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A New Subclass of H-matrices: γ− DZT Matrices

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Abstract. H-matrices have wide applications in numerical analysis, control theory, matrix theory and statistics. An important method to study the properties of H-matrices is to consider its subclasses. H-matrices have wide applications. This paper introduces a new subclass of H-matrices, named $\boldsymbol{\gamma}-$ DZT matrices. We prove that γ-SDD matrices and DZT matrices belong to $\boldsymbol{\gamma}-$ DZT matrices. This paper introduces a new matrix subclass: $\boldsymbol{\gamma}-$ DZT matrices. By constructing the scaling matrix, we prove that the class of $\boldsymbol{\gamma}-$ DZT matrices belongs to H-matrices. Moreover, γ-SDD matrices and DZT matrices belong to $\boldsymbol{\gamma}-$ DZT matrices. This paper successfully introduces a new subclass of matrices, the $\gamma-DZT$ matrix, and proves that the $\gamma-DZT$ matrix belongs to the H-matrix by constructing a scaling matrix. The inclusion relationships among the four types of matrices have been clarified, and H-matrices is clarified. These conclusions provide new theoretical basis and research directions for further studies on the properties and applications of H-matrix, and are of great significance in the development of matrix theory and related application fields.

Keywords: $\boldsymbol{\gamma}$ -DZT matrix, DZT matrix, Scaling matrix

1. Introduction

H-matrices have wide applications in numerical analysis, control theory, matrix theory and statistics [1-4]. An important method to study the properties of H-matrices is to consider its subclasses. H-matrices include many well-known subclasses, such as strictly diagonally dominant (SDD) matrices, doubly strictly diagonally dominant (DSDD) matrices, Σ-SDD matrices, γ-SDD matrices, weakly chained diagonally dominant matrices, Nekrasov matrices, Σ-Nekrasov matrices,

Dashnic Zusmanovich (DZ) matrices and eventually SDD matrices.

The class of Dashnic-Zusmanovich type (DZT) matrices were introduced by Zhao et al. in2018 [5]. They proved that DZT matrices belong to the class of H-matrices. Many results about DZT matrices have been obtained. One can refer to [6-9].

This paper introduces a new matrix subclass: $\boldsymbol{\gamma}-$ DZT matrices. By constructing the scaling matrix, we prove that the class of $\boldsymbol{\gamma}-$ DZT matrices belongs to H-matrices. Moreover, γ-SDD matrices and DZT matrices belong to $\boldsymbol{\gamma}-$ DZT matrices.

2. Preliminaries

We use $C^{n\times n}$ denote all the complex matrices with the order n,and denot $\langle n\rangle=\{1,2,\ldots,n\}$. For any $A\in C^{n\times n}$,define:

$$r_i(A)=\sum_{j\neq i}^{n}|a_{ij}| \tag{2.1}$$

$$c_i(A)=\sum_{j\neq i}^{n}|a_{ji}| \tag{2.2}$$

$$r_i^S(A)=\sum_{j\neq i,j\in S}^{n}|a_{ij}|\left(S\subseteq\langle n\rangle\right) \tag{2.3}$$

$$c_i^S(A)=\sum_{j\neq i,j\in S}^{n}|a_{ji}|\left(S\subseteq\langle n\rangle\right) \tag{2.4}$$

$$\Gamma_i(A)=\left\{j\in\langle n\rangle-\{i\}:\left(\left|a_{ii}\right|-r_i^{\langle n\rangle-\{j\}}(A)\right)|a_{jj}|>\left|a_{ij}\right|r_j(A)\right\},i\in\langle n\rangle \tag{2.5}$$

If $S=\langle n\rangle-\{j\}$,

$r_i^S(A)$ and $c_i^S(A)$ are simplified as $r_i^{\{j\}}(A)$ and $c_i^{\{j\}}(A)$, respectively.

Definition 2.1. Let $A=(a_{ij})\in C^{n\times n}$,We say that A is an SDD matrix if for all $i\in\langle n\rangle$, it holds that $|a_{ii}|>r_i(A)$.

Definition 2.2.Let $A=(a_{ij})\in C^{n\times n}$,if there exists $X=diag(x_1x_2\ldots x_n)$ such that AX is an SDD matrix,we call A is a nonsingular H-matrix. X is called scaling matrix of A.

