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A Review of Weierstrass Functions and Its Dimensions' Calculation

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Abstract: This paper mainly focuses on the Weierstrass Functions. Firstly, several properties of the Weierstrass Functions are introduced. Process of solving the conjecture suggested by Mandelbrot in 1977, that the graphs of Weierstrass type of functions have Hausdorff dimension $D_H = 2 + \frac{\log a}{\log b}$. Since existing studies have proved such result with additional conditions through various approaches, this paper provides necessary information indicating this evolving progress in methodology. Given the intertwined relationship between the two, this paper also includes different methods utilized to prove that the Box-counting dimension of the graphs of Weierstrass type of functions is the given number above. In addition, the possible application of Weierstrass Functions and its relation to, for example, financial market are also included. Providing the summary of works, this paper look forward to the final solution to this long-lasting conjecture.

Keywords: Weierstrass Function, Hausdorff dimension, Turbulence, Financial Market

1. Introduction

First suggested by Weierstrass in 1872, a type of function

$$w(x) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k x) \tag{1}$$

Where

$$0 < a < 1, ab \geq 1 + \frac{3}{2}\pi, b \in 2n + 1, n \in \mathbb{N} \tag{2}$$

is continuous but nowhere differentiable [1]. In 1916, Hardy rigorously proved that for all a and b, such that $0 < a < 1 < b$, and $ab \geq 1$, the above Weierstrass function is nowhere differentiable: first when b is an integer and then b in general cases [2]. In 1977, Mandelbrot pointed out the fractal nature of the Weierstrass functions [3]. Then it has always been conjectured that the Hausdorff dimension of the graph Weierstrass functions is D_H, as mentioned in almost all following papers. This study particularly refers to Falconer's [4]. Another form of nowhere differentiable function, Takagi function, will also be used quite often [5,6], its Hausdorff dimension is 1 [7]. Weierstrass functions are useful in may ways due to their periodicity, continuity, nowhere differentiability, and fractality.

This paper summarizes all the related essay on the topic of Weierstrass function from the theoretical progress on the proof of the Hausdorff dimension of the Weierstrass function to the application in real-life situations. These applications deeply rooted in the outstanding features of the

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function. By stressing the logical connection among essays, this paper points to a potential future direction of theoretical advancement and application.

2. Progress of the Hausdorff dimension of the Weierstrass functions

2.1. Definition

Definition 1 (Hausdorff Measure) Let S be a subset of $\mathbb{R}^n$ and d is a non-negative real number. For any $\delta > 0$, this study defines:

$$H_\delta^d(S) = \inf\{\sum_{i=1}^{\infty}|U_i|^d : \{U_i\} \text{ is a } \delta - \text{cover of } S\} \tag{3}$$

and

$$H^d(S) = \lim_{\delta\to 0} H_\delta^d(S) \tag{4}$$

Definition 2 (Hausdorff-Besicovitch dimension)

$$\dim_H S = \inf\{d \geq 0 : H^d(S) = 0\} = \sup\{d : H^d(S) = \infty\} \tag{5}$$

In 1937, Besicovitch and Ursell [8] proved the following theorem by using the d-measure and Heine-Borel theorem to construct a set of overlapping interval. Also, they studied $\phi_0(x) = \mathrm{dist}(x, \mathbb{Z})$ instead of the cosine curve in $w(x)$; most of the listed studies did not directly deal with Weierstrass functions, instead considering a broader class of functions. On the other hand, they proved that for an appropriate sequence of $\{b_n\}$ such that if $\frac{b_{n+1}}{b_n} \to \infty$ sufficiently slowly as $n \to \infty$, then the Hausdorff dimension is D_H.

Definition 3 The dimensional number d of the curve $y = f(x)$, where $f(x)$ belongs to the Lipschitz δ-class (Lip^δ), satisfies the inequality

$$1 \leq d \leq 2 - \delta \tag{6}$$

In 1992, following Besicovitch and Ursell, Ledrappier [9] also focused on $\phi_0(x) = \mathrm{dist}(x, \mathbb{Z})$ and specify the $\frac{b_{n+1}}{b_n}$ condition in Besicovitch and Ursell to b^n, that is $\frac{b_{n+1}}{b_n} = b$.

