Proceedings of the 2nd International Conference on Computing Innovation and Applied Physics (CONF-CIAP 2023)
DOI: 10.54254/2753-8818/5/20230282

Geometric solution of a six order equation by three-fold origami

Yige Chen

The Second High School Attached to Beijing Normal University, Beijing, China，100192

1020050627@qq.com

Abstract. Robert J. Lang has proposed a theorem that when solving equations using multi-folds origami, general equations of order n can be solved using n-2 simultaneous folds. However, recently Jorge C. Lucero proved that arbitrary five order equations can be solved using two simultaneous folds. Combining this with the fact that one single fold can solve general quartic equations, the writer questions that whether the theorem may be altered into general equations of order n can be solved by n-3 simultaneous folds. Thus, in this paper, a method of geometric graphing -Lill's method is used to try to solve six order equations with three simultaneous folds. By conducting case analysis using theoretical knowledge, it can be found that the six order equations can be possibly solved by three simultaneous folds. Besides, a comparison of solving equations with origami constructions and compass-and-straightedge is carried out. The result will encourage more research on using origami to solve higher order equations and inspire people to pay more attention to origami construction, which is more powerful, accurate, and efficient than the compass-and-straightedge people usually use.

Keywords: origami, analytic geometry, limit, equations, Lill's method.

1. Introduction

Origami originated at about 105 A.D in China as an art form, and flourished in Japan in the 6th century. As it turned to the 20th century, Akira Yoshizawa created the symbols and notations of folding diagrams, which led to the combination of origami and mathematics, forming "origamics" [1]. Six origami axioms were then introduced by Huzita [2], defining distinct ways of constructing single folds by aligning preexisting points and lines. Several years later, Justin discovered the seventh axiom [2]. All of these constitute the seven Huzita-Justin Axioms, which have been proved including all of the basic operations in origami constructions [3].

According to Lill's method, origami construction is able to solve arbitrary equations of order four or less [4]. However, origami cannot solve general quintic equations. It is important to note that the Huzita-Justin Axioms are restricted to single folds. If allowing more than one fold at a time, higher order equations can be solved. In general, it is possible to solve arbitrary nth-degree equations using n-2 simultaneous folds [3]. Still, there are exceptions: it is shown that general quintic equations are able to be solved using two simultaneous folds [5], which raises the question of whether three simultaneous folds are able to solve equations of order higher than 3+2=5, namely, order six.

© 2023 The Authors. This is an open access article distributed under the terms of the Creative Commons Attribution License 4.0 (https://creativecommons.org/licenses/by/4.0/).

DOI: 10.54254/2753-8818/5/20230282

In order to further investigate whether high order equations can be solve by origami using less than n-2 simultaneous folds, a six order equation will be identified, and it will be solved by three simultaneous folds using Lill's method. Through this work, it is demonstrated that there exist six order equations solvable by three simultaneous folds, which questions the existing n-fold axiom and encourages more research in this area.

2. Basic theories

2.1. Origami axioms

In origami, only two kinds of geometric figures can be constructed —— lines and points (intersections of lines), both of which have two degrees of freedom. As a result, there are five alignments (incidences between points, lines, and either of their folded image) in origami [3].

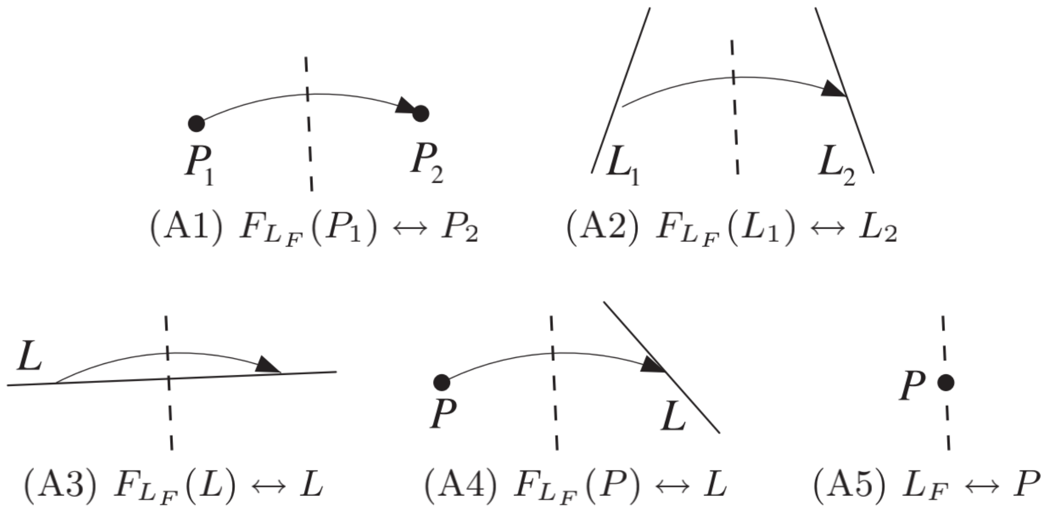

Figure 1. all of the one fold alignments [3].