Definition 2.3. Let $A=(a_{ij})\in C^{n\times n}$,if for all $i\in\langle n\rangle$,either $i\in N^+(A)$,or $\Gamma_i(A)\neq\emptyset$,we call A is a DZT matrix.

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Definition 2.4.Let $A=(a_{ij})\in C^{n\times n}$,we say that A is a γ-SDD matrix if there exists an $\alpha\in[0,1]$ such that A is an α-SDD matrix, i.e.,

$$|a_{ii}|>\alpha r_i(A)+(1-\alpha)c_i(A),\forall i\in\langle n\rangle.$$

3. Main results

Definition 3.1. A matrix $A=[a_{ij}]\in C^{n\times n},n\geq 2$, is called an $\alpha-DZT$ matrix if there exists $\alpha\in[0,1]$, such that for all $i\in\langle n\rangle$, there exists $j\neq i$ satisfying

$$\left(\left|a_{ii}\right|-\alpha\cdot r_i^{\{j\}}\left(A\right)-\left(1-\alpha\right)c_i^{\{j\}}\left(A\right)\right)\left|a_{jj}\right|>\left(\alpha\left|a_{ij}\right|+\left(1-\alpha\right)\left|a_{ji}\right|\right)\left(\alpha\cdot r_j\left(A\right)-\left(1-\alpha\right)c_j\left(A\right)\right), \tag{3.1}$$

We say that A is a $\gamma-$ DZT matrix if there exists an $\alpha\in[0,1]$ such that A is an $\alpha-DZT$ matrix.

It is easy to see that

$$\left|a_{ii}\right|>\frac{\alpha\cdot r_j(A)+(1-\alpha)c_j(A)}{|a_{jj}|}\left(\alpha\left|a_{ij}\right|+\left(1-\alpha\right)\left|a_{ji}\right|\right)+\alpha\cdot r_i^{\{j\}}\left(A\right)+\left(1-\alpha\right)c_i^{\{j\}}\left(A\right). \tag{3.2}$$

Let

$$N_1=\{i:|a_{ii}|\leq\alpha\cdot r_i(A)+(1-\alpha)c_i(A)\},$$

$$N_2=\{i:|a_{ii}|>\alpha\cdot r_i(A)+(1-\alpha)c_i(A)\}.$$

Theorem 3.1. Let $A=[a_{ij}]\in C^{n\times n},n\geq 2$ be an $\alpha-DZT$ matrix, there exists a diagonal matrix $D=diag(d_1,d_2,\ldots,d_i)$ such that DAD is an SDD matrix, where

$$d_i=\begin{cases}1,i\in N_1\\ \frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}+\varepsilon,i\in N_2\end{cases}$$

Proof. Let $B=DAD$. We will show that $|b_{ii}|>\alpha\cdot r_i(B)+(1-\alpha)c_i(B)$ for all $i\in\langle n\rangle$.

Case 1. $i\in N_1$. There must be $j_0\in N_2$, that is $\frac{\alpha\cdot r_{j_0}(A)+(1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|}<1$,and $d_i=1$, then $|b_{ii}|=|a_{ii}|$.

$$\begin{aligned}r_i(B)&=\sum_{j\neq i}|b_{ij}|=\sum_{j\in N_1}|a_{ij}|+\sum_{j\in N_2}|a_{ij}|\cdot d_j\\&=\sum_{j\in N_1}|a_{ij}|+\sum_{j\in N_2}|a_{ij}|\cdot\frac{\alpha\cdot r_j(A)+(1-\alpha)c_j(A)}{|a_{jj}|}\end{aligned}$$

$$\begin{aligned}\alpha\cdot r_i(B)&=\alpha\sum_{j\in N_1}|a_{ij}|+\alpha\sum_{j\in N_2}|a_{ij}|\cdot\frac{\alpha\cdot r_j(A)+(1-\alpha)c_j(A)}{|a_{jj}|}\\&=\alpha\sum_{j\in N_1}|a_{ij}|+\alpha\sum_{j\in N_2\backslash\{j_0\}}|a_{ij}|\cdot\frac{\alpha\cdot r_j(A)+(1-\alpha)c_j(A)}{|a_{jj}|}+\alpha|a_{ij_0}|\cdot\frac{\alpha\cdot r_{j_0}(A)+(1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|}\\&\leq\alpha\sum_{j\in N_1}|a_{ij}|+\alpha\sum_{j\in N_2\backslash\{j_0\}}|a_{ij}|+\alpha|a_{ij_0}|\cdot\frac{\alpha\cdot r_{j_0}(A)+(1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|}\end{aligned}$$