2.2. Alternative definition

In 1980, illustrating the graph of Weierstrass functions at different dimensions, including $D = 1.2$ and etc, Berry and Lewis introduced a "potential" definition of the dimension, which is the electrostatic energy of a unit-density positive charge uniformly covering the x-axis then displaced to the graph of $w(x)$, with a modified Coulomb law [10]. To give a better understanding of this, similar definition was also used by Orey in 1970, to prove that a Gaussian process has stationary increments and satisfies certain scaling properties; then its graph almost surely has a Hausdorff dimension of $2 - \alpha$, where α is the index of the Gaussian process [11]. This study also related to Taylor [12]. Here this definition is introduced with the following falconer [13].

Definition 4 s-energy at a point x of $\mathbb{R}^n$ on the mass distribution μ is

$$I_s(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^s} \tag{7}$$

Theorem 1. Let F be a subset of $\mathbb{R}^n$.

(a) If there is a mass distribution μ on F with $I_a(\mu) < \infty$, then $H^a(S) = \infty$ and $\dim_H F \geq a$.

(b) If F is a Borel set with $H^a(s) > 0$, then there exists a mass distribution μ on F with $I_a(\mu) < \infty$ for all $0 < t < a$.

In 1996, Hunt proved the following theorem, which is a huge step forward [14].

Theorem 2. If each θ_n is chosen independently with respect to the uniform probability measure in $[0,1]$, then the Hausdorff dimension of the graph of $w_{\theta_n}(x)$ is D_H, where

$$w_{\theta_n}(x) = \sum_{k=0}^{\infty} a^k \cos(2\pi(b^k x + \theta_n) \tag{8}$$

The proof of the lower bound utilized the definition of s-energy.

2.3. Attractor in dynamical system

Based on the results of J. Moser [15] in 1968 and Kaplan, Mallet-Paret, and Yorke [16] in 1984, $w(x)$ appears as attractors in dynamical systems. Most of the subsequent studies followed this path. Intuitively, an attractor is a set of the phase space of a system that all paths eventually end up at.

In 1986, Mauldin and Williams [7] proved for modified Weierstrass functions the following theorem stood, with the help of Zygmund's class.

Theorem 3. There exist a constant $C > 0$, the Hausdorff dimension of the graph of $W_b(x)$ is bounded below by $2 - \alpha - \frac{C}{\ln b}$, where

$$W_b(x) = \sum_{n=-\infty}^{\infty} b^{-\alpha n}[\phi(b^n x + \theta_n) - \phi(\theta_n)] \tag{9}$$

where $0 < \alpha < 1 < b$, each θ_n is an arbitrary number, and ϕ has period one.

In 1989, another lower bound was suggested by Przytycki and Urbański [17] that if $\varphi: I \to \mathbb{R}$, then $\dim_H(\text{graph } \varphi) \gg D(\alpha, \frac{C_4}{C_3}) > 1$, where $D(\alpha, \frac{C_4}{C_3})$ is a constant. In the same essay, Przytycki and Urbański also explored that if $b = 2$, and replace the cosine with a Rademacher function, then the Hausdorff dimension of the graph of $w(x)$ is equal to D_H, with other limiting conditions. There are also further discussions on the Hausdorff dimension of the Rademacher functions [18,19]. Similar to this specific passage, there are also several essays exploring self-affine sets where most of them exclude Weierstrass functions, but they are still worth noticing [20-22]. In 1986, Kôno used modified Takagi functions f to show that if f is a nearly self-affine function with other conditions, then the Hausdorff dimension of the graph of f is D_H [23].