In figure 1, A1 to A5 formed by choosing one or two of the basic elements (points and lines) and then combining them. Specifically, A1 and A2 specify 2 equations so that they can specify a single fold line, but A3~A5 only specify one [3]. As a result, these three alignments need to combine to form origami axioms, which include:

Axiom 1 (O1): When there are two points (P1 and P2), a line connecting them can be folded.

Axiom 2 (O2): When there are two points (P1 and P2), P1 can be folded onto P2.

Axiom 3 (O3): When there are two lines (L1 and L2), L1 can be folded onto L2.

Axiom 4 (O4): When there are a point (P) and a line (L), a fold line that pass through P and is perpendicular to L can be made.

Axiom 5 (O5): When there are two points (P1 and P2) as well as a line (L), it is possible to make at least one fold line that put P1 on L and, at the same time, pass trough P2.

Axiom 6 (O6): When there are two points (P1 and P2) as well as two lines (L1 and L2), it is possible to make at least one fold line that put P1 on L1 and, at the same time, P2 on L2.

Axiom 7 (O7): When there are a point (P) and two lines (L1 and L2), it is possible to create a fold line that is perpendicular to L2 and that puts P1 on L1[6].

The ways of combining alignments are listed as in table 1.

DOI: 10.54254/2753-8818/5/20230282

Table 1. All of the possible combinations of alignments involving one equation forming origami axioms [3].

	A3	A4	A5
A3	N/A	O7	O4
A4	O7	O6	O5
A5	O4	O5	O1

2.2. Two folds axioms

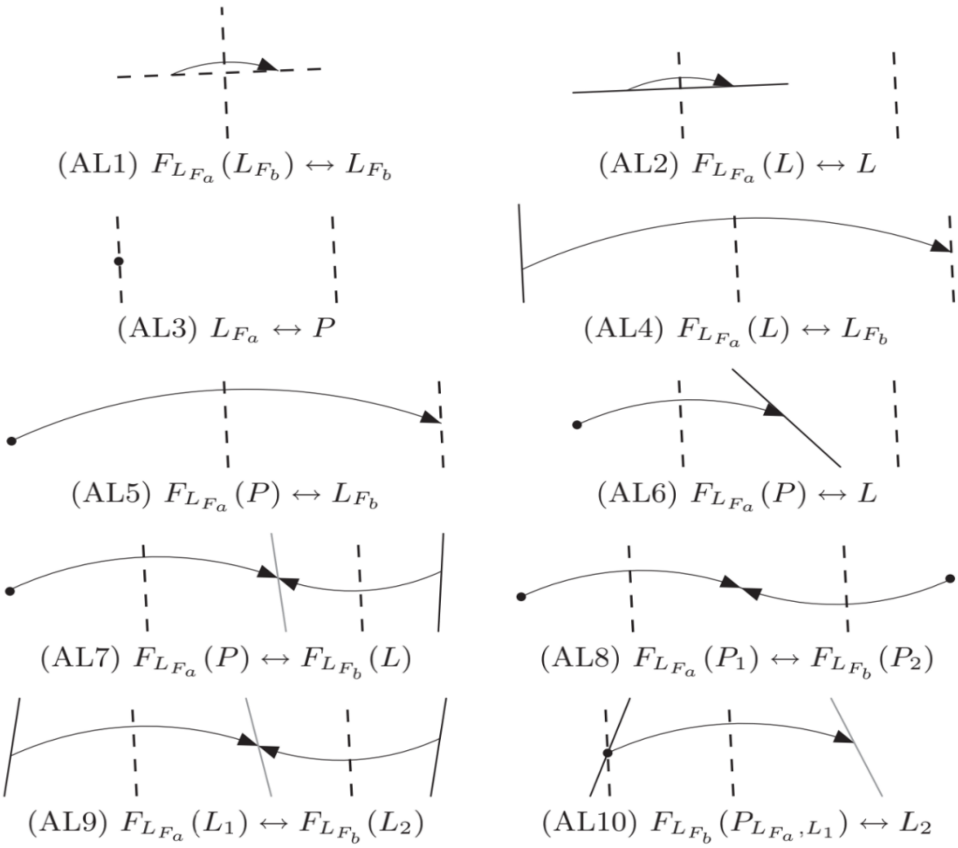

Figure 2. Two folds origami alignments [3].

Similar to the analysis of one fold axioms, figure 2 presents all of the possible two folds alignments. Combining these alignments, 489 two-fold axioms can be formed [3].

