Proceedings of the 4th International Conference on Computing Innovation and Applied Physics
DOI: 10.54254/2753-8818/107/2025.22910

Replication Arguments in the Trinomial Tree Model

Kaiyang Zheng[1*†] , Ziyan Cui[2†] , Weifeng Hu[3†] , Zixuan Shi[4†]

[1] *School of International Education, Wuhan University of Technology, Wuhan, China*
[2] *School of Mathematics Science, Tongji University, Shanghai, China*
[3] *Economics and Management School, Wuhan University, Wuhan, China*
[4] *School of Natural Sciences, University of Manchester, Manchester, United Kingdom*
[*] *Corresponding Author. Email: zhengzky@whut.edu.cn*
[†] *All authors contributed equally to this paper.*

Abstract: Option pricing is a fundamental aspect of the financial derivatives markets, and accurate models are essential for effective risk management and fair market pricing. While the binomial model is widely used due to its simplicity, it has limitations in capturing the complexity of real-world markets, especially when dealing with non-linear payoff functions or multi-asset portfolios. The trinomial model allows the underlying asset price to go up, down, or remain unchanged at each time step, providing a more flexible and accurate framework. This paper discusses the application of the trinomial model in option pricing, focusing on the premise of using the trinomial tree model under the assumption of no arbitrage market, and derives the price boundary of options by constructing a portfolio consisting of bonds, stocks and simple options using the risk-neutral method and the no-arbitrage principle. The results provide valuable insights into option pricing and contribute to the theoretical understanding of option valuation in the discrete-time model. The resulting price boundaries help market participants identify potential arbitrage opportunities, ensure fair pricing, and ultimately enhance market efficiency. This work underscores the importance of considering more sophisticated models like the trinomial tree in option pricing, and sets the stage for further research into multi-period models and real-world market constraints.

Keywords: Option, trinomial model, risk-neutral method, no-arbitrage method

1. Introduction

An option is an important financial derivative that gives the holder the right to buy or sell an asset at a predetermined price within a specified period of time. Accurate pricing of options is essential for traders and financial institutions. Because of its simplicity and intuitiveness, the binomial tree model has long been a common tool for option pricing. However, the limitations of it appear gradually when dealing with nonlinear payoff function options or multi-asset portfolios.

To overcome these limitations, the trinomial tree model was introduced. The model allows the underlying asset price to go up, down, or stay the same at each time step. This improvement makes the trinomial model more flexible and more reflective of actual market conditions, thus providing greater accuracy when pricing options with nonlinear payoff functions.

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This paper aims to explore the application of the trinomial tree model in option pricing, focusing on how to deduce the upper and lower limits of option prices by the risk-neutral method and the no-arbitrage principle. Under the framework of the trinomial model, this paper establishes a reliable method to determine the price boundary of any option, which is essential for effective risk management and fair market pricing.

2. Related works

Since its introduction by Cox, Ross, and Rubinstein [1], the binomial model has been an important tool for option pricing. The core idea is to construct a discrete price path tree by assuming that the underlying asset price can rise or fall at each time step. This method is widely used in option pricing because of its simplicity and ease of understanding. However, the binomial tree model has limitations when dealing with complex option structures.

To overcome these limitations, the trinomial model was developed, introducing a third possible price movement where the underlying asset price can remain unchanged at each time step. This enhancement makes the model more flexible in dealing with actual market conditions and improves computational accuracy, especially for path-dependent options. Boyle [2] applied the trinomial tree model to option pricing and demonstrated that it is more effective in capturing asset price volatility compared to the binomial model.

In the development of option pricing theory, Black and Scholes [3] derived the theoretical price of options by constructing a replicating portfolio and introducing risk-neutral probabilities. Merton [4] further extended the mathematical foundations of the model. Subsequently, Ross [5] developed the arbitrage pricing theory, expanding the application of arbitrage-free pricing in asset valuation.

As research progressed, it was found that the price characteristics of complex options could be more accurately captured by introducing simple nonlinear options, such as standard call options, as additional replication tools. Carr and Madan [6] proposed constructing portfolios using a series of standard call and put options to more precisely price complex options with nonlinear payoffs. Therefore, a standard European call option is utilized in this work to obtain arbitrage-free price bounds for any option under the trinomial tree model.

3. Premise of unique solution in trinomial model

Similar to the actual market situation, in the trinomial model, the stock price is regarded as positive at all periods, and no dividends are paid by the underlying stock throughout the option's life.

