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Darboux Transformation and Exact Solutions for a Semi-discrete Coupled Local Nonlinear Schrödinger Equation

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Abstract. In recent years, significant progress has been made in the study of nonlinear Schrödinger equations within the fields of mathematical physics and nonlinear wave dynamics. However, investigations on the semi-discrete form of coupled local nonlinear Schrödinger equations remain relatively limited. In this work, within the framework of semi-discrete systems, we impose local reduction conditions on the coupled nonlinear Schrödinger equation and propose a new integrable semi-discrete two-component coupled local nonlinear Schrödinger equation. Starting from a semi-discrete 4×4 matrix Lax pair, and with the aid of a gauge transformation, explicit formulas for the Darboux transformation of the system are derived. By selecting appropriate seed solutions, exact solutions of the system on the zero background are obtained through the constructed Darboux transformation. Furthermore, the dynamical properties of these solutions are visualized using mathematical software. The results reveal that the system admits typical breather-type soliton solutions, which exhibit periodic oscillations in the temporal direction and localization in the spatial direction. This study not only deepens the understanding of soliton dynamics in semi-discrete local nonlinear systems but also provides an effective approach and theoretical reference for analyzing similar nonlinear wave equations.

Keywords: Darboux transformation, Semi-discrete coupled local nonlinear Schrödinger system,Explicit exact solution.

1. Introduction

The well-known nonlinear Schrödinger (NLS) equation:

$$iq_t + q_{xx} + 2|q|^2 q = 0, \tag{1}$$

can be regarded as the result of the coupled nonlinear Schrödinger (CNLS) equations:

$$iq_t = -q_{xx} + 2q^2 r, \tag{2}$$

$$ir_t = -r_{xx} - 2r^2 q, \tag{3}$$

under the reduction $r = -q^*$.Equation (2,3) belongs to the well-known AKNS hierarchy, where q and r are two slowly varying complex envelopes of the propagating waves, the subscripts x and t denote the partial derivatives with respect to normalized distance and time respectively.

In 2013, Ablowitz and Mussilimani [1] introduced a "nonlocal" constraint on q and r, given by $q(x,t) = r^*(-x,-t)$, which allows the derivation of the nonlocal nonlinear Schrödinger (NNLS) equation from Equation (2,3):

$$iq_t + q_{xx} + 2q^2(x,t)q^*(-x,t) = 0. \tag{4}$$

Equation (4) exhibits PT symmetry since the nonlinear term $V(x,t) = q(x,t)q^*(-x,t)$ remains invariant under the PT transformation, i.e. $V(x,t) = V^*(-x,t)$. Compared to the NLS Equation (1), the key difference is that the nonlinear term $2|q(x,t)|^2 q(x,t)$ in Equation (1) is replaced by $2q^2(x,t)q^*(-x,t)$ in the NNLS Equation (4), which reflects the nature of the anti-spatiotemporal nonlocal coupling between $q(x,t)$ and $q^*(-x,t)$.

Ablowitz and Ladik [2] discovered the semi-discrete nonlinear Schrödinger (sd-NLS) equation:

$$i\frac{dQ_n}{dt} + \left(Q_{n+1} + Q_{n-1} - 2Q_n\right) + 2\left|Q_n\right|^2\left(Q_{n+1} + Q_{n-1}\right) = 0. \tag{5}$$

In recent years, semi-discrete integrable systems have received increasing attention as mathematical models for various physical phenomena, including nonlinear optics, biology, ladder circuits, and lattice dynamics [3-6].

The studies above primarily focus on single-component continuous and discrete nonlinear Schrödinger equations. However, research on multi-component nonlinear Schrödinger (MNLS) equations has become a major topic of interest. MNLS equations are crucial dynamical systems in optics and mathematical physics, describing the simultaneous propagation of multiple nonlinear waves in a homogeneous medium. They find applications in plasma physics [7], quantum electronics [8], nonlinear optics [9], Bose-Einstein conde-nsates [10], and fluid dynamics [11].