Because

$$\alpha\sum\nolimits_{j\in N_1}|a_{ij}|+\alpha\sum\nolimits_{j\in N_2\backslash\{j_0\}}|a_{ij}|=\alpha\cdot r_i^{\{j_0\}}\left(A\right),$$

then

$$\alpha\cdot r_i\left(B\right)<\alpha\cdot r_i^{\{j_0\}}\left(A\right)+\alpha\left|a_{ij_0}\right|\cdot\frac{\alpha\cdot r_{j_0}(A)+(1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|}.$$

$$\begin{aligned}c_i(B)&=\sum_{j\neq i}|b_{ji}|=\sum_{j\in N_1}|a_{ji}|+d_j\sum_{j\in N_2}|a_{ji}|\\&=\sum_{j\in N_1}|a_{ji}|+\frac{\alpha\cdot r_j(A)+(1-\alpha)c_j(A)}{|a_{jj}|}\sum_{j\in N_2}|a_{ji}|\end{aligned}$$

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$$(1-\alpha)\cdot c_i(B)=(1-\alpha)\sum_{j\in N_1}|a_{ji}|+(1-\alpha)\cdot\frac{\alpha\cdot r_j(A)+(1-\alpha)c_j(A)}{|a_{jj}|}\sum_{j\in N_2}|a_{ji}|$$
$$=(1-\alpha)\sum_{j\in N_1}|a_{ji}|+(1-\alpha)\cdot\frac{\alpha\cdot r_j(A)+(1-\alpha)c_j(A)}{|a_{jj}|}\sum_{j\in N_2\setminus\{j_0\}}|a_{ji}|$$
$$+(1-\alpha)|a_{j_0i}|\cdot\frac{\alpha\cdot r_{j_0}(A)+(1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|}$$
$$\leq(1-\alpha)\sum_{j\in N_1}|a_{ji}|+(1-\alpha)\sum_{j\in N_2\setminus\{j_0\}}|a_{ji}|+(1-\alpha)|a_{j_0i}|\cdot\frac{\alpha\cdot r_{j_0}(A)+(1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|}$$

Because

$$\left(1-\alpha\right)\sum_{j\in N_1}|a_{ji}|+(1-\alpha)\sum_{j\in N_2\setminus\{j_0\}}|a_{ji}|=\left(1-\alpha\right)\cdot c_i^{\{j_0\}}\left(A\right)$$

,

then

$$\left(1-\alpha\right)\cdot c_i\left(B\right)<\left(1-\alpha\right)\cdot c_i^{\{j_0\}}\left(A\right)+\left(1-\alpha\right)\left|a_{j_0i}\right|\cdot\frac{\alpha\cdot r_{j_0}(A)+(1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|}$$,

and because of formula (3.2),
we proved $|b_{ii}|>\alpha\cdot r_i(B)+(1-\alpha)c_i(B)$.
Case 2. $i\in N_2$. There must be $\frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}<1$,
and $d_i=\frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}+\varepsilon$, then $\left|b_{ii}\right|=d_i^2\left|a_{ii}\right|$.
The above formula can be expanded to get

$$\left|b_{ii}\right|=\left(\alpha\cdot r_i\left(A\right)+\left(1-\alpha\right)c_i\left(A\right)+\varepsilon\left|a_{ii}\right|\right)\left(\frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}+\varepsilon\right)$$

$$=\left(\frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}\right)\left(\alpha\cdot r_i\left(A\right)+\alpha\varepsilon\left|a_{ii}\right|\right)+\left(\frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}\right)\left(\left(1-\alpha\right)c_i\left(A\right)+\left(1-\alpha\right)\varepsilon\left|a_{ii}\right|\right)$$

$$=d_i(\alpha\cdot r_i(A)+\alpha\varepsilon|a_{ii}|)+d_i((1-\alpha)c_i(A)+(1-\alpha)\varepsilon|a_{ii}|)$$

