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Calculation of Generalized Dirichlet Integral: From Special Cases to General Formula

Yue Yu

Beijing Royal School, Beijing, China
yuyue@st.brs.edu.cn

Abstract: The Dirichlet integral is widely used in the fields of mathematical analysis, probability theory and physics. This paper explores the calculation of the generalized Dirichlet integral from zero to infinity. The author focuses on transitioning from special cases to deriving a general formula. The methodology used mainly include substitution and integration by parts. In the simplification of formulas, trigonometric identities and Frullani integral are also used. Moreover, the author obtains the general formula by discussing the odd and even power cases respectively. This paper deduces the general formula of Dirichlet integral by using Euler's formula and binomial expansion. The result demonstrates that it uses special cases to find a general formula with the different order power and even the particular case of it, which is the same order power. The formula simplifies calculations. The significance of this paper lies in the calculation of Dirichlet integral general formula and various variations and give the answer. It provides an accurate formula for other studies using the this integral, and enhancing the overall body of knowledge in integral calculus.

Keywords: Dirichlet integral, Integration by part, Frullani integral, Euler's formula

1. Introduction

The development of calculus spans centuries, with its origins dating back to ancient Greece, Egypt and China, and Medieval India, Middle East, and Europe [1]. Early mathematicians laid the groundwork for integration by calculating volume and area. Calculus has greatly promoted the development of many fields. Thus, it became the core of modern mathematics. Among the many applications and studies of definite integrals, the Dirichlet integral is a very integration was first proposed by the German mathematician Dirichlet in his study of celestial mechanics. Dirichlet integral not only plays an important role in mathematical theory, but also has practical applications in signal processing, quantum mechanics and other fields.

Dirichlet integrals play an important role in many fields. First of all, the Dirichlet integral is used as the kernel function of finite impulse response filters [2]. This property makes Dirichlet cores very useful in signal processing, especially in scenarios where precise control of the frequency components of the signal is required. In addition, Dirichlet integrals can be used to deal with partial differential equations with complex geometries and boundary conditions. For example, Dirichlet integrals are used to construct and solve Helmholtz equations with Dirichlet boundary conditions [3]. In statistics, the Dirichlet integral is used to solve the mean distribution of a Dirichlet process [4]. This improves the ability to handle complex random processes and functions. Also, it is also used in many fields of error theory, including deriving the probability limit theorem, improving Laplace's approximation method, and dealing with the asymptotic distribution of the median [5]. Furthermore, the Dirichlet integral is applied to define the gradient measure in a general harmonic space [6]. It is also used to correlate harmonic structures and differential equations in Euclidean domains [7]. This shows that Dirichlet integrals provide important theoretical support for function analysis and differential equations in harmonic spaces. On the other hand, the boundary conditions of Dirichlet integrals are also used in some fields. Dirichlet boundary conditions transform complex partial differential equation problems into boundary integral equation by specifying the value of the wave function on the boundary [8].

This paper mainly shows the derivation of Dirichlet integral. The section 2 shows the numerical value of the definite integral result in the case of some particular example (the order of power). The section 3 shows the derivation of the general formula of Dirichlet integral with the different order of power and its particular case, which is the same order of power. By using the results in section 2 to test the correctness of general formulas demonstrating in section 3.

2. Special cases of dirichlet integral

2.1. Case of same order

By consider the formula of

$$I(n) = \int_0^\infty \left(\frac{\sin\xi}{\xi}\right)^n d\xi\,, \tag{1}$$

the author first takes the specific value of n, in which $n = 1$, $n = 2$, and $n = 3$ are calculated respectively. This takes into account both the cases where n is odd and even. By employing this approach, one can streamline the problem and identify underlying patterns. The following processes are the outcomes when n equals 1, 2, and 3.

