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On Quantum Dense Coding for Bipartite Systems

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Abstract. Quantum communication, as a vital area of quantum information science, aims to leverage quantum phenomena—particularly entanglement—to achieve tasks that are impossible or inefficient in classical systems. It plays a key role in the development of secure communication protocols and advanced information processing. This paper focuses on one such protocol, dense coding, which enables the transmission of two classical bits using only one entangled quantum particle."We introduce the mathematical and physical foundations of quantum communication, such as the entangled state and Kronecker product. Then we propose the quantum communication protocol, namely dense coding using Bell states. It offer unique advantages that are beyond the classical parts. For this purpose, we in detail analyze the quantum measurements using Pauli gates and identity matrix, as well as the transmission of particle from Alice to Bob. They are necessary steps for dense coding. Then we analyze how Bob finishes the protocol using measurement over two qubits.

Keywords: Quantum Entanglement, Dense Coding, Quantum Communication Protocol, Quantum Teleportation, Remote-Controlled Operations

1. Introduction

Quantum entanglement has been applied to various quantum protocols such as teleportation and dense coding assisted by classical channels [1, 2]. References [3, 4] showed showed the operational principles and mathematical foundations of several existing quantum communication protocols. Besides, quantum protocols can also be applied to classical computers [5]. Currently existing classical encryption methods are no longer entirely secure, as methods for rapidly breaking classical encryption have been proposed [6]. After addressing issues such as decoherence and noise [7, 8], the unique advantages of quantum encryption are gradually becoming apparent [9,10]. Subsequently, researchers investigated practical steps for quantum information transmission protocols, aiming to reduce costs and enhance efficiency [11]. Next, the quantum cost of dense coding and teleportation protocols has been evaluated in terms of CNOT gates [12]. Authors in [13] showed that the quantum remote control may be applied to realize the teleportation unitary gates. Furthermore, a similar technique has been used to teleport angles in a secure manner [14]. Further, multi partite scenarios have been considered for a deterministic single qubit. That is, recent advancements in the field of remote implementation of partially unknown quantum operations on multiple qubits have led to significant breakthroughs, rendering practical operations feasible. These developments have substantially propelled the potential applications of quantum communication and networking

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forward [15]. Besides, operation sharing with five-qubit cluster state has been established [16,17]. These operations and protocols provide a good foundation for realizing more complex protocols of transmitting information securely.

In this paper, we extend the above protocols by constructing a multi-party quantum security protocol based on the so-called remote-controlled implementation of operations. In Section II, we delve into the entangled state, inner product, and the Kronecker product. In Section III, we discuss a quantum communication protocol in which Bob sends messages to Charlie, and Alice has the ability to control the message transmission. By redesigning the communication protocol, we achieve the controlled communication among Alice and Bob. Alice measures her qubit and then sends it to the receiver Bob. As a result, Bob can perform a local measurement as a projection, and he manages to decode the secret from Alice. We finally conclude in Section V.

2. Preliminaries

We define the inner product of two n-dimensional vectors $|a\rangle$. and $|b\rangle$. If $|a\rangle = [a1,..., an]^T$ and $\backslash b\rangle = [b1,..., bn]^T$, then their inner product is

$$\langle a|b\rangle := \sum_{j=1}^{n} a_j^* b_j \tag{1}$$

Next we present the Bell state used in the protocol of quantum dense coding. The state is

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2}$$

Where $|00\rangle := |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, because we use the standard orthonormal qubits $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The following are some basics on complex numbers, namely $e^{i\pi}+1=0, e^{i\pi}+1=0$ and $\cos^{-1}\theta;\ \sin^{-1}\theta$;

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3}$$

A general complex number x = a + bi with real a, b can be written as

$$x = a + bi = |x|e^{i\theta} = |x|(\cos\theta + i\sin\theta) = \sqrt{a^2+b^2}(\cos\theta + i\sin\theta) \tag{4}$$

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2.1. Kronecker product and the vec operator

