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Investigating Special Values of Riemann Zeta Function at Even Arguments

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Abstract: This paper discusses the core properties and specific values of the Riemann zeta function, $\zeta(s)$, which is central to analytic number theory and prime number distribution. This paper begins with the Basel Problem, a historic problem about the sum of the reciprocals of the squares of natural numbers. Leonhard Euler's neat solution to the Basel Problem revealed $\zeta(2) = \pi^2/6$ and opened the window of interactions from fundamental geometric constants to infinite series. After providing Euler's intuition, the paper explores the generalization of $\zeta(2n)$ through many avenues, emphasizing the value of Bernoulli numbers for fluid closed-form expressions. The author fully verifies how power series expansions, contour integration, and Fourier series are all converging into the math leading people to the zeta-related identities. The author also discusses where the Riemann zeta function is used outside of pure math. In physics, especially quantum and statistical modeling, the author explores computational techniques. These are essential for advancing people's understanding of prime number distribution and modern cryptography, while also observing connections in math from past to future.

Keywords: Riemann zeta function, Analytic number theory, Basel problem, Bernoulli numbers

1. Introduction

The Riemann zeta function $\zeta(s)$ is fundamental in analytic number theory, linking many fields including complex analysis, number theory, and probability. It first appeared in Euler's 1734 solution to the Basel Problem [1]. This function has played an important role in understanding prime numbers, due to its deep relation to the distribution of primes. The Basel problem asks whether the sum of the reciprocals of the squares of the natural numbers $\sum_{n=1}^{\infty} 1/n^2$ was finite, in 1644, and Euler correctly identified the sum $\frac{\pi^2}{6}$ in his beautiful solution. He opened a never-before-seen connection between seemingly unrelated fields of mathematics [2].

Bernhard Riemann, expanding on Euler's groundwork, mapped the zeta function to the complex plane, which led to the (infamous) Riemann Hypothesis from 1859. This hypothesis states that all of the non-trivial zeros of the zeta function have their real component equal to $1/2$. The conjecture is still open but has extremely far-reaching implications in modern-day mathematics, especially for its views in prime distribution. The zeta function, denoted by the series $\sum_{n=1}^{\infty} 1/n^2$ for Re(s) > 1, has a

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well-defined analytic continuation that exists across the entire complex plane except for a simple pole at $s = 1$ [3].

The literature review highlights the Riemann zeta function being applied in many areas [4]. In physics, particularly in quantum mechanics and statistical mechanics, zeta functions are relevant in explaining certain quantum states or defining thermodynamic quantities. In computational mathematics, efficient algorithms for approximating the zeta function at the critical points are fundamental to modern cryptographic systems and modern computational number theory.

In this paper, the author analyzed some specific methods related to the Riemann zeta function. In Section 2, Both historical and analytical techniques were analyzed for solving the Basel Problem, discussing Euler's original technique and other techniques in addition to his. Section 3 investigates computational techniques for evaluating the zeta function at even integers, including generating functions and their relationship to Bernoulli numbers, leading to a general formula.

2. Techniques for the Basel Problem

The Riemann zeta function is initially defined by the infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{for } Re(s) > 1 \tag{1}$$

For example, $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ can transform this function with prime denominator since every integer can uniquely represent as a product of prime powers:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{s^{2s}} + \frac{1}{(2 \times 3)^s} + \cdots \tag{2}$$

The Basel Problem was first suggested by Italian Mathematician Pietro Mengoli in 1644, and was solved by The God of Math Euler in 1735. It is found that $\zeta(2) = \frac{\pi^2}{6}$.

2.1. Euler's original approach

An interesting fact is when Euler first saw this formula, he immediately spoke out the accurate answer without thinking, just based on his intuition. People will never know how he could just get the answer from nowhere, but people can try to approach and find the answer by calculation. Euler investigated the Basel Problem by employing infinite product expansions of sine functions to link trigonometric identities with infinite series. In this section, the author explores the ways in which Euler cleverly manipulated series expansions.

