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Calculation of the Dirichlet Integrals: From Basic to Generalized

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Abstract: This paper conducts research centered on the integral problems involving trigonometric functions, with a focus on exploring the solution methods in the case of improper integrals. Firstly, it reviews the basic processing methods of definite integrals and improper integrals and introduces the commonly used techniques in solving such integrals, including the application of methods such as variable substitution, integration by parts, and "taking partial derivatives and then integrating". Subsequently, it introduces the definition and typical properties of the Dirichlet integral, providing theoretical support for the subsequent solution of specific problems. Through an in-depth analysis of three specific integral problems, it demonstrates how to transform complex integrals into known forms for solution. The research shows that the rational application of the ideas of function transformation and limits not only helps to simplify the calculation process but also effectively improves the accuracy and efficiency of problem-solving. The discussion in this paper has certain reference value for understanding the structural characteristics and solution paths of improper integrals involving trigonometric functions.

Keywords: Improper integral, Dirichlet integral, Trigonometric function, Integral transformation.

1. Introduction

Regarding the problem of the convergence and divergence of the Dirichlet integral, a large number of research literatures have conducted in-depth discussions on this. For instance, in the wrok by Xu [1], the authors specifically study the convergence problem of this kind of improper Dirichlet integral when the denominator in the integral is in the form of x^p. Specifically, they consider the integral in the form of:

$$\int_0^\infty \frac{\sin(x)}{x^p} dx \tag{1}$$

and prove that when the parameter satisfies $0 < p < 2$, this integral is convergent. This result indicates that within this parameter range, although the integrand exhibits strong oscillatory behavior and singularity near the origin, due to the interaction between the oscillation and decay, the integral can still attain a finite value. Therefore, integrating this function has practical mathematical significance and research value.

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In mathematical analysis, when dealing with integrals where the numerator contains trigonometric functions and the denominator contains powers of the independent variable, especially when the integration interval approaches a singular point (such as 0), one often encounters the dual challenges of function oscillation and singularity. Such integrals are widely found in fields such as Fourier analysis, quantum mechanics, signal processing, and vibration theory. The issues of their convergence and integrability are directly related to the theoretical solvability and computational feasibility of the relevant models. Therefore, in-depth research on such integrals has important theoretical and practical values. For example, Xu has used the value of the Dirichlet integral and combining with the Riemann-Lebesgue lemma [2], to prove that the Fourier series corresponding to a periodic function converges at its piecewise smooth points. What is more, Young revealed some applications of the dirichlet integral to the theory of surfaces [3].

In particular, by using appropriate variable substitutions or limit analyses to transform such integrals into classical integral forms with well-defined known results (such as the integral of $\sin x/x$), not only can the solution process be simplified, but it can also reveal the universal laws and essential structures underlying the complex integral forms. This idea of transformation reflects the strategy of "using simplicity to control complexity" in mathematical analysis and is one of the effective approaches to solving complex integral problems. Therefore, the research carried out around this idea not only helps to expand the scope of application of the theory of improper integrals but also provides a solid theoretical basis for analytical and numerical computations, which is of great significance for promoting the development of related fields.

In this paper, the author will first introduce the basic concepts of integrals and improper integrals. Then, the work will discuss a commonly used method for dealing with improper integrals and briefly review the definition and properties of the Dirichlet integral. Specificly, the author relies of the result of the classical interval result $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$, which is proved in Yu's paper [4]. Finally, through the specific solutions of three typical integral problems, the author demonstrates how to transform integrals involving trigonometric functions into classical forms, thereby simplifying the calculation of their values.

2. Method and theory

2.1. Basic definitions

This subsection aims to introduce some basic definitions in the realm of real analysis. For the Fundamental Theorem of Calculus, it is stated that

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2}$$

The basic formula of integration by parts:

$$\int u\ dv = uv - \int v\ du \tag{3}$$

In addition, one can also solve Definite Integrals with Parameters by Means of the First-order Derivative. The author shall first introduce the Leibniz integral rule. Let $f(x,y)$ and its partial derivative $\partial f(x,y)/\partial y$ be continuous in the rectangular region $a \le x \le b, c \le y \le d$. Then, the function

$$I(y) = \int_a^b f(x,y)dx \tag{4}$$

has a continuous derivative with respect to y in $[c,d]$, and the differentiation is exchanged as

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$$I'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y} dx, c \leq y \leq d \tag{5}$$

This theorem reveals that under some given conditions, an integration operation can be interchanged with partial differentiation, allowing one to first compute the partial derivative and then perform the integration. It provides a new method that can be used in calculating improper integrals.