Reference [12] presents the integrable multi-component form of the semi-discrete coupled nonlinear Schrödinger (sd-CNLS) equations:

$$i\dot{u}_n^{(j)} + \left(u_{n+1}^{(j)} + u_{n-1}^{(j)}\right)\left(1 - \sum_{k=1}^{m} u_n^{(k)} v_n^{(k)}\right) - 2u_n^{(j)} = 0, \tag{6}$$

$$i\dot{v}_n^{(j)} - \left(v_{n+1}^{(j)} + v_{n-1}^{(j)}\right)\left(1 - \sum_{k=1}^{m} v_n^{(k)} u_n^{(k)}\right) + 2v_n^{(j)} = 0, \quad j = 1,2,\ldots,m. \tag{7}$$

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This study investigates the integrability of these equations, derives an infinite set of conservation laws, and constructs their N -soliton solutions via the inverse scattering method (ISM). However, to the best of our knowledge, no soliton solutions for this equation have been obtained using the Darboux transformation.

In this work, we consider the case j and impose the reduction $v_n^{(j)} = -u_n^{(j)*}$,which transforms Equation (6,7) into the following system:

$$i\dot{u}^{(1)} + \left(u_{n+1}^{(1)} + u_{n-1}^{(1)}\right)\left(1 + \left|u_n^{(1)}\right|^2 + \left|u_n^{(2)}\right|^2\right) - 2u_n^{(1)} = 0, \tag{8}$$

$$i\dot{u}^{(2)} + \left(u_{n+1}^{(2)} + u_{n-1}^{(2)}\right)\left(1 + \left|u_n^{(1)}\right|^2 + \left|u_n^{(2)}\right|^2\right) - 2u_n^{(2)} = 0. \tag{9}$$

We first construct the Darboux transformation for Equation (8,9) by employing a gauge transformation and the associated Lax pair. Subsequently, we derive the explicit solutions of Equation (8,9) and analyze their dynamical properties.

2. Darboux transformation

According to Ref [13], the auxiliary linear equations corresponding to the semi-discrete coupled local nonlinear Schrödinger Equation (8,9) are given by:

$$\Psi_{n+1} = N_n \Psi_n, \tag{10}$$

$$\frac{d}{dt}\Psi_n = M_n \Psi_n. \tag{11}$$

The corresponding Lax pair is given as follows:

$$N_n = \begin{pmatrix} \lambda I & Q_n \\ R_n & \lambda^{-1} I \end{pmatrix} \tag{12}$$

$$M_n = i\begin{pmatrix} Q_n R_{n-1} - \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)^2 I & -\lambda Q_n + \frac{1}{\lambda} Q_{n-1} \\ \frac{1}{\lambda} R_n - \lambda R_{n-1} & -V_n U_{n-1} + \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)^2 I \end{pmatrix} \tag{13}$$

We may choose

$$Q_n = \begin{pmatrix} q_n & r_n \\ r_n^* & q_n^* \end{pmatrix}, R_n = \begin{pmatrix} -q_n^* & r_n \\ r_n^* & -q_n \end{pmatrix}, \tag{14}$$

Thus, the Lax pair becomes:

$$N_n = \begin{pmatrix} \lambda & 0 & q_n & r_n \\ 0 & \lambda & r_n^* & q_n^* \\ -q_n^* & r_n & \lambda^{-1} & 0 \\ r_n^* & -q_n & 0 & \lambda^{-1} \end{pmatrix}, \tag{15}$$

$$M_n = i\begin{pmatrix} -\frac{\lambda^2+\lambda^{-2}}{2} - q_n q_{n-1}^* + r_n r_{n-1}^* + 1 & -q_{n-1} r_n + q_n r_{n-1} & \frac{1}{\lambda} q_{n-1} - \lambda q_n & \frac{1}{\lambda} r_{n-1} - \lambda r_n \\ -q_{n-1}^* r_n^* + q_n^* r_{n-1}^* & -\frac{\lambda^2+\lambda^{-2}}{2} - q_{n-1} q_n^* + r_{n-1} r_n^* + 1 & -\lambda r_n^* + \frac{1}{\lambda} r_{n-1}^* & -\lambda q_n^* + \frac{1}{\lambda} q_{n-1}^* \\ -\frac{1}{\lambda} q_n^* + \lambda q_{n-1}^* & \frac{1}{\lambda} r_n - \lambda r_{n-1} & \frac{\lambda^2+\lambda^{-2}}{2} + q_{n-1} q_n^* - r_n r_{n-1}^* - 1 & r_{n-1} q_n^* - r_n q_{n-1}^* \\ \frac{1}{\lambda} r_n^* - \lambda r_{n-1}^* & -\frac{1}{\lambda} q_n + \lambda q_{n-1} & -q_{n-1} r_n^* + q_n r_{n-1}^* & \frac{\lambda^2+\lambda^{-2}}{2} + q_n q_{n-1}^* - r_{n-1} r_n^* - 1 \end{pmatrix}$$