2.1.1. Outline of Euler's argument

At first, take a look at the Taylor expansion of $sin(x)$, which is that

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots.. \tag{3}$$

Since this is a polynomial, people can definitely find a solution by factorization, which means that the equation $sin(x)$ has both an infinite sum form and an infinite product form [5].

When mathematicians try to factorize a function, they usually look at its zeros, and times them together. for example, for function $f(x) = x^2 - 1$, $f(x)$ has a zero of $x = 1$ and -1 when $y = 0$.

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Therefore, one can have the factorization form of $f(x)$ is $(x-1)(x+1)$. People can try the same thing on $sin(x)$. Known that the zeros are $\ldots\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\ldots$ and it goes infinitely. By using the method mentioned above, one can get

$$sin(x) = \cdots \ldots (x+2\pi)(x+\pi)(x)(x-\pi)(x-2\pi) \ldots \ldots \tag{4}$$

It seems like the right solution since all the multiples of π are the solutions to this formula, but if people replace any other number, the result is divergent. To make this formula to converge to $sin(x)$, A coefficient C might need to be employed to the equation. Rearrange the equation one has:

$$sin(x) = Cx\left(x^2-\pi^2\right)\left(x^2-2^2\pi^2\right)\left(x^2-3^2\pi^2\right)\ldots\ldots. \tag{5}$$

Dividing x on both side of the equation, then one can find the limits:

$$\lim_{x\to 0}\frac{sin(x)}{x} = \lim_{x\to 0} C\left(x^2-\pi^2\right)\left(x^2-2^2\pi^2\right)\left(x^2-3^2\pi^2\right)\ldots\ldots \tag{6}$$

Therefore, $1 = C(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots$. Rearrange the equation and one can get $C = \frac{1}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\ldots\ldots}$. Replacing the C, one gets previously:

$$sin(x) = \frac{x\left(x^2-\pi^2\right)\left(x^2-2^2\pi^2\right)\left(x^2-3^2\pi^2\right)\ldots\ldots.}{\left(-\pi^2\right)\left(-2^2\pi^2\right)\left(-3^2\pi^2\right)\ldots\ldots} \tag{7}$$

Or equivalently,

$$sin(x) = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\ldots\ldots. \tag{8}$$

Now equating these two expansions and comparing coefficients of x^2 on both sides $-\frac{1}{3!} = -\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}$. From this equality, Euler concluded

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \tag{9}$$

Even though the notion of Euler was prior to the rigorous definitions of infinite products and uniform convergence, the outline of his idea was later given a solid foundation in the analysis in the 19th and 20th centuries [6]. The key point is to realize that both of the expansions of $sin(\pi x)/(\pi x)$ must represent the same analytic function, so their power series expansion must agree term by term.

2.1.2. Key proof steps for $\zeta(2)$

A more systematic version of this argument goes by defining the Product $f(x)$. Let:

$$f(x) = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) \tag{10}$$

Each factor has zeros at $\pm n$. So $f(x)$ has zeros precisely at $\pm 1, \pm 2, \pm 3, \cdots$.

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The first is to relate $f(x)$ to $sin(\pi x)$. $sin(\pi x)$ has precisely those same zeros. By looking at the leading behavior near$x = 0$, mathematician discovers $sin(\pi x)/(\pi x)$ shares the same zeros with the same multiplicities. The typical statement is that

$$sin(\pi x) = \pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) \tag{11}$$

The second is the Power Series Comparison. Next, by expanding both sides in the series of power, around $x = 0$, it is found that the coefficient of x^2 gets from $sin(\pi x) \approx \pi x - \frac{\pi^3 x^3}{3!} + \cdots$. Dividing by πx, the coefficient of x^2 in $sin(\pi x)/(\pi x)$ is $-\frac{\pi^2}{6}$. On the right side, the coefficient of x^2 gets from expanding each factor. The matching terms yields $-\frac{\pi^2}{6} =- \sum_{n=1}^{\infty}\frac{1}{n^2}$ [7]. Dropping the negative sign on both sides, it is again to find that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, which is $\zeta(2)$. This procedure can be generalized to higher powers, giving insight into $\zeta(2n)$.