Theoretically, a unique solution for replicating an option exists only if the payoff function F is linear. Non-linear payoff functions, such as those for typical options like European calls, can only be replicated if all possible stock prices at maturity are either entirely above or below the strike price, ensuring the linearity of the payoff over the relevant price range.

To apply the trinomial tree model in option pricing, we first need to define the portfolio weights N^S(number of stocks) and N^B (number of bonds). The premise of unique solution of N^S and N^B is:

$$\begin{cases} N^B(1+r) + N^S S_0 u = F(S_0 u), \\ N^B(1+r) + N^S S_0 m = F(S_0 m), \\ N^B(1+r) + N^S S_0 d = F(S_0 d). \end{cases} \tag{1}$$

where S_0 is the stock price at time 0 and r refers to the risk-free interest rate. Additionally, the factors affecting the stock price satisfy the inequality: $u > 1 + r > m > d > 0$, with u representing

DOI: 10.54254/2753-8818/107/2025.22910

the upward movement factor and the figures for middle and downward movement corresponding to m and d respectively.

To ensure that N^B and N^S have a unique set of solutions, the rank of the coefficient matrix must be equal to that of the augmented matrix, which should be 2. This condition leads to:

$$\frac{F(S_0d) - F(S_0m)}{d-m} - \frac{F(S_0m) - F(S_0d)}{m-d} = 0. \tag{2}$$

This implies that F must be linear. If F is concave:

$$\frac{F(S_0u) - F(S_0m)}{u-m} - \frac{F(S_0m) - F(S_0d)}{m-d} > 0,$$

and if F is convex:

$$\frac{F(S_0u) - F(S_0m)}{u-m} - \frac{F(S_0m) - F(S_0d)}{m-d} < 0.$$

Both cases contradict the condition above, which implies that F must be linear. When $F(S) = AS + B$, the option can be replicated exactly using $N^S = A$ stocks and $N^B = \frac{B}{1+r}$ bonds. This exact replication directly determines the option's price.

However, for non-linear payoff functions, a unique solution for N^S and N^B does not exist, meaning exact replication is not possible. In such cases, alternative pricing methods are needed. In the following sections, the work will explore the use of the no-arbitrage principle and the risk-neutral probability approach to determine the range of possible prices for options with non-linear payoffs.

4. No-arbitrage method

In this section, the work superreplicates and subreplicates an arbitrary option to calculate the range of possible option prices consistent with the no-arbitrage principle.

Initially, focus is placed on a specific European call option with a strike price $K = S_0m$. The result obtained will then be generalized to arbitrary options in following.

4.1. Pricing a call option at $K = S_0m$

Theorem 1. *In the trinomial model, the price of an European call option with $K = S_0m$ is bounded by:*

$$\left[S_0 - \frac{S_0m}{1+r}, \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0\right].$$

Proof: The call option payoff under the condition of $K = S_0m$ is given by:

$$N^B(1+r) + N^S S_0u = F(S_0u) = S_0(u-m) \tag{3}$$

$$N^B(1+r) + N^S S_0m = F(S_0m) = 0 \tag{4}$$

$$N^B(1+r) + N^S S_0d = F(S_0d) = 0. \tag{5}$$

The system of equations is overdetermined, which means that it is usually impossible to get an exact solution for N^B and N^S that satisfies all three equations, but only the price range. The next step is to prove the necessity and sufficiency of option price boundaries to ensure that accurate price ranges are obtained.

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4.1.1. Necessary proof

By using (3) and (5), the N^B and N^S can be figured out as:

$$N^S = \frac{u-1}{u-d}, \qquad N^B = \frac{1}{1+r}\frac{-dS_0(u-m)}{u-d}.$$

Substituting these into the payoff for S_0m:

$$F(S_0m)(ud) = N^B(1+r) + N^S S_0 = \frac{S_0(u-m)(m-d)}{u-d} \geq 0 = F(S_0m).$$

This indicates that the portfolio is more valuable than the actual payoff when $S = S_0m$, implying that the calculated option price $C_0(ud)$ is an overestimate, leading to the upper bound:

$$C_0 \leq \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0.$$

Similarly, with formula (3) and (4), a lower bound is derived:

$$C_0 \geq S_0 - \frac{S_0m}{1+r}.$$

In summary, the upper and lower bounds are:

$$S_0 - \frac{S_0m}{1+r} \leq C_0 \leq \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0.$$