2.2. Integration by part

Chen (2009) proved that $\int_0^\infty \sin x / x \, dx$ is convergent by applying integration by part [5]. It is worth taking a look at its method of transforming the original improper intergral into a new form using integration by parts.

First of all, the expression can be transformed, so that the integration by part can be applied. In view of $\frac{d}{dx}(-\cos x) = \sin x$, the author can transform the equation into $\sin x \, dx = d(1 - \cos x)$. Hence, $I = \int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{1}{x} \cdot d(1 - \cos x)$.

Secondly, the author applies the integration by part. Let $u = \frac{1}{x}$, $dv = d(1 - \cos x)$. It can be calculated that $du = \frac{d}{dx}\frac{1}{x}dx = -\frac{1}{x^2}dx$ and $v = 1 - \cos x$. The author then uses the basic formula of integration by parts Eq(a) and gets:

$$I = \left[\frac{1 - \cos x}{x}\right]_0^\infty + \int_0^\infty \frac{1 - \cos x}{x^2} dx \tag{6}$$

The author then tries to get the result of $A = \left[\frac{1-\cos x}{x}\right]_0^\infty$. The author first analyzes the limitation of $\frac{1-\cos x}{x}$ given that $x \to \infty$. Since $-1 < \cos x < 1$, $-2 < |1 - \cos x| < 2$. Then, the author gets $-\frac{2}{x} \leq \frac{1-\cos x}{x} \leq \frac{2}{x}$. According to the Squeeze Theorem, since when $x \to \infty$, $-\frac{2}{x} \to 0$ and $\frac{2}{x} \to 0$, the author gets $\lim_{x\to\infty} \frac{1-\cos x}{x} = 0$.

The author then analyzes the limitation of $\frac{1-\cos x}{x}$ given that $x \to 0$. Since when $x \to 0$, the numerator $x \to 0$ and the denominator $1 - \cos x \to 0$, it belongs to the indeterminate form of $\frac{0}{0}$. Therefore, the L'Hopital's Rule is needed to be applied. The author differentiates the numerator and the denominator respectively and obtains $\frac{d}{dx}(1 - \cos x) = \sin x$, $\frac{d}{dx}x = 1$. According to the L'Hopital's Rule, the author gets $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{\sin x}{1} = 0$. Then, it is calculated that

$$I = 0 + \int_0^\infty \frac{1 - \cos x}{x^2} dx = \int_0^\infty \frac{1 - \cos x}{x^2} dx \tag{7}$$

2.3. Introduction to a special form of improper integral

The author would like to introduce the Dirichlet integral. The basic form of Dirichlet integral, and the classic result of the integral is shown below:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \tag{8}$$

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The result of this interval has been proved in many ways, such as Chen used the methods of real analysis [5], Bradley (2021) applied the fundamental theorems in complex analysis [6], and Wang (2014) showed some traditional ways to prove the result [7]. To simplify the notation, the function is also written as $\operatorname{sinc}(x) = \frac{\sin x}{x}$. There is a conclusion can be derived based on this result, which is shown as follows

$$\int_0^\infty \frac{\sin(Ax)}{x} dx = \frac{\pi}{2} \tag{9}$$

To prove the equation, let $u = Ax$ then $x = \frac{x}{A}, dx = \frac{du}{A}$. The integral will change into $\int_0^\infty \frac{\sin(Ax)}{x} dx = \int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}$. There is also a classical result about the integral of the square of the integral of $\operatorname{sinc}(x)$, which is proved in Oliveria's study [8].

$$\int_0^\infty \operatorname{sinc}^2(x)\, dx = \frac{\pi}{2} \tag{10}$$

3. Results and applications

3.1. Solution to the integral $I_{x,z}$

Consider the integral [9]:

$$I_{x,z} = \int_0^\infty \frac{\sin(xt)\, e^{-zt}}{t} dt \tag{11}$$

To find this integral, the author first calculates the partial derivative of $I_{x,z}$ with respect to x, i.e., $\frac{\partial}{\partial x} I_{x,z} = \int_0^\infty \cos(xt)\, e^{-zt} dt$. To proceed further, one can calculate it by applying integration by parts. Let $u = \cos(xt)\,, dv = e^{-zt} dt$. It can be calculated that $du = \frac{d}{dt}\cos(xt)\, dt = -x \sin xt\, dt$, and $v = \int dv = \int e^{-zt} dt = -\frac{1}{z} e^{-zt}$. According to the basic formula of integration by parts Eq(a), Thus, the author gets:

$$\frac{\partial}{\partial x} I_{x,z} = \left[-\frac{1}{z}\cos(xt)\, e^{-zt}\right]_0^\infty - \frac{x}{z}\int_0^\infty \sin(xt)\, e^{-zt} dt = A + B \tag{12}$$