2.2. Additional approaches to the Basel Problem

Euler's solution is the classic. However, there are still lots of alternative paths to the same result that illustrate broader concepts in analysis.

The first is the Fourier series approach. The Fourier series approach provides another powerful method of analysis. Consider a periodic function $f(x) = x^2$, defined on the interval $[-\pi, \pi]$ and periodically extended [8]. Its Fourier expansion is:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{cos(nx)}{n^2} \tag{12}$$

To determine the Basel sum, series at $x = 0$ were evaluated: $0^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$. Hence,

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} =- \frac{\pi^2}{12} \tag{13}$$

This alternating series can be related to the original series using algebraic manipulations and leading back to the Basel Problem.

The second is the contour integration methods (complex analysis). Using residues and carefully chosen contours, one can sometimes evaluate sums like $\sum 1/n^2$. This method is a bit more advanced but elegantly ties in the integral calculus of complex functions with real series.

The third is the polynomials and partial fractions. Intelligently developing polynomials out of the roots that are integers, then studying the partial-fraction decomposition, one can extract terms such as $1/n^2$,keeping in line with basically what Euler did. It also shows how much a mathematician can do with partial fractions in summation problems. All of these approaches provide the same purpose, that they all demonstrated that $\zeta(2)$ is linked to deep structures in analysis, with π appearing naturally.

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3. Computational techniques for evaluating $\zeta(2n)$

Although the Basel Problem pertains to $\zeta(2)$, Euler noted that similar methods may solve or at least give closed forms for $\zeta(2)$with n a positive integer. Next, show how generating functions, Bernoulli numbers, and one particular series converge to a large formula:

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}. \tag{14}$$

where B_{2n} are Bernoulli numbers [9].

3.1. Bernoulli numbers: definition and basic properties

Bernoulli numbers $\{B_k\}$ form a sequence appearing in expansions of many classical functions, especially those related to power sums, usually show in the Taylor series expansion of the function $x/(e^x - 1)$.

Specifically, one can write

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \tag{15}$$

with the B_k being constants determined by the Maclaurin series expansion. For instance, $B_0 = 1$, $B_1 =-\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 =-\frac{1}{30}$, $B_6 = \frac{1}{42}$, and so on, where most odd-indexed Bernoulli numbers beyond B_1 are zero.

However, how Bernoulli numbers relate to ζ? One of Euler's breakthroughs was uncovering that the sums of powers of integers like $\sum_{k=1}^{N} k^p$, can be expressed through Bernoulli numbers. Taking the limit as $N \to \infty$ (if it converges or is interpreted suitably) provides expressions for $\zeta(p)$. In particular, for even $p = 2n$, there is a known closed form.

3.2. Generating functions that lead to $\zeta(2n)$

Another perspective is the generating function approach. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n \tag{16}$$

while not quite $\zeta(2)$, examining power series of related forms for different exponents can lead back to the Riemann zeta function in terms of $\lim_{x \to 1^-}$. Methods of this nature are often quite useful for numerical approximations or expansions for $\zeta(2n)$.

The author shall construct a related series and define

$$S_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m} \tag{17}$$

For $|x| < 1$, this series converges absolutely. As $x \to 1$ from below (i.e., $x \nearrow 1$), $S_m(x)$ approaches $\zeta(m)$ if $m > 1$. If one can amend $S_m(x)$, one may sometimes be able to turn it into integrals of simpler functions, or into a summation that involves Bernoulli numbers, since it is

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possible to connect expansions for $\frac{1}{1-x}$ (a geometric series) and its consecutive derivatives of integrals, in particular higher-order ones, to sums of integer powers.

The carefully taking the limit as $x \to 1^{-}$ yields

$$\lim_{x \to 1^{-}} S_m(x) = \sum_{n=1}^{\infty} \frac{1}{n^m} = \zeta(m) \tag{18}$$

Although this is a more of a general outline than a proof for the closed-form Bernoulli formula for even arguments, the fundamental idea is here: expansions in series plus the expansions for rational functions or (...) exponential-related expressions.

