journal of Zhaoqing University, 39(5), 4-6+23.*

[6] *Ycart, B. (2018). A case of mathematical eponymy: the Vandermonde determinant. Revue D Histoire Des Mathématiques. 19(1), 43-47.*

[7] *Pak, K., & Trybulec, A. (2007). Laplace Expansion. Formalized Mathematics, 15(3), 143–150.*

DOI: 10.54254/2753-8818/2025.22744

[8] Lecture 18: Properties of determinants. (n.d.). MIT OpenCourseWare. Retrieved March 18, 2025, from https://ocw.mit.edu/
[9] Upper and lower triangular matrices. (n.d.). Krista King Math | Online Math Help. Retrieved March 19, 2025, from https://www.kristakingmath.com/blog/upper-and-lower-triangular-matrices
[10] Li, Y., & Ding, X. (2023). Vandermonde Determinant and its applications. Journal of Education and Culture Studies, 7(4), p16.