2.1.1. Case of $n = 1$

When $n = 1$, $I(n)$ can be written as $\int_0^\infty \frac{\sin\xi}{\xi} d\xi$. Since $\int_0^\infty e^{-a\xi} da = \frac{1}{\xi}$ where a is a variable, while ξ is a constant in the formula. Thus $\int_0^\infty \frac{\sin\xi}{\xi} d\xi$ can be written like this:

$$\int_0^\infty \frac{\sin\xi}{\xi} d\xi = \int_0^\infty \sin\xi \int_0^\infty e^{-a\xi} da\, d\xi = \int_0^\infty \int_0^\infty \sin\xi\, e^{-a\xi} d\xi\, da \tag{2}$$

Then, the author assumes $\int_0^\infty \sin\xi\, e^{-a\xi} d\xi$ is J. By using the integration by part, one part of the formula (2) can be integrand as

$$J = \left[\frac{-1}{a} e^{-a\xi} \sin\xi\right]_0^\infty + \frac{1}{a}\int_0^\infty \cos\xi\, e^{-a\xi} d\xi \tag{3}$$

Since $\lim_{\xi\to\infty} \frac{-1}{a} e^{-a\xi} \sin\xi = 0$ and $\lim_{\xi\to 0} \frac{-1}{a} e^{-a\xi} \sin\xi = 0$. Thus, the formula can be simplified and also use integration by part:

$$J = \frac{1}{a}\int_0^\infty \cos\xi\, e^{-a\xi} d\xi = \frac{1}{a}\left[-\frac{1}{a} e^{-a\xi} \cos\xi\right]_0^\infty - \frac{1}{a}\int_0^\infty \sin\xi\, e^{-a\xi} d\xi \tag{4}$$

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This can be further simplified to $\frac{1}{a}\left(\frac{1}{a}-\frac{1}{a}\int_0^\infty \sin\xi\, e^{-a\xi}d\xi\right)=\frac{1}{a^2}-\frac{1}{a^2}\int_0^\infty \sin\xi\, e^{-a\xi}d\xi$. It is obvious that $J=\frac{1}{a^2}-\frac{1}{a^2}J$. Then, $J=\frac{1}{a^2+1}$ and substitute it into the formula (2), the author can find that

$$\int_0^\infty \frac{\sin\xi}{\xi}d\xi=\int_0^\infty \frac{1}{a^2+1}da=[\arctan a]_0^\infty=\frac{\pi}{2} \tag{5}$$

2.1.2. Case of $n = 2$

When $n=2$, $I(n)$ can be written as $\int_0^\infty \frac{(\sin\xi)^2}{\xi^2}d\xi$. In order to simplify this integral, the author first adopts the integration by parts method. In this example, the author expresses the integral as:

$$\int_0^\infty \frac{(\sin\xi)^2}{\xi^2}d\xi=\left[-\frac{(\sin\xi)^2}{\xi}\right]_0^\infty+\int_0^\infty \frac{2\sin\xi\cos\xi}{\xi}d\xi=\int_0^\infty \frac{\sin 2\xi}{\xi}d\xi \tag{6}$$

After using the integration by part, the author uses the trigonometric identity $2\sin\xi\cos\xi=\sin 2\xi$ to simplify the integral. Next, the authors further simplify the integral by substituting $u=2\xi$ and then $\frac{1}{2}du=d\xi$. With this substitution, the integral can be expressed as:

$$\int_0^\infty \frac{\sin 2\xi}{\xi}d\xi=\int_0^\infty \frac{\sin u}{u}du \tag{7}$$

Obviously, its expression is the same as that of $n=1$. According to Eq. (5), the result is

$$\int_0^\infty \frac{(\sin\xi)^2}{\xi^2}d\xi=\frac{\pi}{2} \tag{8}$$

2.1.3. Case of $n = 3$

When $n=3$, $I(n)$ can be written as $\int_0^\infty \frac{(\sin\xi)^3}{\xi^3}d\xi$. To simplify this integral, the author uses trigonometric $\sin 3\xi=3\sin\xi-4(\sin\xi)^3$. Thus, $(\sin\xi)^3=\frac{3\sin\xi-\sin 3\xi}{4}$ and $\int_0^\infty \left(\frac{\sin\xi}{\xi}\right)^n d\xi$ can be written in the form as

$$\int_0^\infty \left(\frac{\sin\xi}{\xi}\right)^3 d\xi=\frac{1}{4}\int_0^\infty \frac{3\sin\xi-\sin 3\xi}{\xi^3}d\xi \tag{9}$$

By using integration by part,

$$\frac{1}{4}\int_0^\infty \frac{3\sin\xi-\sin 3\xi}{\xi^3}d\xi=\frac{1}{4}\left(\left[-\frac{3\sin\xi-\sin 3\xi}{2\xi^2}\right]_0^\infty+\frac{1}{2}\int_0^\infty \frac{3\cos\xi-\cos 3\xi}{\xi^2}d\xi\right) \tag{10}$$