Definition 1. Let A be an n × p matrix and B an m × q matrix. The mn × pq matrix

$$A\otimes B=\begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,p}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,p}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,p}B \end{bmatrix} \tag{5}$$

is called the Kronecker product of A and B. It is also called the direct product or the tensor product.

$$I_n=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{6}$$

$$\mathrm{diag}(x_1,...,x_n)=\begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix} \tag{7}$$

$$\mathrm{diag}(x_1,...,x_n)=I_n\cdot \mathrm{diag}(x_1,...,x_n)=\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots 1 \end{bmatrix}\begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix} \tag{8}$$

$$I_n\cdot \mathrm{diag}(x_1,...,x_n)=\mathrm{diag}(x_1,...,x_n) \tag{9}$$

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2.2. Gates and states

We introduce the Hadamard gate as follows.

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{10}$$

The Hadamard operation can transform the qubits $|0\rangle$ and $|1\rangle$ into their superposition with equal weights. The specific formulas are as follows.

$$H\,|0\rangle=\frac{|0\rangle+|1\rangle}{\sqrt{2}} \tag{11}$$

$$H\,|1\rangle=\frac{|0\rangle-|1\rangle}{\sqrt{2}} \tag{12}$$

Next, the CNOT gate is a quantum gate capable of transforming quantum states in the form of $|a, b\rangle$ into $|a, a \oplus b\rangle$ where a, b = 0, 1. The operating principle of the CNOT gate is to take the first qubit as the control qubit, and perform the opposite transformation on the second qubit based on the state of the control qubit. It is important to note that the CNOT gate cannot operate on individual quantum states, as it automatically considers the first qubit as the decision maker. The CNOT gate has the matrix form as follows.

$$CNOT=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix} \tag{13}$$

There are four forms of Bell states, each of which is a maximally entangled state of two quantum bits. The properties of entanglement are used to transfer information in the following quantum communication. The specific formula of four Bell states are as follows.

$$|\Phi^{+}\rangle=\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right) \tag{14}$$

$$|\Phi^{-}\rangle=\frac{1}{\sqrt{2}}\left(|00\rangle-|11\rangle\right) \tag{15}$$

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$$|\Psi^{+}\rangle=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle) \tag{16}$$

$$|\Psi^{-}\rangle=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle) \tag{17}$$

3. Result on dense coding

In this section, we introduce the basics of quantum dense coding protocol, see a simple description in Figure 1. As the first step, we generate a pair of entangled EPR states through the source S, shared between Alice and Bob. The state of the EPR pair can be represented as the Bell state (37). To obtain the Bell state $|\phi\rangle$, a Hadamard gate is first applied to the initial state $|0\rangle$, namely on the first qubit. This is followed by a controlled-NOT (CNOT) operation. Mathematically, this process is as follows:

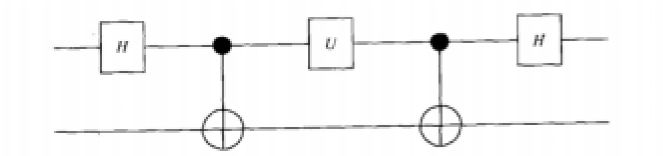

Figure 1. A quantum circuit that performs a dense coding scheme

In the computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, the EPR pair and the gate operations can be represented using matrices. The result shows that the obtained $|\phi\rangle$ can be expressed in matrix form as,

$$|\phi^{1}\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \tag{18}$$

In the computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, (Note: The first digit refers to Alice's half of the EPR pair, and the second digit refers to Bob's half), the transformation (4.20) has the following matrix representation,

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \tag{19}$$

The value of U is determined by the classical comparison of the two bits that Alice wants to transmit to Bob. If she wants to transmit $|00\rangle$, she performs the following operation on her half of