,

$$r_i(B)=\sum_{j\neq i}|b_{ij}|=\sum_{j\in N_1}d_i|a_{ij}|+\sum_{j\in N_2}d_i|a_{ij}|\cdot d_j$$
$$=d_i(\sum_{j\in N_1}|a_{ij}|+\sum_{j\in N_2}|a_{ij}|\cdot d_j)$$

$$\alpha\cdot r_i\left(B\right)=d_i\alpha\left(\sum_{j\in N_1}|a_{ij}|+\sum_{j\in N_2}|a_{ij}|\cdot d_j\right)$$

$$=d_i\alpha\left(\sum_{j\in N_1}|a_{ij}|+\sum_{j\in N_2}|a_{ij}|\left(\frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}+\varepsilon\right)\right)$$

$$=d_i\alpha\left(\sum_{j\in N_1}|a_{ij}|+\sum_{j\in N_2}|a_{ij}|\frac{\alpha\cdot r_i(A)+(1-\alpha)c_i(A)}{|a_{ii}|}+\varepsilon\sum_{j\in N_2}|a_{ij}|\right)$$

$$\leq d_i\alpha\left(\sum_{j\in N_1}|a_{ij}|+\sum_{j\in N_2}|a_{ij}|+\varepsilon\sum_{j\in N_2}|a_{ij}|\right)$$

$$=d_i\left(\alpha\cdot r_i\left(A\right)+\alpha\cdot\varepsilon\sum_{j\in N_2}|a_{ij}|\right)$$

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$$< d_i(\alpha \cdot r_i(A) + \alpha \cdot \varepsilon |a_{ii}|)$$

And,

$$c_i(B) = \sum_{j \neq i} |b_{ji}| = \sum_{j \in N_1} |a_{ji}| d_i + \sum_{j \in N_2} d_j |a_{ji}| \cdot d_i$$
$$= (\sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} d_j |a_{ji}|) \cdot d_i$$

$$\left(1-\alpha\right) \cdot c_i\left(B\right) = \left(1-\alpha\right)\left(\sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} d_j |a_{ji}|\right) \cdot d_i$$

$$= d_i\left(1-\alpha\right)\left(\sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2}\left(\frac{\alpha \cdot r_i(A) + (1-\alpha)c_i(A)}{|a_{ii}|} + \varepsilon\right)\left|a_{ji}\right|\right)$$

$$= d_i\left(1-\alpha\right)\left(\sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} \frac{\alpha \cdot r_i(A) + (1-\alpha)c_i(A)}{|a_{ii}|}\left|a_{ji}\right| + \varepsilon \sum_{j \in N_2} |a_{ji}|\right)$$

$$\leq d_i\left(1-\alpha\right)\left(\sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2}\left|a_{ji}\right| + \varepsilon \sum_{j \in N_2} |a_{ji}|\right)$$

$$= d_i\left(\left(1-\alpha\right)c_i\left(A\right) + \left(1-\alpha\right)\varepsilon \sum_{j \in N_2} |a_{ji}|\right)$$

$$< d_i((1-\alpha)c_i(A) + (1-\alpha)\varepsilon |a_{ii}|)$$

Solving simultaneously, we get

$$\alpha \cdot r_i(B) + (1-\alpha) \cdot c_i(B) < d_i(\alpha \cdot r_i(A) + \alpha \cdot \varepsilon |a_{ii}|) + d_i((1-\alpha)c_i(A) + (1-\alpha)\varepsilon |a_{ii}|) ,$$

then $|b_{ii}| > \alpha \cdot r_i(B) + (1-\alpha)c_i(B)$ .we compete the proof.

Theorem 3.2. A matrix $A = [a_{ij}] \in C^{n \times n}, n \geq 2$, then the matrix A satisfies the following relation:

$$\begin{cases} A \in SDD \Rightarrow A \in DZT \Rightarrow A \in \gamma - DZT \\ A \in SDD \Rightarrow A \in \gamma - SDD \Rightarrow A \in \gamma - DZT \end{cases}$$

Proof. If A is a DZT matrix, then A is a 1-DZT matrix. Now we show that if A is an $\alpha - SDD$ matrix, then A is an $\alpha - DZT$ matrix.