4.1.2. Sufficiency proof

Assume the option price lies within the interval $[S_0 - \frac{S_0m}{1+r}, \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0]$. Let N^C denote the number of call options bought in the portfolio. Consider whether an arbitrage portfolio exists under these conditions, where the cost is zero and the payoff is always positive. These conditions can be expressed as:

$$N^B + N^S S_0 + N^C C_0 = 0, \tag{6}$$
$$N^B(1+r) + N^S S_0 u + N^C(S_0(u-m)) > 0, \tag{7}$$
$$N^B(1+r) + N^S S_0 m > 0, \tag{8}$$
$$N^B(1+r) + N^S S_0 d > 0. \tag{9}$$

In the first case, consider $N^C > 0$

Eliminating N^S using (6), and substituting the lower bound of C_0 into (7), the inequality still stands. Finally, it is obtained:

$$N^B(1+r-u) + N^C S_0 m(\frac{u-1-r}{1+r}) > 0, \tag{10}$$
$$\frac{(1+r-m)N^B}{C_0 m} > N^C, \tag{11}$$
$$\frac{(1+r-d)N^B}{C_0 d} > N^C. \tag{12}$$

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Given that $N^C > 0$ and $1 + r > d$, (12) implies $N^B > 0$. Furthermore, from (11), $1 + r - m > 0$. Simplifying, we get:

$$\frac{(1+r)N^B}{S_0 m} > N^C > \frac{(1+r)N^B}{S_0 m},$$

which is a contradiction.

In the second case, consider $N^C < 0$

If N^C is negative, similar reasoning shows that substituting the upper bound for C_0 into the inequalities results in:

$$\frac{(1+r)(u-d)N^B}{S_0 d(u-m)} > N^C > \frac{(1+r)(u-d)N^B}{S_0 d(u-m)},$$

which is also a contradiction.

In the third case, consider $N^C = 0$

If $N^C = 0$, the portfolio consists only of bonds and stocks, making it impossible to construct a risk-free arbitrage portfolio.

Therefore, the necessary condition is also sufficient for determining the price bounds of the option. This completes the proof.

4.2. Pricing an arbitrary option

Based on the results above, bonds, stocks, and a call option with $K = S_0 m$ can be used to replicate an arbitrary option and construct various portfolios.

According to the no-arbitrage principle, a portfolio can be constructed as follows:

$$\begin{cases} N^B(1+r) + N^S S_0 u + N^C C(S_0 u) = F(S_0 u) \\ N^B(1+r) + N^S S_0 m + N^C C(S_0 m) = F(S_0 m) \\ N^B(1+r) + N^S S_0 d + N^C C(S_0 d) = F(S_0 d) \end{cases}. \tag{13}$$

Where $C(S_0 u) = S_0(u - m)$, $C(S_0 m) = 0$, and $C(S_0 d) = 0$, $C(s)$ denotes the payoff of call option when stock price is s. This system can be rewritten in matrix form as:

$$\begin{pmatrix} (1+r) & S_0 u & S_0(u-m) \\ (1+r) & S_0 m & 0 \\ (1+r) & S_0 d & 0 \end{pmatrix} \begin{pmatrix} N^B \\ N^S \\ N^C \end{pmatrix} = \begin{pmatrix} F(s_0 u) \\ F(s_0 m) \\ F(s_0 d) \end{pmatrix}.$$

First, we calculate the determinant det of the coefficient matrix:

$$\det = (1+r)S_0^2(u-m)(d-m).$$

For Cramer's Rule to hold, the determinant must be non-zero:

$$\det \neq 0 \implies u \neq m \quad \text{and} \quad d \neq m.$$

Then, Using the Cramer's Rule, the solution to the system is:

$$\begin{cases} N^B = \frac{mF(S_0 d) - dF(S_0 m)}{(m-d)(1+r)} \\ N^S = \frac{F(S_0 m) - F(S_0 d)}{S_0(m-d)} \\ N^C = \frac{(m-d)F(S_0 u) - (u-d)F(S_0 m) - (m-u)F(S_0 d)}{(m-d)S_0(u-m)} \end{cases}.$$