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the EPR pair,

$$(I \otimes I)|\phi^{+}\rangle = |\phi^{+}\rangle \tag{20}$$

If she wants to transmit |01〉 , she applies the Pauli matrix σx on her half of the EPR pair,

$$(I\sigma_x \otimes I)|\phi^{+}\rangle = |\psi^{+}\rangle \tag{21}$$

In the computational basis, the state is represented as follows,

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

If she wants to transmit |10〉 , she applies σz on her half of the EPR pair,

$$(I\sigma_z \otimes I)|\phi^{+}\rangle = |\phi^{+}\rangle \tag{23}$$

and the matrix form is shown as,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{24}$$

Finally, if she wants to transmit |11〉 , she applies $i\sigma_y$ on her half of the EPR pair,

$$(i\sigma_y \otimes I)|\phi^{+}\rangle = |\psi^{-}\rangle \tag{25}$$

and the matrix form is shown as,

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$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \tag{26}$$

We calculate the tensor products $\sigma_x \otimes \sigma_y$ and $I \otimes \sigma_x$.

$$\sigma_x \otimes \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$$I \otimes \sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{28}$$

Since

$$A^{\dagger} = A; \; B^{\dagger} = B \tag{29}$$

we have

$$\{i[A,B]\}^{\dagger} = \{iAB - iBA\}^{\dagger} = -iB^{\dagger}A^{\dagger} + iA^{\dagger}B^{\dagger} = i[A,B] \tag{30}$$

$$\sigma_x \otimes \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

$$I \otimes \sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{32}$$

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We also need decode the message by inverse operation. That is,

$$(\mathrm{CNOT}(\mathrm{H}\otimes\mathrm{I}))^{-1}=(\mathrm{H}\otimes\mathrm{I})\mathrm{CNOT} \tag{33}$$

We record this operator as B. By taking the decoding operator, Bob can finally obtain the correct message.

$$\mathrm{B}\left|\psi^{+}\right\rangle=|01\rangle,\mathrm{B}\left|\psi^{-}\right\rangle=|11\rangle,\mathrm{B}\left|\phi^{+}\right\rangle=|00\rangle,\mathrm{B}\left|\phi^{-}\right\rangle=|10\rangle \tag{34}$$

Any unitary operator U is orthogonal and thus can be diagonalized. Therefore, its spectral decomposition can be written as

$$\mathrm{U}=\sum_{\mathrm{j}}\lambda_{\mathrm{j}}\langle\mathrm{j}\rangle\langle\mathrm{j}| \tag{35}$$

Since a unitary operator preserves the inner product of vectors, we have

$$\langle\mathrm{j}|\mathrm{U}^{\dagger}\mathrm{U}|\mathrm{j}\rangle=\lambda_{\mathrm{j}}\lambda_{\mathrm{j}}^{*}=1 \tag{36}$$

This implies that $|\lambda_j| = 1$, so Equation (19) can be rewritten as,

$$\mathrm{U}=\sum_{\mathrm{j}}\mathrm{e}^{\mathrm{i}\alpha_{\mathrm{j}}}|\mathrm{j}\rangle\langle\mathrm{j}| \tag{37}$$

where α_j are real numbers. Now, we define the operator

$$\mathrm{A}=\sum_{\mathrm{j}}\alpha_{\mathrm{j}}\langle\mathrm{j}\rangle\langle\mathrm{j}| \tag{38}$$

Since A is already in diagonal form, the operator e^{iA} can be directly written as

$$\mathrm{e}^{\mathrm{iA}}=\sum_{\mathrm{j}}\mathrm{e}^{\mathrm{i}\alpha_{\mathrm{j}}|\mathrm{j}\rangle\langle\mathrm{j}|} \tag{39}$$

Thus, e^{iA} = U. Next, we prove that A is Hermitian. First, we have

$$\mathrm{U}=\sum_{\mathrm{n}=0}^{\infty}\frac{(\mathrm{iA})^{\mathrm{n}}}{\mathrm{n}!} \tag{40}$$

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$$U^{\dagger}=\sum_{n=0}^{\infty}\left[\frac{(iA)^{n}}{n!}\right]^{\dagger}=\sum_{n=0}^{\infty}\frac{(-iA)^{n}}{n!}=e^{-iA^{\dagger}} \tag{41}$$

