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An Analysis of the Proper Time Variation of Rigid Bodies Extending Along Inertial Motion in Space

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Abstract. Proper time and time dilation, though well-established in special relativity, are predominantly analysed for point particles. Much less attention has been given to how proper time behaves across spatially extended rigid bodies in inertial motion. Existing studies have explored differential aging effects under relativistic rigid motion, but primarily in scenarios involving acceleration. Therefore, this paper aims to investigate how proper time varies across different points of an extended object experiencing purely inertial motion, focusing on how the different clock synchronisation conventions affect the overall desynchronisation in proper time. This paper uses classical formulations of special relativity in order to contribute to a new perspective on the role of simultaneity in distributed time frames. This paper discovered that the two primary clock synchronization conventions— the Einstein convention and slow clock transport—produce comparable desynchronization in proper time; however, each method presents distinct advantages and disadvantages in practical contexts, necessitating a complementary approach to effectively address desynchronization in applications like satellite communication.

Keywords: Proper Time, Clock Synchronisation, Spatially Extended Bodies, Inertial Frames, Relativity of Simultaneity

1. Introduction

The treatment of proper time, which is defined as the integral of an observer's worldline (the observer's trace on a space-time diagram), has traditionally focused on point-like observers or isolated clocks. One example is the study of the twin paradox, in which the aging of two observers, often modelled as particles with point-based clocks, is compared. In these studies, little attention has been given to systems composed of spatially extended bodies. While Ben-Ya'acov's work on accelerated rigid motion advanced differential aging analysis, it neglected inertial motion scenarios where simultaneity alone drives proper time variations [1]. This paper seeks to fill this gap by analysing proper time within a spatially extended rigid body in inertial motion, specifically investigating how standard clock synchronization conventions influence proper time readings of clocks distributed throughout the extended body and determining whether these clocks can register differing elapsed proper times for events deemed simultaneous in a specific inertial frame. This investigation may be relevant not only to the conceptual understanding of special relativity but also to practical situations, such as satellites and relativistic spacecrafts, where a precise calculation of

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clock synchronisation across the spatially separated system is essential. The paper aims to use theoretical analysis based on the Lorentz transformations. An ideal Born-rigid body in uniform motion with ideal clocks distributed along its spatial extension is modelled. Using Lorentz transformations and concepts of simultaneity, the paper will examine how time intervals change between a stationary frame and the frame of the rigid body, with extra attention on comparing proper time intervals measured by different clocks. The paper aims to rely solely on Lorentz derivations and spacetime diagrams to ensure the simplicity of the explanations.

2. Theoretical background

2.1. Lorentz transformations and simultaneity

Einstein's theory of special relativity relies on the postulates of the principle of relativity and the constant value of the speed of light across all inertial frames [2]. Using two facts: both S and S′ are inertial frames and S′ is traveling at speed v relative to S such that an observer sitting at the origin of S′ ($x' = 0$) would move along the trajectory of $x = vt$ in S, the function can be written as the following in its simplest form.

$$x' = \gamma(x - vt) \tag{1}$$

The reverse transformation of $x' = \gamma(x - vt)$ can be written when considering the same restrictions but from the other frame.

Using the postulate that the speed of light c is the same in both S and S′, $x = ct$ and $x' = ct'$ can be substituted into the above to find γ, which is the Lorentz factor.

$$\gamma = \sqrt{\frac{1}{1-v^2/c^2}} \tag{2}$$

This then allows the formulation of the second transformation by combining $x' = \gamma(x - vt)$ and $x = \gamma(x' + vt')$.