Where Ψ_n is a 4×4 matrix, and λ is the spectral parameter, which is independent of both n and t.
Substituting the Lax Equation (15,16) into the zero-curvature equation:

$$\frac{d}{dt} N_n = M_{n+1} N_n - N_n M_n, \tag{17}$$

we obtain the equations under investigation:

$$iq_{n,t} = q_{n+1} - 2q_n + q_{n-1} + \left(|q_n|^2 - |r_n|^2\right)(q_{n+1} - q_{n-1}), \tag{18}$$

$$ir_{n,t} = r_{n+1} - 2r_n + r_{n-1} + \left(|q_n|^2 - |r_n|^2\right)(r_{n+1} + r_{n-1}). \tag{19}$$

The Darboux transformation is an effective tool for constructing exact solutions of integrable nonlinear equations. To derive the Darboux transformation for Equation (18,19), we introduce a gauge transformation:

$$\Psi_n[1] = T_n \Psi_n, \tag{20}$$

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where the transformation matrix T_n is given by:

$$T_n = \lambda A + \lambda^{-1} B + C \tag{21}$$

Where

$$A = \begin{pmatrix} a_{11,n} & a_{12,n} & a_{13,n} & a_{14,n} \\ a_{21,n} & a_{22,n} & a_{23,n} & a_{24,n} \\ a_{31,n} & a_{32,n} & a_{33,n} & a_{34,n} \\ a_{41,n} & a_{42,n} & a_{43,n} & a_{44,n} \end{pmatrix}, B = \begin{pmatrix} b_{11,n} & b_{12,n} & b_{13,n} & b_{14,n} \\ b_{21,n} & b_{22,n} & b_{23,n} & b_{24,n} \\ b_{31,n} & b_{32,n} & b_{33,n} & b_{34,n} \\ b_{41,n} & b_{42,n} & b_{43,n} & b_{44,n} \end{pmatrix}, C = \begin{pmatrix} c_{11,n} & c_{12,n} & c_{13,n} & c_{14,n} \\ c_{21,n} & c_{22,n} & c_{23,n} & c_{24,n} \\ c_{31,n} & c_{32,n} & c_{33,n} & c_{34,n} \\ c_{41,n} & c_{42,n} & c_{43,n} & c_{44,n} \end{pmatrix}.$$

By applying the gauge transformation, a spectral problem can be converted into another of the same type, transforming the spectral problem Equation (10,11) into:

$$\Psi_{n+1}[1] = N_n[1]\Psi_n[1], \tag{22}$$

$$\frac{d}{dt}\Psi_n[1] = M_n[1]\Psi_n[1]. \tag{23}$$

Combining Equations (10,11) and (20), we obtain:

$$N_n[1] = T_{n+1} N_n T_n^{-1}, \tag{24}$$

$$M_n[1] = (T_{n,t} + T_n M_n) T_n^{-1}. \tag{25}$$

Here, $N_n[1]$ and $Mn[1]$ share the same structure as N_n and M_n . By comparing the coefficients of like powers of λ on both sides of Equation (24,25), the matrix T_n can be simplified as follows:

$$T_n = \begin{pmatrix} \lambda + \frac{1}{\lambda} b_{11,n} & \frac{1}{\lambda} b_{12,n} & c_{13,n} & c_{14,n} \\ \frac{1}{\lambda} b_{12,n}^* & \lambda + \frac{1}{\lambda} b_{11,n}^* & c_{14,n}^* & c_{13,n}^* \\ c_{31,n} & c_{32,n} & a_{33,n}\lambda + \frac{1}{\lambda} & a_{34,n}\lambda \\ c_{32,n}^* & c_{31,n}^* & a_{34,n}^*\lambda & a_{33,n}^*\lambda + \frac{1}{\lambda} \end{pmatrix}. \tag{26}$$