4.2.1. Compute by range of C_0

By replicating the complex option, its price can be determined using the portfolio values obtained from the solutions:

$$\begin{aligned} V_0 &= N^B + N^S S_0 + N^C C_0 \\ &= [\frac{C_0}{S_0(u-m)}]F(S_0 u) \\ &+ [\frac{S_0(u-m)(1+r-d)-(1+r)(u-d)C_0}{(1+r)(m-d)(u-m)S_0}]F(S_0 m) \\ &+ [\frac{(m-1-r)S_0+(1+r)C_0}{(m-d)(1+r)S_0}]F(S_0 d). \end{aligned}$$

Substitute the lower bound $C_0 = S_0 - \frac{mS_0}{1+r}$ into V_0:

$$\begin{aligned} \underset{\sim}{V_0}\Big|_{C_0 = S_0 - \frac{mS_0}{1+r}} &= \left[\frac{1}{u-m} - \frac{m}{(u-m)(1+r)}\right]F(S_0 u) \\ &+ \left[\frac{(u-m)(1+r-d)-(1+r)(u-d)+m(u-d)}{(1+r)(m-d)(u-m)}\right]F(S_0 m) \\ &+ 0 \cdot F(S_0 d). \end{aligned} \tag{14}$$

Substitute the upper bound $C_0 = \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0$ into V_0:

$$\tilde{V_0}\Big|_{C_0 = \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0} = \left[\frac{1+r-d}{(1+r)(u-d)}\right]F(S_0 u) + 0 \cdot F(S_0 m) + \left[\frac{1+r-u}{(d-u)(1+r)}\right]F(S_0 d). \tag{15}$$

By considering the sign of N^C, we can determine the upper and lower bounds:

- When F is concave, N^C is positive, and the price of the complex option is minimized at $\underset{\sim}{V_0}$ with $C_0 = S_0 - \frac{mS_0}{1+r}$ and maximized at $\tilde{V_0}$ with $C_0 = \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0$.
- When F is convex, N^C is negative, and the price is maximized at $\underset{\sim}{V_0}$ and minimized at $\tilde{V_0}$.
- When F is linear, assume $F(s) = As + B$. Substituting into $\tilde{V_0}$ and $\underset{\sim}{V_0}$ yields $AS_0 + \frac{B}{1+r}$.

Because the arbitrary option's price is a linear function of C_0, the range of the arbitrary option's price obtained is both necessary and sufficient as C_0.

5. Risk-neutral method

Based on the principle of risk-neutral valuation [7, 8], the price of an asset should be equal to the present value of its expected future payoff under the risk-neutral probability measure. This is expressed as:

$$V_0(S_0) = \frac{1}{1+r}E^Q[F(S_1)].$$

Where Q represents the risk-neutral probabilities associated with the different possible outcomes of the asset's future value. The risk-neutral probabilities, denoted as q_u, q_m, and q_d for the up, middle, and down states, must satisfy the following equation:

$$S_0 = \frac{1}{1+r}[q_u F(S_0 u) + q_m F(S_0 m) + q_d F(S_0 d)].$$

Solve this equation set, the q_u and q_m represented by q_m as following:

$$\begin{cases} q_d = \frac{1+r-u-q_m(m-u)}{d-u} \\ q_u = \frac{d-1-r+q_m(m-d)}{d-u} \end{cases}.$$

The values of q_u, q_m, and q_d must all lie within the interval $[0,1]$, which constrains q_m to:

$$0 \leq q_m \leq \frac{1+r-u}{m-u}. \tag{16}$$

5.1. Pricing an at-the-money call

With the use of the risk-neutral probabilities, the price of a call option with strike price $K = S_0 m$ is given by:

$$\frac{1}{1+r}E^Q[F(S_1)] = \frac{(d-1)S_0}{(u-d)(1+r)}q_m + \frac{(1+r-d)(u-1)S_0}{(u-d)(1+r)} = Aq_m + B. \tag{17}$$

Since $\frac{d-1}{(u-d)(1+r)} < 0$, the original function decreases monotonically. The result is obtained:

· **Upper Bound**: When $q_m = 0$,

$$C_0 = \frac{1}{1+r}[\frac{1+r-d-q_m(m-d)}{u-d}S_0(u-m)] = \frac{(1+r-d)(u-m)}{(1+r)(u-d)}S_0. \tag{18}$$

· **Lower Bound**: When $q_m = \frac{1+r-u}{m-u}$,

$$C_0 = \frac{1}{1+r}[\frac{1+r-d-q_m(m-d)}{u-d}S_0(u-m)] = S_0 - \frac{mS_0}{1+r}. \tag{19}$$

Therefore, it is apparent that the conclusion of two methods is consistent.