First of all, by definition, it is obvious that if matrix A belongs to SDD matrix, then A must belong to $\alpha - SDD$ matrix. In the same way, if matrix A belongs to DZT matrix, then A must belong to the $\alpha - DZT$ matrix. So now it is only necessary to prove that when A belongs to SDD matrix and $\alpha - SDD$ matrix, A must belong to DZT matrix and $\alpha - DZT$ matrix respectively.

By definition, if the hypothesis $A \in SDD \subseteq A \in DZT$ is true, then

$$r_i\left(A\right) > \frac{r_{j_0}(A)}{|a_{j_0 j_0}|}\left|a_{ij_0}\right| + r_i^{\{j_0\}}\left(A\right)$$

$$r_i^{\{j_0\}}\left(A\right) + \left|a_{ij_0}\right| > \frac{r_{j_0}(A)}{|a_{j_0 j_0}|}\left|a_{ij_0}\right| + r_i^{\{j_0\}}\left(A\right)$$

$$\left|a_{ij_0}\right| > \frac{r_{j_0}(A)}{|a_{j_0 j_0}|}\left|a_{ij_0}\right|$$

Because $\frac{r_{j_0}(A)}{|a_{j_0 j_0}|} < 1$, therefore, the above inequality is true, that is, the hypothesis is true.

For the argument, we also use the reverse order method, assuming that the conclusion $A \in \alpha - SDD \subseteq A \in \alpha - DZT$ is true, then by definition,

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$$\alpha \cdot r_i\left(A\right) + \left(1-\alpha\right)c_i\left(A\right) > \frac{\alpha \cdot r_{j_0}(A) + (1-\alpha)c_{j_0}(A)}{|a_{j_0 j_0}|}\left(\alpha\left|a_{ij_0}\right| + \left(1-\alpha\right)\left|a_{j_0 i}\right|\right)$$

$$+\alpha \cdot r_i^{\{j_0\}}\left(A\right) + \left(1-\alpha\right)c_i^{\{j_0\}}\left(A\right)$$

$$\alpha \cdot \left[r_i^{\{j_0\}}\left(A\right) + \left|a_{ij_0}\right|\right] + \left(1-\alpha\right)\left[c_i^{\{j_0\}}\left(A\right) + \left|a_{j_0 i}\right|\right] > \frac{\alpha \cdot r_{j_0}(A) + (1-\alpha)c_{j_0}(A)}{|a_{j_0 j_0}|}\left(\alpha\left|a_{ij_0}\right| + \left(1-\alpha\right)\left|a_{j_0 i}\right|\right)$$

$$+\alpha \cdot r_i^{\{j_0\}}\left(A\right) + \left(1-\alpha\right)c_i^{\{j_0\}}\left(A\right)$$

$$\alpha\left|a_{ij_0}\right| + \left(1-\alpha\right)\left|a_{j_0 i}\right| > \frac{\alpha \cdot r_{j_0}(A) + (1-\alpha)c_{j_0}(A)}{|a_{j_0 j_0}|}\left(\alpha\left|a_{ij_0}\right| + \left(1-\alpha\right)\left|a_{j_0 i}\right|\right)$$

Eliminate the common terms on both sides of the inequality at the same time.

In the same way, because $\frac{\alpha \cdot r_{j_0}(A) + (1-\alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} < 1$, therefore the hypothesis is true. In summary, theorem 2 is proved.

For the proof of theorem 2, we mainly use the core idea of reverse order. We start with the result we want to prove, analyze what conditions are needed to get that result, and then work our way forward to see how we can obtain these conditions from what we already know. Thus, we prove the relationship between SDD, DZT, $\gamma - SDD$ and $\gamma - DZT$ matrix.

4. Conclusion

This paper successfully introduces a new subclass of matrices, the $\gamma - DZT$ matrix, and proves that the $\gamma - DZT$ matrix belongs to the H-matrix by constructing a scaling matrix. The inclusion relationships among the four types of matrices have been clarified, and H-matrices is clarified. These conclusions provide new theoretical basis and research directions for further studies on the properties and applications of H-matrix, and are of great significance in the development of matrix theory and related application fields.

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