In 1992, Ledrappier [9] introduced dynamical system and Markov partition into this question. Ledrappier specified to $\Gamma(\{b^n\}, \phi, s) = \{(x, y) y = \sum_{n=0}^{\infty} b^{n(s-2)} \phi(b^n x)$, where $b = 2, \phi = \phi_0, s = 1.5$ whose Hausdorff dimension is s. Also, he proved the following Corollary that is important with relation to Erdös number [24].

Theorem 4. Let 2^{1-s} be an Erdös number, then $\dim_H \Gamma_{s,\phi} = s$.

2.4. Recent progress of the Hausdorff dimension of the Weierstrass functions

In 2001, Liu [25] showed for the subsets of the graph of some similar functions has Hausdorff dimension equal to one. Without specifying to the case of b^n, in 2011, Baranski [26] focused instead on that if $\frac{b_{n+1}}{b_n} \to \infty$ as $n \to \infty$, the essay proved for f(x), that is $w(x)$ with its cosine curve replaced by some specific Lipschitz function, then

$$\dim_H(\text{graph } f) = \underline{\dim_B}(\text{graph } f) = 1 + \lim_{n\to\infty} \inf \frac{\log^+ d_n}{\log\left(\frac{b_{n+1} d_n}{d_{n+1}}\right)} \tag{10}$$

$$\overline{\dim_B}(\text{graph } f) = 1 + \lim_{n\to\infty} \inf \frac{\log^+ d_n}{\log b_n} \tag{11}$$

where

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$$d_n = a_1 b_1 + \cdots + a_n b_n \tag{12}$$

This conclusion is based on Carvalho [27], a subsequent important study was by Baranski, Barany, and Romanowska [28], using the Ladrappier-Young theory, based on the result by Tsujii [29], they proved that for integer $b \geq 2$, $\dim_H \mu_{a,b} = D_H$, for every a close enough to 1. In 2017, Keller pushed forward their conclusions [30]. A survey of previous results could also be found in Barański's paper [31]. These results clearly followed the study by Ledrappier on dynamical system.

In 2018, Shen proved the following theorem [32].

Theorem 5. For any integer $b \geq 2$, any and $a \in (b^{-1}, 1)$, the Hausdorff dimension of the graph of the Weierstrass function $w(x)$ is equal to D_H.

This also follows Ledrappier's theorem[9], that is

Theorem 6. Let $\phi: \mathbb{R} \to \mathbb{R}$,be a continuous, piecewise $C^{1+\alpha}$and $\mathbb{Z}$-periodic funciton. Assume that $\dim(m_x) = 1$ holds for Lebesgue a.e.x $\in (0,1)$, Then the Hausdorff dimension of the graph $f^{\phi}_{\lambda,b}$ is equal to D.

In 2021, Ren and Shen himself [33] proved a another theorem that went deeper than [28].

3. Alternative dimension

Definition 5. (Box-counting Dimension)

$$\dim_B(S) = \lim_{\delta \to 0} \frac{\log M(\delta)}{-\log \delta} \tag{13}$$

where $M(\delta)$ is the the number of boxes of length side δ to cover S.

Theorem 7. For a>1, the Weierstrass functions have Box-counting dimension $D_B = 2 + \frac{\log a}{\log b}$.

An straightforward proof of the theorem could be found in Falconer's Fractal Geometry: Mathematical Foundations and Applications; the proof of this theorem follows two propositions [13]. Also, for a rigorous and more complicated proof with connection to ergodic theory, this paper refer to Yorke [16], where the authors proved that the Box-counting dimension on the attracting torus is equivalent to the Lyapunov dimension.

The reason why the Box-counting dimension of the Weierstrass functions interest us so much is the following theorem.

Theorem 8.

$$\dim_H(S) \leq \dim_B(S) \tag{14}$$

Indeed, intuitively, Hausdorff dimension covers the set S with varied length covering, whereas Box-counting dimension fixes the length. Therefore, future discussion of the Hausdorff dimension of the graph of Weierstrass functions will almost certainly focus on its lower bounds.

Without doubt, there are other form of dimension definition, such as packing dimension and k-dimension, witch leads to [34-36].

4. Applications

In this section this paper demonstrate the process of using Weierstrass-Mandelbrot functions in application.