By direct calculation, the following relation between the new and old potentials is obtained:

$$q_n[1] = q_n a_{33,n+1}^* - r_n a_{34,n+1}^* - c_{31,n+1}^*, \tag{27}$$

$$r_n[1] = r_n a_{33,n+1} - q_n a_{34,n+1} + c_{32,n+1}. \tag{28}$$

Clearly, $\det T$ is a quartic polynomial in λ , implying the existence of $\lambda_j\ (j = 1,2,3,4)$ such that $\det T = 0$.
Moreover, when $\lambda = \lambda_1$, the vector

$$\varphi_n(\lambda_1) = \left(\varphi_n^1(\lambda_1),\ \varphi_n^2(\lambda_1),\ \varphi_n^3(\lambda_1),\ \varphi_n^4(\lambda_1)\right)^\top$$

is a solution of Equation (10,11). Similarly, when $\lambda = \lambda_1^*$, the vector

$$\varphi_n(\lambda_1^*) = \left(\varphi_n^{2*}(\lambda_1^*),\ \varphi_n^{1*}(\lambda_1^*),\ \varphi_n^{4*}(\lambda_1^*),\ \varphi_n^{3*}(\lambda_1^*)\right)^\top$$

also satisfies Equation (10,11). Therefore, for $\lambda = \lambda_j\ (j = 1,2)$, a fundamental set of solutions to Equation (10,11) is given by:

$$\varphi_n(\lambda_1) = \left(\varphi_n^1(\lambda_1), \varphi_n^2(\lambda_1), \varphi_n^3(\lambda_1), \varphi_n^4(\lambda_1)\right)^\top, \quad \varphi_n(\lambda_1^*) = \left(\varphi_n^{2*}(\lambda_1^*), \varphi_n^{1*}(\lambda_1^*), \varphi_n^{4*}(\lambda_1^*), \varphi_n^{3*}(\lambda_1^*)\right)^\top$$

$$\psi_n(\lambda_2) = \left(\psi_n^1(\lambda_2), \psi_n^2(\lambda_2), \psi_n^3(\lambda_2), \psi_n^4(\lambda_2)\right)^\top, \quad \psi_n(\lambda_2^*) = \left(\psi_n^{2*}(\lambda_2^*), \psi_n^{1*}(\lambda_2^*), \psi_n^{4*}(\lambda_2^*), \psi_n^{3*}(\lambda_2^*)\right)^\top \tag{29}$$

Thus, Equation (20) can be rewritten as:

$$\Psi_n\begin{bmatrix}1\end{bmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix} \begin{pmatrix} \varphi_n^1 & \varphi_n^{2*} & \psi_n^1 & \psi_n^{2*} \\ \varphi_n^2 & \varphi_n^{1*} & \psi_n^2 & \psi_n^{1*} \\ \varphi_n^3 & \varphi_n^{4*} & \psi_n^3 & \psi_n^{4*} \\ \varphi_n^4 & \varphi_n^{3*} & \psi_n^4 & \psi_n^{3*} \end{pmatrix}. \tag{30}$$

Therefore,when $\lambda = \lambda_j (j = 1,2)$ the vectors

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$$\varphi_n[1](\lambda_1) = T\varphi_n(\lambda_1),\ \varphi_n[1](\lambda_1^*) = T\varphi_n(\lambda_1^*),$$

$$\psi_n[1](\lambda_2) = T\psi_n(\lambda_2),\ \psi_n[1](\lambda_2^*) = T\psi_n(\lambda_2^*)$$

are linearly dependent. There exist constants $\alpha_j^{(1)}, \alpha_j^{(2)}, \alpha_j^{(3)}$, not all zero, such that:

$$\begin{gathered}
T_{11}\varphi_n^1 + T_{12}\varphi_n^2 + T_{13}\varphi_n^3 + T_{14}\varphi_n^4 + \alpha_j^{(1)}\left(T_{11}\varphi_n^{2*} + T_{12}\varphi_n^{1*} + T_{13}\varphi_n^{4*} + T_{14}\varphi_n^{3*}\right) \\
+\alpha_j^{(2)}\left(T_{11}\psi_n^1 + T_{12}\psi_n^2 + T_{13}\psi_n^3 + T_{14}\psi_n^4\right) + \alpha_j^{(3)}\left(T_{11}\psi_n^{2*} + T_{12}\psi_n^{1*} + T_{13}\psi_n^{4*} + T_{14}\psi_n^{3*}\right) = 0, \\
T_{21}\varphi_n^1 + T_{22}\varphi_n^2 + T_{23}\varphi_n^3 + T_{24}\varphi_n^4 + \alpha_j^{(1)}\left(T_{21}\varphi_n^{2*} + T_{22}\varphi_n^{1*} + T_{23}\varphi_n^{4*} + T_{24}\varphi_n^{3*}\right) \\
+\alpha_j^{(2)}\left(T_{21}\psi_n^1 + T_{22}\psi_n^2 + T_{23}\psi_n^3 + T_{24}\psi_n^4\right) + \alpha_j^{(3)}\left(T_{21}\psi_n^{2*} + T_{22}\psi_n^{1*} + T_{23}\psi_n^{4*} + T_{24}\psi_n^{3*}\right) = 0, \\
T_{31}\varphi_n^1 + T_{32}\varphi_n^2 + T_{33}\varphi_n^3 + T_{34}\varphi_n^4 + \alpha_j^{(1)}\left(T_{31}\varphi_n^{2*} + T_{32}\varphi_n^{1*} + T_{33}\varphi_n^{4*} + T_{34}\varphi_n^{3*}\right) \\
+\alpha_j^{(2)}\left(T_{31}\psi_n^1 + T_{32}\psi_n^2 + T_{33}\psi_n^3 + T_{34}\psi_n^3\right) + \alpha_j^{(3)}\left(T_{31}\psi_n^{2*} + T_{32}\psi_n^{1*} + T_{33}\psi_n^{4*} + T_{34}\psi_n^{3*}\right) = 0, \\
T_{41}\varphi_n^1 + T_{42}\varphi_n^2 + T_{43}\varphi_n^3 + T_{44}\varphi_n^4 + \alpha_j^{(1)}\left(T_{41}\varphi_n^{2*} + T_{42}\varphi_n^{1*} + T_{43}\varphi_n^{4*} + T_{44}\varphi_n^{3*}\right) \\
+\alpha_j^{(2)}\left(T_{41}\psi_n^1 + T_{42}\psi_n^2 + T_{43}\psi_n^3 + T_{44}\psi_n^3\right) + \alpha_j^{(3)}\left(T_{41}\psi_n^{2*} + T_{42}\psi_n^{1*} + T_{43}\psi_n^{4*} + T_{44}\psi_n^{3*}\right) = 0
\end{gathered} \tag{31}$$

Rearranging these equations, we obtain:

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix} \begin{pmatrix} 1 \\ \beta_j^{(1)} \\ \beta_j^{(2)} \\ \beta_j^{(3)} \end{pmatrix} = 0 \tag{32}$$

where

$$\begin{aligned}
\beta_j^{(1)} &= \frac{\varphi_n^2 + \alpha_j^{(1)}\varphi_n^{1*} + \alpha_j^{(2)}\psi_n^2 + \alpha_j^{(3)}\psi_n^{1*}}{\varphi_n^1 + \alpha_j^{(1)}\varphi_n^{2*} + \alpha_j^{(2)}\psi_n^1 + \alpha_j^{(3)}\psi_n^{2*}}, \\
\beta_j^{(2)} &= \frac{\varphi_n^3 + \alpha_j^{(1)}\varphi_n^{4*} + \alpha_j^{(2)}\psi_n^3 + \alpha_j^{(3)}\psi_n^{4*}}{\varphi_n^1 + \alpha_j^{(1)}\varphi_n^{2*} + \alpha_j^{(2)}\psi_n^1 + \alpha_j^{(3)}\psi_n^{2*}}, \\
\beta_j^{(3)} &= \frac{\varphi_n^4 + \alpha_j^{(1)}\varphi_n^{3*} + \alpha_j^{(2)}\psi_n^4 + \alpha_j^{(3)}\psi_n^{3*}}{\varphi_n^1 + \alpha_j^{(1)}\varphi_n^{2*} + \alpha_j^{(2)}\psi_n^1 + \alpha_j^{(3)}\psi_n^{2*}}.
\end{aligned} \tag{33}$$