Due to $\lim_{\xi\to 0}-\frac{3\sin\xi-\sin 3\xi}{2\xi^2}=0$ and $\lim_{\xi\to\infty}-\frac{3\sin\xi-\sin 3\xi}{2\xi^2}=0$. So $\left[-\frac{3\sin\xi-\sin 3\xi}{2\xi^2}\right]_0^\infty=0$ and Eq. (10) continue to use integration by part. As shown:

$$\frac{1}{8}\int_0^\infty \frac{3\cos\xi - \cos 3\xi}{\xi^2} d\xi = \frac{1}{8}\left(\left[-\frac{3\cos\xi - \cos 3\xi}{\xi}\right]_0^\infty + \int_0^\infty \frac{-3\sin\xi + 9\sin 3\xi}{\xi} d\xi\right) \tag{11}$$

Also, $\left[-\frac{3\cos\xi - \cos 3\xi}{\xi}\right]_0^\infty = 0$. So, the solution is

$$\int_0^\infty \left(\frac{\sin\xi}{\xi}\right)^3 d\xi = \frac{1}{8}\left(-3\int_0^\infty \frac{\sin\xi}{\xi} d\xi + 9\int_0^\infty \frac{\sin 3\xi}{\xi} d\xi\right) = \frac{3\pi}{8} \tag{12}$$

2.2. Case of different orders

In previous research, an analysis was conducted for the specific case where $m = n$ of

$$f(m,n) = \int_0^\infty \frac{(\sin\xi)^m}{\xi^n} d\xi \tag{13}$$

for $m, n \in \mathbb{Z}^+$. The current research will extend to the scenario where $m \neq n$. In delving into the general formula for the integral $f(m,n)$, it is beneficial to initially examine specific values of m and n, particularly satisfying the condition $1 \leq n \leq m$. This methodological approach facilitates a deeper understanding of the integral's properties and provides a foundational basis for deriving more comprehensive solutions. The author will specifically analyze the following cases: $m = 3$ and $n = 1$, $m = 3$ and $n = 2$. The analysis of these specific examples will help people better understand the behavior of the integral and provide references for deriving a general formula. Extending this analysis to other values of m and n that meet the condition $1 \leq n \leq m$ allows for further validation and derivation of generalized results.

Case 1: When $m = 3$ and $n = 1$. $f(m,n)$ can be written as $\int_0^\infty \frac{(\sin\xi)^3}{\xi} d\xi$. By using trigonometric identities, it can be transformed into the form of

$$\int_0^\infty \frac{3\sin\xi - \sin 3\xi}{4\xi} d\xi = \frac{1}{4}\left(3\int_0^\infty \frac{\sin\xi}{\xi} d\xi - \int_0^\infty \frac{\sin 3\xi}{\xi} d\xi\right) \tag{14}$$

For the section $\int_0^\infty \frac{\sin 3\xi}{\xi} d\xi$. By substitution, let 3ξ be u. It can be $\int_0^\infty \frac{\sin u}{u} du$. For both $\int_0^\infty \frac{\sin\xi}{\xi} d\xi$ and $\int_0^\infty \frac{\sin u}{u} du$ are the same form of formula (5). So, they are equal to $\frac{\pi}{2}$. Then,

$$\int_0^\infty \frac{(\sin\xi)^3}{\xi} d\xi = \frac{1}{4}\left(3\int_0^\infty \frac{\sin\xi}{\xi} d\xi - \int_0^\infty \frac{\sin 3\xi}{\xi} d\xi\right) = \frac{\pi}{4} \tag{15}$$

Case 2: When $m = 3$ and $n = 2$. $f(m,n)$ can be written as $\int_0^\infty \frac{(\sin\xi)^3}{\xi^2} d\xi$. By using trigonometric identity, it can be transformed into the form of

$$\int_0^\infty \frac{\sin\xi\left(1 - (\cos\xi)^2\right)}{\xi^2} d\xi \tag{16}$$

According to the integration by part,

$$\int_0^\infty \frac{\sin\xi\left(1-(\cos\xi)^2\right)}{\xi^2}d\xi = \left[-\frac{\sin\xi\left(1-(\cos\xi)^2\right)}{\xi}\right]_0^\infty + \int_0^\infty \frac{3\cos\xi - 3(\cos\xi)^3}{\xi}d\xi \quad (17)$$