$$x = \gamma\left(\gamma\left(x - vt\right) + vt'\right)$$

$$x = \gamma^2\left(x - vt\right) + \gamma vt'$$

$$\gamma vt' = x - \gamma^2\left(x - vt\right)$$

$$\gamma vt' = x\left(1 - \gamma^2\right) + \gamma^2 vt$$

$$\gamma vt' = x\left(1 - \frac{1}{1-v^2/c^2}\right) + \gamma^2 vt$$

$$\gamma vt' = x\left(1 - \frac{c^2}{c^2-v^2}\right) + \gamma^2 vt$$

$$\gamma vt' = x\left(\frac{-v^2}{c^2-v^2}\right) + \gamma^2 vt$$

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$$\gamma v t\prime = -v^2 x\left(\frac{1}{c^2 - v^2}\right) + \gamma^2 v t$$

$$t\prime = -vx\left(\frac{\frac{1}{c^2 - v^2}}{\gamma}\right) + \gamma t$$

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}} = \sqrt{\frac{c^2}{c^2 - v^2}}$$

$$c^2 - v^2 = \frac{c^2}{\gamma^2}$$

$$t\prime = -vx\left(\frac{\frac{\gamma^2}{c^2}}{\gamma}\right) + \gamma t$$

$$t\prime = -vx\left(\frac{\gamma}{c^2}\right) + \gamma t$$

$$t\prime = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$ct\prime = \gamma\left(ct - \frac{vx}{c}\right) \tag{3}$$

The Lorentz transformation results in the relativity of simultaneity as it entails that equal t is not identical to equal t′, suggesting that two events that are simultaneous in one inertial frame are not simultaneous in any other inertial frames with relative constant velocity—as long as v is non-zero, the lines of simultaneity (given by: time value in the host frame=constant) will always be shifted when transforming to different frames.

2.2. Proper time

The invariant interval of two events, a measure of the separation of two events in spacetime that is the same for all inertial observers, is given by the following (for simplicity, one spatial dimension x is assumed) [2,3].

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 \tag{4}$$

For a particle at rest at the origin of S′, the invariant interval of two points on the particle's worldline is simply given by $c^2\Delta t^2$, and, instead of t, τ (proper time) can be used to entail the time experienced by the particle in its trajectory.

$$\Delta \tau = \frac{\Delta s}{c} \tag{5}$$

If the particle is moving at v, the proper time experienced between an infinitesimal separation of its trajectory is given by the following.

$$d\tau = \sqrt{dt^2 - \frac{dx^2}{c^2}} = dt\sqrt{1 - \frac{v^2}{c^2}} = dt \cdot \frac{1}{\gamma} \tag{6}$$

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2.3. Born rigidity

In special relativity, the speed of signal propagation is limited by the speed of light. Max Born addressed this in 1909 by introducing the concept of Born rigidity, which deemed a body as Born-rigid if the distance between infinitesimally close points remains constant in the instantaneous rest frame of the body [4]. Importantly, Born rigidity differs from classical rigidity in its relativistic compatibility. A classical rigid body cannot deform in any way, but this is incompatible with special relativity, which states that no information can travel faster than the speed of light. Born's framework leverages the invariant interval, defining rigidity as constant proper distance between infinitesimal points in the instantaneous rest frame—a condition observer-independent.

3. Modelling

A Born-rigid body can be modelled, with rest frame S′, moving at v in the x direction relative to a frame S, with its extension being along the x direction of S as well. There are two clocks placed along this body: one at the front, one at the rear.

4. Clock synchronisation methods

To examine the variation of proper time among different clocks, the clocks must first be synchronized inside their own frame, indicating that they should exhibit the same time value for an event occurring in that identical inertial frame. Various ways exist to accomplish this, each yielding distinct conclusions regarding the fluctuation of correct time between the two clocks within the same rigid body model [5].

4.1. Einstein convention

The Einstein convention is a method that relies on the isotropy of the speed of light (i.e., same in all directions for all frames). Using this, a light signal can be sent from one clock, reflected back from a second clock, and the second clock can be adjusted so that the time of event of signal reflection is halfway between emission and reception at the first clock [6,7].

Say that the rigid body has length L in its rest frame S′, then the two clocks, A and B, have coordinates $x'_A = 0$ and $x'_B = L$ respectively. In S′, the clocks are synchronised and are at rest, so the proper time of each clock equal t′.

$$\Delta\tau_A = \Delta\tau_B = \Delta t' \tag{7}$$

Assume two events, 1 and 2, that are simultaneous in frame S′. Event 1 occurs at coordinates ($t' = 0$, $x' = 0$) and ($t' = 0$, $x' = L$) for clocks A and B in S′, respectively. In S, event 1 occurs at ($t = 0$, $x = 0$) for clock A; for clock B, event 1 occurs at time coordinates $t' = \gamma\left(t - \frac{vx}{c^2}\right)$, and since $t' = 0$, $t = \gamma\frac{vx'}{c^2} = \gamma\frac{vL}{c^2}$ for clock B in S.