By substituting Equation (26) into Equation (32), the following linear algebraic system is obtained:

$$\begin{pmatrix} \lambda_j + \frac{1}{\lambda_j} b_{11,n} & \frac{1}{\lambda_j} b_{12,n} & c_{13,n} & c_{14,n} \\ \frac{1}{\lambda_j} b_{12,n}^* & \lambda_j + \frac{1}{\lambda_j} b_{11,n}^* & c_{14,n}^* & c_{13,n}^* \\ c_{31,n} & c_{32,n} & a_{33,n}\lambda_j + \frac{1}{\lambda_j} & a_{34,n}\lambda_j \\ c_{32,n}^* & c_{31,n}^* & a_{34,n}^*\lambda_j & a_{33,n}^*\lambda_j + \frac{1}{\lambda_j} \end{pmatrix} \begin{pmatrix} 1 \\ \beta_j^{(1)} \\ \beta_j^{(2)} \\ \beta_j^{(3)} \end{pmatrix} = 0. \tag{34}$$

By appropriately selecting $\lambda_j, \alpha_j^{(1)}, \alpha_j^{(2)}, \alpha_j^{(3)} \left(j = 1,2,3,4\right)$, the determinant of the coefficient matrix in the linear algebraic system (33) can be ensured to be nonzero. Under this condition, the variables $a_{33,n},\ a_{34,n},\ b_{11,n},\ b_{12,n},\ c_{13,n},\ c_{14,n},\ c_{31,n},\ c_{32,n}$ can be uniquely determined by (34).

3. Explicit exact solutions

This section focuses on obtaining exact solutions of Equation (18,19) through the application of the Darboux transformation (27,28).

Initially, we select the seed solution $q_n = 0, r_n = 0$. In this case, the matrices N_n and M_n are given by:

$$N_n = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & {\lambda^*}^{-1} \end{pmatrix}, \tag{35}$$

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$$M_n = i\begin{pmatrix} -\frac{\lambda^2+\lambda^{-2}}{2}+1 & 0 & 0 & 0 \\ 0 & -\frac{\lambda^{*2}+\lambda^{*-2}}{2}+1 & 0 & 0 \\ 0 & 0 & \frac{\lambda^2+\lambda^{-2}}{2}-1 & 0 \\ 0 & 0 & 0 & \frac{\lambda^{*2}+\lambda^{*-2}}{2}-1 \end{pmatrix}. \tag{36}$$

Thus, the spatial spectral problem $\Psi_{n+1} = N_n\Psi_n$ and the temporal spectral problem $\frac{d}{dt}\Psi_n = M_n\Psi_n$ admit the following four fundamental solutions:

$$\varphi_n\left(\lambda\right) = \begin{pmatrix} \lambda^n e^{i\left(-\frac{\lambda^2+\lambda^{-2}}{2}+1\right)t} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_n\left(\lambda\right) = \begin{pmatrix} 0 \\ \lambda^{*n} e^{i\left(-\frac{\lambda^{*2}+\lambda^{*-2}}{2}+1\right)t} \\ 0 \\ 0 \end{pmatrix}$$
$$x_n\left(\lambda\right) = \begin{pmatrix} 0 \\ 0 \\ \lambda^{-n} e^{i\left(\frac{\lambda^2+\lambda^{-2}}{2}-1\right)t} \\ 0 \end{pmatrix}, \quad y_n\left(\lambda\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lambda^{*-n} e^{i\left(\frac{\lambda^{*2}+\lambda^{*-2}}{2}-1\right)t} \end{pmatrix}. \tag{37}$$