For $\lim_{\xi\to 0}\frac{\sin\xi(1-(\cos\xi)^2)}{\xi} = \lim_{\xi\to 0}\frac{(\sin\xi)^3}{\xi} = 0$, for $\lim_{\xi\to\infty}\frac{\sin\xi(1-(\cos\xi)^2)}{\xi}$, since $\sin\xi\,(1-(\cos\xi)^2)$ is bounded and $\frac{1}{\xi}\to 0$. So $\left[-\frac{\sin\xi(1-(\cos\xi)^2)}{\xi}\right]_0^\infty = 0$. By using trigonometric identity again for $3(\cos\xi)^3$, it is observed that

$$\int_0^\infty \frac{3\cos\xi - 3(\cos\xi)^3}{\xi}d\xi = \frac{3}{4}\int_0^\infty \frac{\cos\xi - \cos 3\xi}{\xi}d\xi \quad (18)$$

Since the integral satisfies the form of 'Frullani integral' $\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = f(0)ln\frac{b}{a}, (a, b > 0)$ [9], thus the solution of $\int_0^\infty \frac{(\sin\xi)^3}{\xi^2}d\xi$ is

$$\frac{3}{4}\int_0^\infty \frac{\cos\xi - \cos 3\xi}{\xi}d\xi = \frac{3}{4}\ln 3. \quad (19)$$

3. General cases of dirichlet integral

3.1. General formula

In order to obtain the general formula of the integral $f(m,n)$, this paper will use the method of classification discussion, respectively for the case of ever power and odd power, to derive the general formula of the integral in detail [10].

The author firstly starts with the integral of even powers in both numerator and denominator. Let $g(m,n) = \int_0^\infty \frac{(\sin\xi)^{2m}}{\xi^{2n}}d\xi$. First, the authors expand $(\sin\xi)^{2m}$ to make it easier to integrate. According to the Euler's formula $e^{ix} = \cos x + i\sin x$, $(\sin\xi)^{2m}$ can be expressed as $\left(\frac{e^{i\xi}-e^{-i\xi}}{2i}\right)^{2m}$. Then, through the binomial theorem, the expression of the complex function is expanded as

$$\left(\frac{e^{i\xi}-e^{-i\xi}}{2i}\right)^{2m} = \frac{1}{(2i)^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}\left(e^{i\xi}\right)^{2m-k}\left(-e^{-i\xi}\right)^k \quad (20)$$

In order to simplify $\frac{1}{(2i)^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}\left(e^{i\xi}\right)^{2m-k}\left(-e^{-i\xi}\right)^k$ further, the author breaks it down and simplifies it to $\frac{1}{(-4)^{2m}}\sum_{k=0}^{m-1}\binom{2m}{k}(-1)^k e^{-i(2m-2k)\xi}$. Such a resolution is conducive to further simplification by complex conjugation properties. Then, according to the Euler's formula $e^{ix} = \cos x + i\sin x$ again, and another form $e^{-ix} = \cos x - i\sin x$. The author combines two above forms as $2\cos\xi = e^{-i\xi} + e^{i\xi}$. Thus, the formula can be expressed as

$$\frac{1}{(-4)^{2m}} \times 2\sum_{k=0}^{m-1}\binom{2m}{k}(-1)^k\cos(2m-2k)\xi + (-1)^m\binom{2m}{m} \quad (21)$$

Since $(-4)^{2m} = (-1)^m 2^{2m}$, and then the author collated and substituted into the formula, the expression of $(\sin\xi)^{2m}$ can be obtained as

$$(\sin\xi)^{2m} = \frac{1}{2^{2m-1}(-1)^m}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\cos(2m-2k)\xi + \frac{1}{2^{2m}}\binom{2m}{m} \qquad (22)$$

Now, the author differentiates $(\sin^{2m})^{2n-1}$. For the term $\frac{1}{2^{2m-1}(-1)^m}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\cos(2m-2k)\xi$ of the expansion of $(\sin\xi)^{2m}$, its derivative is

$$\frac{1}{2^{2m-1}(-1)^m}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}(2m-2k)^{2n-1}\cos\left((2m-2k)\xi + \frac{(2n-1)\pi}{2}\right) \qquad (23)$$