3.3. Derivation of the formula for $\zeta(2n)$

Now let the author outline a more direct argument for Eq. (14). The author then mentions that $sin(\pi x) = \pi x \prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2})$ and a closely related function is $\pi cot(\pi x)$. A known expansion is

$$\pi x sin(\pi x) = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \cdots .... + 2\sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2} \tag{19}$$

Such expansions can be manipulated to isolate series like $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$.

The expansions for $\pi x cot(\pi x)$ can be written in terms of Bernoulli polynomials or Bernoulli numbers:

$$\pi x cot(\pi x) = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} (\pi x)^{2k} \tag{20}$$

The precise formulation for the summations $\sum 1/n^{2k}$ is obtained via comparison of coefficients in those expansions or by carefully evaluating at appropriate values of x, yielding the familiar closed-form expression in terms of Bernoulli numbers [10].

For the explicitly obtained $\zeta(2n)$, the result states that $\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$. For example, when $n = 1, B_2 = \frac{1}{6}$, one gets $\zeta(2) = \frac{\pi^2}{6}$, which is consistent with the Basel Problem result.

In contemporary mathematics, the researchers often use these closed forms to compute $\zeta(2n)$ to high accuracy. It is easy to compute π^2 and π^4 to many decimal places and to compute factorials and the Bernoulli numbers for moderately large n. This is important in different areas. In prime-related computations, zeta values at large arguments can sometimes be part of bounding prime gaps or analyzing advanced prime distribution functions. In cryptography, while $\zeta(2n)$ specifically may not be the direct focus, the broader toolbox for approximating $\zeta(s)$ at large or complex s underpins computations that show up in certain primality tests or zero verifications in cryptography.

4. Conclusion

Historically, the Riemann zeta function, originally a fundamental part of the Basel Problem, has become a major link between many parts of mathematics and physics. Euler's amazing result that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ was the motivating force behind this development. In addition, the expansions that result in closed form of $\zeta(2n)$—in terms of Bernoulli numbers—execute the same deep relation for infinite series and special constants while also revealing the geometry of this part of analysis. In addition to these classical values, $\zeta(s)$ is an extremely complex function, exhibiting fascinating behavior throughout the complex plane. The renowned Riemann Hypothesis remains unsolved and highlights the centrality of this function in understanding primes. From a computational perspective, the capacity to evaluate $\zeta(s)$ at specific points is the basis of research in cryptanalysis, random matrix theory, prime testing, and other advanced topics.

In summary, the Basel problem showed a remarkable convergence to $\pi^2/6$, bridging infinite sums and geometry. In analytical continuation, Riemann extended $\zeta(s)$ to all complex $s \neq 1$, and this extension is at the heart of modern number theory. For the closed forms at even integers, the relationship with Bernoulli numbers gives a neat expression for $\zeta(2n)$. Finally, for the computational & physical significance, zeta functions appear in quantum field regularizations, thermodynamic sums, and advanced prime distribution studies.

This paper provides readers with proofs and derivations built-in steps - spanning from the product expansions of Euler to the general formula where Bernoulli numbers appear, and also serves as encouragement for those with a lesser mathematics background to see how and why it is these (and others) are beautiful formulas. The author has also hinted to the computational methods that permit the exploration of zeta values far beyond what has been researched in this field, and I use the term field rather liberally, field of continued study in a field where details of many of those same methods is not typical in the study of zeta values or what is somewhat a derivatives study of complex variables between the two methods. As mathematics becomes progressively more sophisticated (and complex) in relation to applications in various areas, including but not limited to, cryptography, quantum theory, and computational science, knowledge in relation to the special values of ζ(s) remains a centre of study and interest. In continuing studies, Euler and Riemann similarly remain content to examine the study of odd numbers, the quiet and unassuming distribution of primes amongst overall integers, and the chaotic complexity of the complex plane.

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