Substituting these fundamental solutions into Equation (33), we obtain:

$$\beta_j^{(1)} = \frac{\alpha_j^{(1)}\lambda_j^{*n} e^{i\left(-\frac{\lambda_j^{*2}+\lambda_j^{*-2}}{2}+1\right)t}}{\lambda_j^n e^{i\left(-\frac{\lambda_j^2+\lambda_j^{-2}}{2}+1\right)t}} = \alpha_j^{(1)}\lambda_j^{*n}\lambda_j^{-n} e^{i\left(-\frac{\lambda_j^{*2}+\lambda_j^{*-2}}{2}+\frac{\lambda_j^2+\lambda_j^{-2}}{2}\right)t},$$
$$\beta_j^{(2)} = \frac{\alpha_j^{(2)}\lambda_j^{-n} e^{i\left(\frac{\lambda_j^2+\lambda_j^{-2}}{2}-1\right)t}}{\lambda_j^n e^{i\left(-\frac{\lambda_j^2+\lambda_j^{-2}}{2}+1\right)t}} = \alpha_j^{(2)}\lambda_j^{-2n} e^{i\left(\lambda_j^2+\lambda_j^{-2}-2\right)t},$$
$$\beta_j^{(3)} = \frac{\alpha_j^{(3)}\lambda_j^{*-n} e^{i\left(\frac{\lambda_j^{*2}+\lambda_j^{*-2}}{2}-1\right)t}}{\lambda_j^n e^{i\left(-\frac{\lambda_j^2+\lambda_j^{-2}}{2}+1\right)t}} = \alpha_j^{(3)}\left(\lambda_j^{*-n}\lambda_j^{-n}\right) e^{i\left(\frac{\lambda_j^{*2}+\lambda_j^{*-2}}{2}+\frac{\lambda_j^2+\lambda_j^{-2}}{2}-2\right)t}. \tag{38}$$

Thus, the system of Equation (34) transforms into:

$$\begin{cases} \left(\lambda_j + b_{11,n}\lambda_j^{-1}\right) + b_{12,n}\lambda_j^{-1}\beta_j^{(1)} + c_{13,n}\beta_j^{(2)} + c_{14,n}\beta_j^{(3)} = 0 \\ b_{12,n}^*\lambda_j^{-1} + \left(\lambda_j + b_{11,n}^*\lambda_j^{-1}\right)\beta_j^{(1)} + c_{14,n}^*\beta_j^{(2)} + c_{13,n}^*\beta_j^{(3)} = 0 \\ c_{31,n} + c_{32,n}\beta_j^{(1)} + \left(a_{33,n}\lambda_j + \lambda_j^{-1}\right)\beta_j^{(2)} + a_{34,n}\lambda_j\beta_j^{(3)} = 0 \\ c_{32,n}^* + c_{31,n}^*\beta_j^{(1)} + a_{34,n}^*\lambda_j\beta_j^{(2)} + \left(a_{33,n}^*\lambda_j + \lambda_j^{-1}\right)\beta_j^{(3)} = 0 \end{cases} \tag{39}$$

Rewriting the third and fourth equations, we obtain:

$$c_{31,n} + \beta_j^{(1)}c_{32,n} + \lambda_j\beta_j^{(2)}a_{33,n} + \lambda_j\beta_j^{(3)}a_{34,n} = -\lambda_j^{-1}\beta_j^{(2)}. \tag{40}$$

$$c_{31,n}\beta_j^{(1)*} + c_{32,n} + a_{33,n}\lambda_j^*\beta_j^{(3)*} + a_{34,n}\lambda_j^*\beta_j^{(2)*} = -\lambda_j^{*-1}\beta_j^{(3)*}. \tag{41}$$

Using Cramer's rule, we obtain:

$$c_{31,n} = \frac{\Delta_{2,n}}{\Delta_{1,n}}, \tag{42}$$

$$c_{32,n} = \frac{\Delta_{3,n}}{\Delta_{1,n}}, \tag{43}$$

Where $\Delta_{1,n}$ and $\Delta_{2,n}, \Delta_{3,n}$, are determinants given by:

$$\Delta_{1,n}=\begin{vmatrix}1 & \beta_1^{(1)} & \lambda_1\beta_1^{(2)} & \lambda_1\beta_1^{(3)}\\ 1 & \beta_2^{(1)} & \lambda_2\beta_2^{(2)} & \lambda_2\beta_2^{(3)}\\ \beta_1^{(1)^*} & 1 & \lambda_1^*\beta_1^{(3)^*} & \lambda_1^*\beta_1^{(2)^*}\\ \beta_2^{(1)^*} & 1 & \lambda_2^*\beta_2^{(3)^*} & \lambda_2^*\beta_2^{(2)^*}\end{vmatrix} \tag{44}$$

$$\Delta_{2,n}=\begin{vmatrix}-\lambda_1^{*-1}\beta_1^{(2)} & \beta_1^{(1)} & \lambda_1\beta_1^{(2)} & \lambda_1\beta_1^{(3)}\\ -\lambda_2^{*-1}\beta_2^{(2)} & \beta_2^{(1)} & \lambda_2\beta_2^{(2)} & \lambda_2\beta_2^{(3)}\\ -\lambda_1^{*-1}\beta_1^{(3)^*} & 1 & \lambda_1^*\beta_1^{(3)^*} & \lambda_1^*\beta_1^{(2)^*}\\ -\lambda_2^{*-1}\beta_2^{(3)^*} & 1 & \lambda_2^*\beta_2^{(3)^*} & \lambda_2^*\beta_2^{(2)^*}\end{vmatrix}$$

(45)

$$\Delta_{3,n}=\begin{vmatrix}1 & -\lambda_1^{-1}\beta_1^{(2)} & \lambda_1\beta_1^{(2)} & \lambda_1\beta_1^{(3)}\\ 1 & -\lambda_2^{-1}\beta_2^{(2)} & \lambda_2\beta_2^{(2)} & \lambda_2\beta_2^{(3)}\\ \beta_1^{(1)^*} & -\lambda_1^{*-1}\beta_1^{(3)^*} & \lambda_1^*\beta_1^{(3)^*} & \lambda_1^*\beta_1^{(2)^*}\\ \beta_2^{(1)^*} & -\lambda_2^{*-1}\beta_2^{(3)^*} & \lambda_2^*\beta_2^{(3)^*} & \lambda_2^*\beta_2^{(2)^*}\end{vmatrix}. \tag{46}$$

By substituting the seed solution into Equation (27,28), the new solution is obtained as follows:

$$r_n\left[1\right]=c_{32,n+1}=\frac{\Delta_{3,n+1}}{\Delta_{1,n+1}}. \tag{47}$$

$$q_n\left[1\right]=-c_{31,n+1}^*=-\frac{\Delta_{2,n+1}}{\Delta_{1,n+1}}. \tag{48}$$

In order to gain a clearer understanding of the obtained solutions and to investigate their dynamical properties, we define $\alpha_j^{(i)}=e^{h_{ji}}$ and choose the parameters as: $\lambda_1=1,\lambda_2=1,\lambda_3=1+i,\lambda_4=-1+i$ and $h_{11}=-1,h_{21}=1,h_{31}=1,h_{41}=1,h_{12}=1,h_{22}=-1,h_{32}=1,h_{42}=1,h_{13}1,h_{23}=1,h_{33}=-1,h_{43}=1$ .Under this choice of parameters, a one-soliton solution of the two-component semi-discrete coupled nonlinear Schrödinger equation on the zero background is obtained, as demonstrated in Figure 1. This solution corresponds to a breather, exhibiting periodic oscillations in the temporal direction, with its amplitude alternating between growth and decay. Along the n -axis, the solution decays exponentially, which can be interpreted as the interaction between a soliton and its counterpart. Furthermore, the solution maintains its localized profile throughout the propagation process.

Figure 1. Single-soliton solution of the semi-discrete coupled local nonlinear Schrödinger equation on zero background

4. Conclusion

In this paper, we focus on the investigation of a two-component integrable semi-discrete coupled local nonlinear Schrödinger equation. Based on the Lax pair, the Darboux transformation (DT) of the system is constructed, and its validity is established through a formal proposition. By choosing appropriate parameters, breather solutions on the zero background are derived. Furthermore, the dynamical behaviors of these solitons are analyzed in detail. The results of this work further reveal novel dynamical distributions of the nonlinear coupled local Schrödinger equation. The proposed approach can also be applied to soliton equations arising from nonlinear local problems in physics and mathematics.

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