4.1. Turbulence

There have been many studies that apply fractal geometry to turbulence. In 1975, Mandelbrot [37] stated that turbulent scalar fields exhibit a fractal nature, with a fractal dimension of some iso-surfaces falling between 2 and 3 [38-42].

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The Weierstrass function captured the fractal nature of turbulence. There is no direct connection between studies, but this paper listed them below. In 1992, Hemphrey, Schuler, and Rubinsky argued that $w(x)$ represent the fractal component of turbulent velocity in both isotropic and anisotropic flows, such as applications in rotating disk flows. The choice of the Weierstrass function gave them the ideal irregularity in a turbulent velocity record [43]. In 1999, based on the result of Mauldin and Wiliams [7], Rocco and West showed that generalized Weierstrass functions are a solution to a fractional differential stochastic equation of motion [44]. In 2022, Liu, Shi and Hu applied $w(x)$ to the simulation of atomospheric scalar turbulence [45]. In 2025, Cai et al. utilized $w(x)$ to simulate typhoon wind speeds. Based on the Weierstrass function, they also provided the comparison between different methods of calculating the dimension, including box-counting method [46]. In the essay, they also summarized the applied method of several studies in calculating the dimension of the wind. To be more specific, in 1994, Sarkar conducted a comparison among existing methods in calculating the fractal dimension of an image and suggested an efficient differential box-counting approach [47].

4.2. Financial market

In 2007, Mandelbrot [48] discussed detailedly in his book The Misbehavior of Markets: A fractal view of financial turbulence that to measure the market behavior, fractals could be a more effective way than traditional methods [49-51]. A detailed example of using fractal analysis could be found in Banerjee and Mulligan's paper [52].

In 2005, based on the Lomb analysis of a Weierstrass-type function [53], Bartolozzi et al., argued that the spectral pattern of the daily closure of the four most important indexes can be captured by Weierstrass function [54]. This research was based on the results given by Zhou and Sornette in 2003, that the Weierstrass function could capture some specific features of the stock market since 2000 [55]. In 2015, Zhang, Yu, and Sun suggested that $w(x)$ could capture the tendency and variation of actual stock market indexes with different Hausdorff dimensions, by changing the values of a and b. The dimension here played an important roles in simulating the market behavior [56]. In 2023, Zhang explored the effect of disturbance on the economic and financial system using $w(x)$ [57].

4.3. Other studies

In 1990, Majumdar and Tien applied $w(x)$ with different Hausdorff dimensions to measure the roughness of both Brownian and non-Brownian rough surfaces, since $w(x)$ demonstrated both features of continuity, non-differentiability, and self-affinity, which are desirable [58]. In 2012, Jiang and Zheng used the Weierstrass fractal function to analyze the thermal contact resistance of rough surfaces [59].

4.4. Algorithm

In 2017, Dong, Ju, and Gao suggested that the cuckoo search algorithm could be adopted to determine the dimension since, as this paper suggested, the Hausdorff dimension of the Weierstrass function is still not proved. Specifically, the cuckoo algorithm could avoid using the box-counting method that fixs the length of each cover; instead, it gives a more “Hausdorff-like” measure. Therefore, it provides a more precise figure of the Weierstrass function [60].

5. Conclusion

This paper noticed the peapers by Qiu Y. and Liang Y. [61], they provide a general summary of all fractal dimensions related to this problem in chronological order but not that specific and logical, whereas our essay provides a much more in-depth understanding of the logic between results and

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focuses more on Hausdorff dimension. The conjecture has not been proved up till now; yet the Weierstrass function was widely used in various fields of physics and finance. It is hard to find a function with such distinctive features as the Weierstrass function. It is this versatile nature of the function that promotes its appearance in and relation to various areas and different approaches in solving the conjecture. These approaches are all connected and demonstrate a diversity. Also, different measures or dimensions give rise to different results when calculating the dimension of the Weierstrass function, rendering a huge disparity among the difficulty of calculation and application in reality.

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