5.2. Pricing an arbitrary option

With regards to an arbitrary option, the price is similarly determined by:

$$\begin{aligned} V_0 &= \frac{1}{1+r}\left[q_u F(S_0 u) + q_m F(S_0 m) + q_d F(S_0 d)\right] \\ &= \frac{1}{1+r}\left[\frac{d-1-r}{d-u}F(S_0 u) + \frac{1+r-u}{d-u}F(S_0 d)\right] \\ &+ \left[\frac{(m-d)F(S_0 u) - (m-u)F(S_0 d) + F(S_0 m)(d-u)}{(1+r)(d-u)}\right] q_m = Cq_m + D. \end{aligned}$$

Where C and D are constants derived from the specific payoff function F.

Then, substitute the lower bound $q_m = 0$ into V_0:

$$\underset{\sim}{V_0} = \frac{1}{1+r}\left[\frac{d-1-r}{d-u}F(S_0u) + \frac{1+r-u}{d-u}F(S_0d)\right]. \tag{20}$$

Substitute the lower bound $q_m = \frac{1+r-u}{m-u}$ into V_0:

$$\tilde{V_0} = \frac{1}{1+r}\left[\frac{d-1-r+(1+r-u)\left(\frac{m-d}{m-u}\right)}{d-u}F(S_0u) + \left(\frac{1+r-u}{m-u}\right)F(S_0m)\right]. \tag{21}$$

By considering the sign of the coefficient associated with q_m, this step can determine the upper and lower bounds for the option price:

· When F is concave, the coefficient of q_m is negative. Thus, the price of the complex option is maximized at $\underset{\sim}{V_0}$ when $q_m = 0$ and minimized at $\tilde{V_0}$ when $q_m = \frac{1+r-u}{m-u}$.
· When F is convex, the coefficient of q_m is positive. Consequently, the price is minimized at $\underset{\sim}{V_0}$ when $q_m = 0$ and maximized at $\tilde{V_0}$ when $q_m = \frac{1+r-u}{m-u}$.
· When F is linear, the price remains constant regardless of q_m, as the function is affine. In this case, substituting into $\tilde{V_0}$ and $\underset{\sim}{V_0}$ results in the same value, confirming consistency with the earlier derived bounds.

It should be noted that the key difference between the no arbitrage principle and the risk-neutral probability approach lies in their assumptions and applications. The principle of no arbitrage ensures that there is no risk-free profit by enforcing price consistency across markets. However, in real markets, transaction costs are unavoidable, and in some countries and regions, investors often cannot find an accurate portfolio to replicate derivatives due to short selling restrictions and liquidity constraints.

The risk-neutral probability method prices derivatives by using an assumed risk-neutral probability. The problem with this approach is that it does not provide investors with a portfolio to arbitrage when derivatives are overvalued or undervalued. Moreover, most investors are generally risk-averse rather than risk-neutral, which is reflected in the concavity of the utility function, and they usually require a certain premium to compensate for possible risk.

6. Conclusion

This paper examines the application of the no-arbitrage principle and risk-neutral probability method in determining the price range of options within the trinomial model framework. It is demonstrated that for a linear payoff function, an option can be precisely replicated by a combination of stocks and bonds, thus determining its price. However, for non-linear payoff functions, the upper and lower bounds of option prices are derived using two methods.

The determination of these price ranges provides derivatives market participants with a reasonable price expectation range, thereby mitigating the investment risk caused by price uncertainty. By clarifying the upper and lower bounds of the price, investors can more intuitively determine whether the option is overvalued or undervalued, and better control the risk.

On the other hand, these price ranges also help maintain the stability and efficiency of the market. When the option price deviates from the theoretical range, arbitrageurs in the market can correct the price deviation by buying or selling the relevant assets. This mechanism maintains the fairness and efficiency of the market to a certain extent, and also provides investors with more flexible and less risky investment choices.

DOI: 10.54254/2753-8818/107/2025.22910

However, there are also some limitations in this paper. The model only considers the price range of options under the one-period trinomial model. Future research should explore the boundaries over longer periods. In addition, the model assumes that investors are risk neutral, and there are no transaction costs and restrictions, which is still different from the real market. These simplifications highlight the need for further refinement and adaptation of the model to better capture the difference of practical market conditions.

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