The first term is calculated as

$$A = \left[-\frac{1}{z}\cos(xt)\, e^{-zt}\right]_0^\infty = \lim_{t\to\infty}\left[-\frac{1}{z}\cos(xt)\, e^{-zt}\right] - \lim_{t\to 0}\left[-\frac{1}{z}\cos(xt)\, e^{-zt}\right] \tag{13}$$

Thus, $A = 0 - \left(-\frac{1}{z}\right) = \frac{1}{z}$. Likewise, to calculate the second term, one can apply the substitution method which yields

$$B = -\frac{x}{z}\int_0^\infty \sin(xt)\, e^{-zt} dt = -\frac{x}{z}\int_0^\infty [\sin(xt)]\left(-\frac{1}{z}\right)[(-z)\, e^{-zt} dt]$$

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$$= -\frac{x}{z}\int_0^\infty [\sin(xt)]\left(-\frac{1}{z}\right)\ de^{-zt} = \frac{x}{z^2}\int_0^\infty \sin(xt)\ d\,e^{-zt} \tag{14}$$

For further calculation, the author applies integration by parts. Let $u = \sin(xt)\,, dv = \ de^{-zt}$. Then, the author gets $du = \frac{d}{dt}\sin(xt) = x cos(xt)dt$, and $v = \int dv = \int\ de^{-zt} = e^{-zt}$. According to the basic formula of integration by parts Eq(a), Thus, it can be calculated that:

$$B = \left[\frac{x}{z^2}\sin(xt)\,e^{-zt}\right]_0^\infty - \frac{x^2}{z^2}\int_0^\infty \cos(xt)e^{-zt}\ dt \tag{15}$$

Then the author evaluates a definite integral using limits:

$$B = \lim_{t\to\infty}\left[\frac{x}{z^2}\sin(xt)\,e^{-zt}\right] - \lim_{t\to 0}\left[\frac{x}{z^2}\sin(xt)\,e^{-zt}\right] - \frac{x^2}{z^2}\int_0^\infty \cos(xt)e^{-zt}\ dt$$

$$= 0 - 0 - \frac{x^2}{z^2}\int_0^\infty \cos(xt)e^{-zt}\ dt = -\frac{x^2}{z^2}\int_0^\infty \cos(xt)e^{-zt}\ dt \tag{16}$$

As a result of the simplification, the author obtains that:

$$\frac{\partial}{\partial x}I_{x,z} = A + B = \frac{1}{z} - \frac{x^2}{z^2}\int_0^\infty \cos(xt)e^{-zt}\ dt \tag{17}$$

Then the equation can be rewritten as:

$$F = n + aF, \left(F = \frac{\partial}{\partial x}I_{x,z}, n = \frac{1}{z}, a = -\frac{x^2}{z^2}\right) \tag{18}$$

Isolating I on one side, the author gets $(1-a)F = n$. It is clear that $a \neq 1$ in this case. Hence, it can be traslated into $F = \frac{n}{1-a}$. To get the answer, the author substitutes the values, then it can be calculated that:

$$\frac{\partial}{\partial x}I_{x,z} = \frac{\frac{1}{z}}{1+\frac{x^2}{z^2}} = \frac{z}{x^2+z^2} \tag{19}$$

Since integration and differentiation are inverse operations, the author integrates the above equation:

$$I_{x,z} = \int_0^x \frac{\partial}{\partial x}I_{x,z}\,dx = z\int_0^x \frac{z}{x^2+z^2}dx = z\int_0^x \frac{1}{z}\arctan\left(\frac{x}{z}\right) = \int_0^x \arctan\left(\frac{x}{z}\right) \tag{20}$$

For the final step, the author applies the Fundamental Theorem of Calculus Eq. (2):

$$I_{x,z} = \arctan\left(\frac{x}{z}\right) - \arctan(0) = \arctan\left(\frac{x}{z}\right) \tag{21}$$

3.2. Case of m=4 & n=2

Consider the integral [10]:

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$$I = \int_0^\infty \frac{\sin^4(x)}{x^2} dx \tag{22}$$

According to $\sin^2 A + \cos^2 A = 1$, the author gets:

$$I = \int_0^\infty \frac{\sin^2(x) \cdot \sin^2(x)}{x^2} dx = \int_0^\infty \frac{\sin^2(x)}{x^2} dx - \int_0^\infty \frac{\sin^2(x)\cos^2(x)}{x^2} dx \tag{23}$$