Thus, the condition $U^{\dagger}$ = U-1 is satisfied only if $A^{\dagger}$ = A, i.e., A is a Hermitian operator.

$$\Delta\sigma_{x}\Delta\sigma_{y}\geq\frac{1}{2}\left|\langle 0|\left[\left[\sigma_{x},\sigma_{y}\right]\right]0\rangle\right| \tag{42}$$

Computing the commutator $\left[\sigma_{x},\sigma_{y}\right]$, we obtain

$$\Delta\sigma_{x}\Delta\sigma_{y}\geq\frac{1}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}2i & 0\\ 0 & -2i\end{bmatrix}\begin{bmatrix}1\end{bmatrix}=1 \tag{43}$$

The first instrument measures the final state |+⟩z = |1⟩. The particle in state |1⟩ then enters the second instrument and is prepared in the state

$$|+\rangle_{y}=\frac{1}{\sqrt{2}}\left(-i|0\rangle+|1\rangle\right) \tag{44}$$

Finally, the third instrument analyzes the state |+⟩y, measuring σx. With equal probabilities p0 and p1, the two possible output states |0⟩x = |0⟩ and |1⟩x = |1⟩ are obtained,

$$p_{0}=|\langle 0|\rangle_{+}|_{y}|^{2}=\frac{1}{2},\ p_{1}=|\langle 1|\rangle_{+}|_{y}|^{2}=\frac{1}{2} \tag{45}$$

The solution to the Schrodinger equation is given by

$$|\psi(t)\rangle=U(t)|\psi(0)\rangle \tag{46}$$

where the unitary time evolution operator is

$$U(t)=\exp\left(-\frac{i}{\hbar}Ht\right) \tag{47}$$

and

$$H=-\mu\cdot\sigma \tag{48}$$

Expanding U(t), we obtain

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$$U(t)=e^{i\alpha n\cdot\sigma}=\cos\alpha I+i\sin\alpha(n\cdot\sigma) \tag{49}$$

Finally, in matrix representation,

$$U(t)=\begin{bmatrix}\cos\alpha+i\sin\alpha n_z & \sin\alpha(n_y+in_x)\\ \sin\alpha(-n_y+in_x) & \cos\alpha-i\sin\alpha n_z\end{bmatrix} \tag{50}$$

4. Quantum teleportation

Quantum invisible transmission of states through quantum entanglement of Bell states allows the transmission of quantum information at both ends solely through the classical channel. The principles and methods are as follows. Suppose Alice wants to convey to Bob the two-energy-level quantum information that is in a superposition state $\alpha|0\rangle+\beta|1\rangle$. She can use the second method which is shown as follows.

Step 1. We build the EPR pair.

The source S generates an EPR pair namely a Bell state, and sends the first half to Alice and the second half to Bob. Make sure the pair of quanta is in an entangled state $|\psi^+\rangle=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$. To build this EPR pair, S should carry out the physical operation as follows.

$$CNOT(H\otimes I)|00\rangle=|\psi^+\rangle \tag{51}$$

By sending the EPR pair to Alice and Bob, Alice has two particles, which are respectively half of the EPR pair and the message qubit she wants to transmit to Bob. On the other hand, Bob only has one qubit, namely the other half of EPR pair.

The state of the three qubits are given by the tensor product of $|\psi\rangle$ and $|\psi^+\rangle$ as follows.

$$\begin{aligned}|\psi\rangle\otimes|\psi^+\rangle &=(\alpha|0\rangle+\beta|1\rangle)\otimes\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)\\ &=\frac{\alpha}{\sqrt{2}}(|001\rangle+|010\rangle)+\frac{\beta}{\sqrt{2}}(|101\rangle+|110\rangle).\end{aligned} \tag{52}$$

Step 2. Joint measurement by Alice.