Therefore, in frame S, for an event simultaneous in S′, clock B is delayed by $\Delta t = \gamma\frac{vL}{c^2}$, which is unsurprising given the relativity of simultaneity.

To examine the proper time accumulated by each clock, both clocks can be allowed to reach $t' = T$. Now, during this interval, both clocks accumulate the same amount of proper time in S as they move at v in S. However, since it is established that, in S, clock B always start later than A by

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$\Delta t = \gamma \frac{vL}{c^2}$, the desynchronisation in proper time of two clocks can be calculated by putting this Δt into the proper time formula to find the desynchronisation in proper time between the two clocks.

$$\Delta\tau_{dsync} = \frac{1}{\gamma} \cdot \Delta t = \frac{1}{\gamma} \cdot \gamma \frac{vL}{c^2} = \frac{vL}{c^2} \tag{8}$$

This reflects that simultaneity (clock agreement) is frame-dependent in uniform inertial motion.

4.2. Slow clock transport

First, the two clocks in the exact same location, say at $x\prime = 0$. Clock B is moved at some speed u away from A along the body, where $u \ll c$ (u is defined form the frame S) [8]. During this, B's proper time is dilated compared to t′, given by $d\tau = dt\prime \cdot \gamma$, where γ is a function of u. Given that $u \ll c$, $\frac{u^2}{c^2} \ll 1$, meaning that γ is extremely close to 1. Therefore, it can be concluded that the time dilation effect on B relative to frame S′ (A and the rigid body) is minimal, proving that the slow clock transport method ensures A and B to be closely synchronised without relying on the isotropy of light.

Now, say there are another two events, 1 and 2, where event 1 is when B starts moving away from A with u and event 2 is when B reaches L (the end of the body). Unlike the situation under the Einstein convention, the slow clock transport results in a very small difference in proper time accumulation from $t\prime = 0$ to $t\prime = T$ in S because the presence of u.

$$d\tau_A = dt\sqrt{1 - \frac{v^2}{c^2}} = dt\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \approx dt\left(1 - \frac{v^2}{2c^2}\right)$$

$$d\tau_B = dt\sqrt{1 - \frac{(v+u)^2}{c^2}} = dt\left(1 - \frac{(v+u)^2}{c^2}\right)^{\frac{1}{2}} \approx dt\left(1 - \frac{v^2+2vu}{2c^2}\right) \tag{9}$$

For simplicity, both proper times are approximated using binomial approximations (given that $\left|-\frac{v^2}{c^2}\right| < 1$ and $\left|-\frac{(v+u)^2}{c^2}\right| < 1$) to the first order. Higher orders of these binomial series can be ignored since the next orders for the first expansion are: $-\frac{v^4}{8c^4} + \frac{v^6}{16c^6} - \frac{5v^8}{128c^8}$ …. For common Low Earth Orbit satellites, v is around 10,000 ms^{-1}, which, compared to the speed of light, give values of magnitude 10^{-18} for the $-\frac{v^4}{8c^4}$ term, and the magnitude would get smaller for higher orders. Therefore, it is sufficient to say that higher magnitudes can be ignored. It is also established that u is extremely small, so the second expression possess a similar nature and can also be approximated. The u^2 term in the second expansion is also ignored since u is small.

Therefore, the desynchronisation in proper time of clocks A and B is given by:

$$\Delta\tau_{dsync} = \int_0^T \left[\left(1 - \frac{v^2}{2c^2}\right) - \left(1 - \frac{v^2+2vu}{2c^2}\right)\right] dt$$

$$\Delta\tau_{dsync} = \int_0^T \left(-\frac{v^2}{2c^2} + \frac{v^2+2vu}{2c^2}\right) dt = \int_0^T \frac{vu}{c^2} dt = \frac{vu}{c^2} \cdot T \tag{10}$$

In frame S, length L would change by a factor of $\frac{1}{\gamma}$ given length contraction, so $T = \frac{L}{\gamma u}$. Therefore, the proper time desynchronisation is given by the following.