According to $\sin\ 2A = 2\sin A \cos\ A$, the author gets:

$$I = \int_0^\infty \frac{\sin^2(x)}{x^2} dx - \int_0^\infty \frac{1}{4}\frac{\sin^2(2x)}{x^2} dx = \int_0^\infty \frac{\sin^2(x)}{x^2} dx - \int_0^\infty \frac{\sin^2(2x)}{(2x)^2} dx \tag{24}$$

According to $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$, the author gets:

$$I = \int_0^\infty \mathrm{sinc}^2(x)\, dx - \int_0^\infty \mathrm{sinc}^2(2x)\, dx \tag{25}$$

Let $u = 2x$, then $du = 2dx$. The author gets $dx = \frac{du}{2}$. Then, the author substitutes into the integral:

$$I = \int_0^\infty \mathrm{sinc}^2(x)\, dx - \int_0^\infty \mathrm{sinc}^2(u) \frac{du}{2} = \int_0^\infty \mathrm{sinc}^2(x)\, dx - \frac{1}{2}\int_0^\infty \mathrm{sinc}^2(u)\, du \tag{26}$$

Note that u is just a substitution variable, and the limits of integration remain from 0 to ∞. Therefore, one can rewrite it in standard notation $I = \int_0^\infty \mathrm{sinc}^2(x)\, dx - \frac{1}{2}\int_0^\infty \mathrm{sinc}^2(x)\, dx$. Then, according Eq. (3), the author gets:

$$I = \frac{1}{2}\int_0^\infty \mathrm{sinc}^2(x)\, dx = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \tag{27}$$

3.3. Case of m=2 & n=2

Consider the integral [9]:

$$I = \int_0^\infty \frac{\sin^2(x)}{x^2} dx \tag{28}$$

The author introduces the parameter t in the trigonometric part of the integral.

$$I(t) = \int_0^\infty \frac{\sin^2(tx)}{x^2} dx \tag{29}$$

From the above equation, the author has $I(0) = 0$. For further calculation, the author differentiates with respect to t:

$$I'(t) = \int_0^\infty \frac{1}{x^2} \cdot \frac{d}{dt}\left[\sin^2(tx)\right] dx = \int_0^\infty \frac{2\sin(tx)\cos(tx)}{x} dx \tag{30}$$

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According to $\sin 2A = 2\sin A \cos A$, the author gets $I'(t) = \int_0^\infty \frac{\sin(2tx)}{x} dx$. This integral is a standard result $\int_0^\infty \frac{\sin(Ax)}{x} dx = \frac{\pi}{2}$ $(A = 2t)$ with $I'(t) = \frac{\pi}{2}$. Then, the author performs integration on the above equation

$$I(t) = \int_0^t I'(t)\,dt + I(0) = \int_0^t \frac{\pi}{2} dt + 0 = \frac{\pi}{2}\int_0^t 1\,dt + 0 = \frac{\pi}{2}[t]_0^t = \frac{\pi}{2}t \tag{31}$$

Finally, the author sets $t = 1$ to get the original integral $I = I(1) = \frac{\pi}{2}$.

4. Conclusion

This paper systematically studies the integral problems involving trigonometric functions, especially the solution methods in the case of improper integrals. By reviewing the basic concepts of definite integrals and improper integrals, it introduces classical techniques such as variable substitution, integration by parts, and "taking partial derivatives and then integrating". Based on the Dirichlet integral theory, it analyzes the convergence characteristics under the interaction of oscillation and decay. Combining three specific typical integral problems, this paper demonstrates how to transform integrals with complex forms into classical integral forms with well-defined known results, thus effectively simplifying the calculation process and improving the solution efficiency. The research shows that the rational utilization of function transformation and the concept of limits is the key strategy for dealing with the problem that it is difficult to find the corresponding antiderivative in such integrals, making it impossible to use the conventional Newton-Leibniz method to solve the integral problem, which further reflects the idea of simplifying complexity.

Although this paper has achieved certain results in theoretical discussion and case analysis, there is still room for improvement. For example, the treatment of improper integrals for more general trigonometric function integrals (such as more complex situations like adding parametric functions, in the complex number field, etc., which cannot be covered by the author currently) still needs further exploration. In addition, this paper mainly focuses on analytical methods and does not involve the role and precision control of numerical methods in such integrals. In the future, numerical analysis tools can be combined to conduct in-depth research on the numerical solvability and error estimation of different types of improper integrals. Looking ahead, further exploration of the practical applications of such integrals in physical modeling and engineering calculations, as well as their extended forms in Fourier analysis and high-order vibration systems, will provide more theoretical support and practical paths for the interdisciplinary integration of mathematics and engineering sciences.

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