The reason why Alice needs to do the joint measurement is that if Alice only determines message qubit, this quantum will collapse to $|0\rangle$ or $|1\rangle$. The way out of this problem is to make measurements on the Bell state, i.e., $\langle\phi^+\rangle,\langle\phi^-\rangle,\langle\psi^+\rangle,\langle\psi^-\rangle$

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$$\begin{cases} |00\rangle = \frac{1}{\sqrt{2}}\left(|\phi^{+}\rangle + |\phi^{-}\rangle\right), \\ |11\rangle = \frac{1}{\sqrt{2}}\left(|\phi^{+}\rangle - |\phi^{-}\rangle\right), \\ |01\rangle = \frac{1}{\sqrt{2}}\left(|\psi^{+}\rangle + |\psi^{-}\rangle\right), \\ |10\rangle = \frac{1}{\sqrt{2}}\left(|\psi^{+}\rangle - |\psi^{-}\rangle\right). \end{cases} \tag{53}$$

Then we insert Eq. (37) into (36), by transforming the computational basis into the Bell basis.

$$\begin{aligned} |\psi\rangle \otimes |\psi^{+}\rangle \quad &= \frac{\alpha}{2}\left(|\phi^{+}| + |\phi^{-}\right)|1\rangle + \frac{\alpha}{2}\left(|\psi^{+}| + |\psi^{-}\right)|0\rangle \\ &+ \frac{\beta}{2}\left(|\psi^{+}\rangle - |\psi^{-}\rangle\right)|1\rangle + \frac{\beta}{2}\left(|\phi^{+}\rangle - |\phi^{-}\rangle\right)|0\rangle \\ &= \frac{1}{2}|\psi^{+}\rangle(\alpha|0\rangle + \beta|1\rangle + \frac{1}{2}|\psi^{-}\rangle(\alpha|0\rangle - \beta|1\rangle \\ &+ \frac{1}{2}|\phi^{+}\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|\phi^{-}\rangle(\alpha|1\rangle - \beta|0\rangle)\ . \end{aligned} \tag{54}$$

In this case, if Alice does a Bell Measurement, she has the same chance to obtain every state of $\langle\phi^{+}\rangle, \langle\phi^{-}\rangle, \langle\psi^{+}\rangle, \langle\psi^{-}\rangle$. By using suitable unitary operation (see the next subsection), Alice can change these four measurement results to some message which can be sent by classical channels.

$$\begin{aligned} &\frac{1}{2}|01\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|0\rangle - \beta|1\rangle) \\ &+ \frac{1}{2}|\langle 00\rangle\left(\alpha\langle 1\rangle + \beta\langle 0\rangle\right) + \frac{1}{2}|\langle 10\rangle\left(\alpha\langle 1\rangle - \beta\langle 0\rangle\right)\rangle\ . \end{aligned} \tag{55}$$

Step 3. Unitary operation is carried out by Bob.

Alice sends the result of the Bell measurements to Bob by classical messages. Bob can use this message to determine which of the quantum states is this message corresponds to.

$$|00\rangle \to (\alpha|1 + \beta|0\rangle) \to \sigma_{x} \tag{56}$$

$$|10\rangle \to (\alpha|1\rangle - \beta|0\rangle) \to i\sigma_{y} \tag{57}$$

$$|01\rangle \to (\alpha|0 + \beta|1\rangle) \to I \tag{58}$$

$$|11\rangle \to (\alpha|0\rangle - \beta|1\rangle) \to \sigma_{z} \tag{59}$$

By taking this operation, Bob can obtain the message qubit $\alpha|0\rangle + \beta|1\rangle$. So we have completed the protocol of quantum teleportation.

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5. Conclusions

We have shown the dense coding protocol of controlled remote implementation of operations in two participants. They show the quantum security guaranteed by the basic rules of quantum mechanics by using entanglement and measurement. An open problem unsolved in this paper is to extend the results to implementation of operations of many parties. Whether entanglement in this protocol could be decreased is also an interesting issue for the next step.

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