According to the induction formula of trigonometric $\cos\left(\alpha + \frac{(2n-1)\pi}{2}\right) = (-1)^{n-1}\sin\alpha$, the author simplifies the formula above and obtains

$$g(m,n) = \frac{(-1)^{n-1}}{2^{2m-1}(-1)^m}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}(2m-2k)^{2n-1}\sin(2m-2k)\xi \qquad (24)$$

For another term of the expansion of $(\sin\xi)^{2m}$, $\frac{1}{2^{2m}}\binom{2m}{m}$ is a constant term. So, its derivative is zero. Thus, the whole expression of the derivative of $(\sin^{2m})^{2n-1}$ is (24). Then,

$$g(m,n) = \int_0^\infty \frac{(-1)^{n-1}}{2^{2m-1}(-1)^m}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}(2m-2k)^{2n-1}\sin(2m-2k)\xi\, d\xi$$
$$= \frac{(-1)^{m-n}}{2^{2m-1}(2n-1)!}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}(2m-2k)^{2n-1}\int_0^\infty \frac{\sin(2m-2k)\xi}{\xi}d\xi \qquad (25)$$

From the conclusion is the section two, $\int_0^\infty \frac{\sin a\xi}{\xi}d\xi = \frac{\pi}{2}$. Thus, $g(m,n)$ can be still simplified as

$$g(m,n) = \frac{(-1)^{m-n}\pi}{2^{2m}(2n-1)!}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}(2m-2k)^{2n-1} \qquad (26)$$

Now, the author will discuss the integral with odd power, which is $u(m,n) = \int_0^\infty \frac{(\sin\xi)^{2m+1}}{\xi^{2n+1}}d\xi$. By the similar deduction of the expansion will even power. $(\sin\xi)^{2m+1} = \frac{1}{2^{2m}(-1)^m}\sum_{k=0}^{m-1}(-1)^k\binom{2m+1}{k}\sin(2m+1-2k)\xi$. Next, the author differentiates $((\sin\xi)^{2m+1})^{2n}$, which is equal to

$$\frac{(-1)^{m-n}}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}(2m+1-2k)^{2n}\sin(2m+1-2k)\xi \qquad (27)$$

Then, the author plugs the $2n^{th}$ derivative in the same way as even power term and explores the expression as

$$u(m,n) = \frac{(-1)^{m-n}}{2^{2m}(2n)!}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}(2m+1-2k)^{2n}\int_0^\infty \frac{\sin(2m+1-2k)\xi}{\xi}d\xi \tag{28}$$

In addition, it is well-known that $\int_0^\infty \frac{\sin(2m+1-2k)\xi}{\xi}d\xi$ is the form of $\int_0^\infty \frac{\sin a\xi}{\xi}dx = \frac{\pi}{2}$. Thus $u(m,n)$ can be simplifies as

$$u(m,n) = \frac{(-1)^{m-n}\pi}{2^{2m}(2n)!}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}(2m+1-2k)^{2n} \tag{29}$$

The cases of odd power and even power have been discussed separately and the corresponding integral results have been obtained. Then, the general formula of $f(m,n)$ can be deduced by combining the expression (26) and (29). It can be concluded in a single formula that

$$\int_0^\infty \frac{(\sin\xi)^m}{\xi^n}d\xi = \frac{(-1)^{\frac{m-n}{2}}\pi}{2^m(n-1)!}\sum_{k=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(-1)^k\binom{m}{k}(m-2k)^{n-1} \tag{30}$$

3.2. Special cases

The first case is $m \neq 1$. However, the general formula for $I(n)$ is actually a particular case of $f(m,n)$ with $m = n$. Therefore, by analyzing the properties of $f(m,n)$ and substituting $m = n$, the author can derive a concrete general formula for $I(n)$ as

$$\int_0^\infty \left(\frac{\sin\xi}{\xi}\right)^n d\xi = \frac{\pi}{2^n(n-1)!}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^k\binom{n}{k}(n-2k)^{n-1} \tag{31}$$

The other case is $n \neq 1$. Through the above derivation, the author successfully derived the general formula of $f(m,n)$ and the expression of its particular case $I(n)$. the author deeply studies the problem of integrating $\sin x$ functions with different powers, especially for integrals $f(m,n)$ where m and n are positive integers. The author first uses the binomial theorem to express the ever power and odd power of $\sin x$ function as the linear combination of $\cos x$ function, and simplifies the integration process by integration by parts. In particular, the author pays attention to the particular case of $m = n$, $I(n)$. Thus, a simplified integral form is deduced.