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$$\Delta\tau_{\text{dsync}} = \frac{vu}{c^2} \cdot \frac{L}{\gamma u} = \frac{vL}{\gamma c^2} \tag{11}$$

This is proper time desynchronisation is similar to that of the Einstein convention. However, for this expression to be true, u must be sufficiently small such that higher orders of the binomial approximation can be ignored for $u \ll c$.

Considering the actual desynchronisation of this method, this proper time alignment reflects a different simultaneity convention, emphasizing the fact that simultaneity is not an absolute feature but depends on the synchronisation process chosen. The advantage of using slow clock transport is that, unlike the Einstein convention, it avoids relying on propagation of light signals, which can be easily delayed in non-inertial frames (accelerating or rotating). Slow clock transport can also act as a secondary method to ensure that synchronisation with the Einstein convention is done correctly, since the desynchronisation resulting from both methods are extremely similar. That said, because u needs to be small, slow clock transport is unsuitable for large distance synchronisation.

5. Applications

In general, for any extended inertial systems, like spacecraft or satellite constellations, establishing a consistent temporal framework becomes essential for coherent operations. The choice of synchronisation convention has direct implications for communication protocols within these systems. Understanding and accounting for any desynchronisation are thus crucial in the design and operation of distributed relativistic systems. There are several existing applications of clock synchronisation conventions:

The Global Positioning System (GPS) employs Einstein synchronization as the primary protocol to maintain synchronisation across its satellite constellation. However, due to the rotating frame of Earth, constant corrections are needed to keep clock readings the same. One example of such corrections is slow clock transport, which is sometimes used to verify synchronisation under the Einstein convention. This is inherently constrained, as the GPS requires precision to nanoseconds, necessitating that the transmitted clock operates at an exceedingly minimal velocity. GPS engineers establish a specific temporal reference called GPS time, which serves as a theoretical global time scale synchronized with the Earth-Centered Inertial (ECI) frame. This coordinate time serves as a common basis for clock synchronisation across satellites, allowing engineers to incorporate aspects of both special relativity (desynchronisation under Einstein convention and time dilation due to relative velocity to the Earth) and general relativity (gravitational time dilation) to coordinate corrections [9,10].

Experimental setups involving networks of atomic clocks serve as critical platforms for verifying special relativistic predictions, with time dilation being the primary testable phenomenon Notable experiments include the Hafele–Keating experiment, where atomic clocks sent flying around the world exhibited measurable time differences compared to stationary clocks. More recent experiments have achieved increased precision, detecting time dilation effects between clocks separated by millimetres. These experiments often employ different synchronisation methods to establish a common temporal reference. The choice of the synchronisation method can lead to observable variations in the measured time differences [11].

Very-Long-Baseline Interferometry (VLBI), a cornerstone technique in radio astronomy, demands nanosecond-level synchronization across intercontinental telescope arrays. The telescopes, which are often located on different continents, simultaneously observe the same astronomical radio waves and record signals with the help of precise time stamps provided by atomic clocks. Because

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the signals are later combined to reconstruct the incoming wavefronts, any miscalculation in timing down to nanoseconds can lead to reduced imaging accuracy. Methods including Einstein synchronization, supplemented by corrections from slow clock transport methods, and satellite-based time standards like the GPS time discussed earlier are employed. This enables VLBI systems to achieve angular resolutions fine enough to image the event horizon of black holes, as shown by the Event Horizon Telescope [12].