In order to prove Eq. (30) and (31) are correct, the author substitutes the real case in the previous sectsion of the article for verification. The author substitutes the different order with $m = 3$ and $n = 1$, and the same order with $n = 2$ into the general formula respectively. The result demonstrates that when $n = 2$, $\int_0^\infty \left(\frac{\sin\xi}{\xi}\right)^2 d\xi = \frac{\pi}{4(2-1)!}\sum_{k=0}^{1}(-1)^k\binom{2}{k}(2-2k)^1 = \frac{\pi}{2}$. When $m = 3$ and $n = 1$, $\int_0^\infty \frac{(\sin\xi)^3}{\xi^1}d\xi = \frac{-\pi}{2^3}\sum_{k=0}^{1}(-1)^k\binom{3}{k} = \frac{\pi}{4}$. This is consistent with real case, so it passes the test.

4. Conclusion

The paper has systematically deduced the generalized Dirichlet integral. The paper finds the definite Dirichlet integral from zero to infinite for a particular case by first using some special examples. The author discusses and obtains the general formula of Dirichlet integral by diving even and odd power cases. The main body of the paper has presented various methods and advanced techniques for evaluating these integrals, emphasizing their importance in mathematical analysis. Through an in-depth examination of special cases, the author has derived a general formula that not only streamlines

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the calculation process but also broadens the applicability of the Dirichlet integral to a wider range of problems. In conclusion, this paper has provided valuable deriving process of the generalized Dirichlet integral and a foundation for further study and practical application in calculus.

Furthermore, there are still parts of the paper that could be improved. The paper hopes to use multiple methodologies to complete the proof, such as complex analysis and Fourier change. In addition, the results from different proving methods should be compare and analysis the difference and application situation between them, so as to make the conclusion more reliable. In addition, the paper can also discuss some real applications including in mathematics or other fields. By engaging with researchers in fields such as quantum mechanics, fluid dynamics and materials science, new avenues for applying the integral may emerge. It can show in detail how the Dirichlet integral is used and its importance.

References

[1] *Thompson, L. (2023). The History and Applications of Calculus (Doctoral dissertation, University of Oregon).*

[2] *Zamaruiev, V. V. (2013, April). The use of the dirichlet kernel in the control systems of active filters for industrial power line. In 2013 IEEE XXXIII International Scientific Conference Electronics and Nanotechnology (ELNANO) (pp. 363-366). IEEE.*

[3] *Ehrhardt, T., Nolasco, A. P., & Speck, F. O. (2011). Boundary integral methods for wedge diffraction problems: the angle 2π/n, Dirichlet and Neumann conditions. Operators and Matrices, 5, 1-40.*

[4] *Regazzini, E., Guglielmi, A., & Di Nunno, G. (2002). Theory and numerical analysis for exact distributions of functionals of a Dirichlet process. The Annals of Statistics, 30(5), 1376-1411.*

[5] *Fischer, H. (1994). Dirichlet's contributions to mathematical probability theory. Historia mathematica, 21(1), 39-63.*

[6] *Maeda, F. Y. (1977). Dirichlet integrals on general harmonic spaces. Hiroshima Mathematical Journal, 7(1), 119-133.*

[7] *Ostrowski, A. M. (1949). On some generalizations of the Cauchy-Frullani integral. Proceedings of the National Academy of Sciences, 35(10), 612-616.*

[8] *Chen, W., Li, J. P., & Fu, Z. J. (2017). Simulation of two-dimensional Dirichlet boundary scalar wave equation using time-dependent fundamental solution in the Singular Boundary Method. Chinese Journal of Computational Mechanics, 34(02), 231-237.*

[9] *Larson, C. (2017). An Appreciation of Euler's Formula. Rose-Hulman Undergraduate Mathematics Journal, 18(1), 17.*

[10] *Aleksanyan, H., Shahgholian, H., & Sjölin, P. (2015). Applications of Fourier analysis in homogenization of the Dirichlet problem: L p estimates. Archive for Rational Mechanics and Analysis, 215, 65-87.*