6. Conclusion

In this paper, the variation in proper time assignment across spatially extended Born-rigid bodies in inertial motion is examined with a focus on the dependence of synchronisation on the convention chosen. Under Einstein synchronisation, clocks positioned at different locations along the body are desynchronised by $\Delta\tau_{\mathrm{dsync}} = \frac{vL}{c^2}$, where L is the spatial separation in the rest frame and v is the inertial velocity relative to the observer. This desynchronisation does not imply a difference in accumulated proper time as all points share the same constant velocity, but it does affect how simultaneity surfaces are defined in different frames. In contrast, the slow clock transport method, which uses gradually moving clocks, yields $\Delta\tau_{\mathrm{dsync}} = \frac{vL}{\gamma c^2}$, which is similar to Einstein's to the first order but differ more when expanded up to higher orders. This highlights the convention-dependent nature of simultaneity, even in purely inertial scenarios. In actual scenarios, such as with systems like GPS, where precise global timekeeping is crucial, the selection of synchronization convention affects the implementation of timing adjustments. Comprehending the impact of synchronization on accurate time assignments is essential in distributed relativistic systems (e.g., spacecraft) and experimental networks of atomic clocks. Future research may improve precision timing technologies and advance relativistic models to get greater accuracy in positioning systems for increasingly complicated engineering applications.

References

[1] Ben-Ya'acov, U. (2016). The 'twin paradox' in relativistic rigid motion. European Journal of Physics, 37(5), 055601. https: //doi.org/10.1088/0143-0807/37/5/055601
[2] Einstein, A. (1905). On the electrodynamics of moving bodies (W. Perrett & G. B. Jeffery, Trans.). The principle of relativity (pp. 37–65). Dover Publications. (Original work published 1905).
[3] Rao, A. V. G., Mallesh, K. S., & Rao, K. N. S. (2015). On time-interval transformations in special relativity. arXiv. https: //doi.org/10.48550/arXiv.1105.4085
[4] Born, M. (1909). The theory of the rigid electron in the kinematics of the principle of relativity. Annalen der Physik, 29(3), 571–584. https: //doi.org/10.1002/andp.19093230306
[5] Bhadra, A., Chakraborty, A., Ghose, S., & Raychaudhuri, B. (2021). Synchronization gauge field, standing waves and one-way-speed of light. Physica Scripta, 96(5), 055301. https: //doi.org/10.1088/1402-4896/abe37b
[6] Rizzi, G., Ruggiero, M. L., & Serafini, A. (2004). Synchronisation gauges and the principles of special relativity. Foundations of Physics, 34(12), 1835–1887. https: //doi.org/10.1007/s10701-004-1624-3
[7] Rothenstein, B., & Popescu, S. (2007). Lorentz transformations: Einstein's derivation simplified. arXiv. https: //doi.org/10.48550/arXiv.physics/0702157
[8] Mitaroff, W. A. (2023). Aspects of clock synchronisation in relativistic kinematics—A tutorial. arXiv. https: //doi.org/10.48550/arXiv.2306.09582
[9] Mozota Frauca, Á. (2024). GPS observables in Newtonian spacetime or why we do not need 'physical' coordinate systems. Classical and Quantum Gravity, 41(4), 045002. https: //doi.org/10.1088/1361-6382/ad208a
[10] Dalmazzone, C., Guigue, M., Mellet, L., Popov, B., Russo, S., Voisin, V., Abgrall, M., Chupin, B., Lim, C. B., Pottie, P.-É., & Ulrich, P. (2024). Precise synchronization of a free-running Rubidium atomic clock with GPS Time for applications in experimental particle physics. arXiv. https: //arxiv.org/abs/2407.20825

DOI: 10.54254/2753-8818/2025.DL25453

[11] Arneth, B., & Borros, B. (2024). Hafele and Keating revisited: A novel interpretation of the results of the Hafele–Keating experiment. Physics Essays, 37(2), 150–152. https: //doi.org/10.4006/0836-1398-37.2.150
[12] Krásná, H., Baldreich, L., Böhm, J., Böhm, S., Gruber, J., Hellerschmied, A., Jaron, F., Kern, L., Mayer, D., Nothnagel, A., Panzenböck, O., & Wolf, H. (2023). VLBI celestial and terrestrial reference frames VIE2022b. Astronomy & Astrophysics, 679, A53. https: //doi.org/10.1051/0004-6361/202245434