3. Solution of equation

Origami constructions belongs to geometric methods of solving equations, which, drawing equations in one way or another, solve equations by measuring the length of the graphed segments.

3.1. Experiments of solving higher order equations

3.1.1. Principle description. Analyzing origami axioms, it can be understood that origami constructions can form lines and parabolas (when fold a point to a line), which can help to solve equations in the coordinate plane.

One of the geometric methods of solving equations is Lill's method, which is perfect for solving equations using origami constructions. Lill's method uses reflections to solve real polynomial equations. When solving equations, the equation will be first constructed by forming right angled path from the origin (point O) to a terminal point (point T), in which the segment lengths are the absolute

DOI: 10.54254/2753-8818/5/20230282

values of the coefficient of equations [7], and directions depend on the sign of the coefficient as well as the reminder of the sequence of the coefficient modular four. When the sign is positive, the direction will be the same as the arrow for that reminder; while when the sign is negative, the direction will be the opposite. Specifically, When the remainder of the sequence of coefficient is 0, 1, 2, or 3, the direction of the graph in Lill's method is shown in Figure 3 [8] .

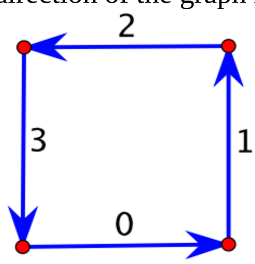

Figure 3. The direction of the graph in Lill's method when the remainder of the sequence of coefficient is 0, 1, 2, or 3 [8].

After that, from O will a line be propelled at a certain angle. The line will make a right-angled turn at each line toward the following one, altering the slant until the ultimate turn strikes point T. The intersection point is the right solution, when the condition is fulfilled [7] .

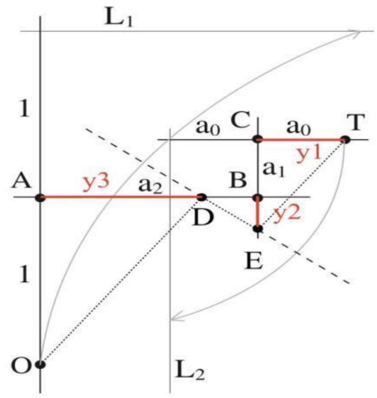

Figure 4. Solving cubic equation using Lill's method and single fold axiom [3].

Figure 4 is the representation of$x^3 + a_2x^2 - a_1x - a_0 = 0$using Lill's method. This equation can be solved using O6 (When there are two points (P1 and P2) as well as two lines (L1 and L2), it is possible to make at least one fold line that put P1 on L1 and, at the same time, P2 on L2.): fold O to L1 and, at the same time, T to L2.

Apparently, the fold line is perpendicular to the line formed by the existing point and the newly created point. Therefore, DE is perpendicular to OD and ET. Thus, it is easy to show that $\triangle AOD \sim \triangle BDE \sim \triangle CET$.

Now, let $x = \tan\theta$ (θ is the angle formed by OD and OA), then

$$\tan\theta = \frac{y_3}{1} = \frac{y_2}{a_2 - y_3} = \frac{y_1}{a_1 + y_2} \tag{1}$$

Therefore:

$$y_3 = \tan\theta = -x \tag{2}$$

$$y_2 = (a_2 - y_3)(\tan\theta) = (a_2 + x)(-x) = -x^2 - a_2x \tag{3}$$

$$y_1 = (a_1 + y_2)(\tan\theta) = (-x)(a_1 - x^2 - a_2x) = x^3 + a_2x^2 - a_1x = a_0 \tag{4}$$

Therefore, $x^3 + a_2x^2 - a_1x - a_0 = 0$, so $-\tan\theta$ is a root of the equation $x^3 + a_2x^2 - a_1x - a_0 = 0$

In fact, using a single fold, it is possible to solve all of the cubic equations. Also, since all of the quartic equations can be converted to cubic equations, it can be concluded that single fold origami constructions are possible to solve general equations of order four or less.

3.1.2. Case analysis. As shown in figure 5, the six order equation $x^6 + 4x^5 + 2x^4 - 3x^3 + 7x^2 + x = 0$ can be solved by the three-folds axiom that fold A to L1 and G to L2 and let LK to be perpendicular to the resulting fold lines ML and BK, and then align J to A and M to G.
Let $x = \tan\theta$ (θ is the angle formed by AB and AJ), then

$$\tan\theta = \frac{y_5}{1} = \frac{y_4}{4-y_5} = \frac{y_3}{2-y_4} = \frac{y_2}{3+y_3} = \frac{y_1}{7+y_2} \tag{5}$$

Therefore,

$$y_5 = \tan\theta = -x \tag{6}$$

$$y_4 = (\tan\theta)(4 - y_5) = (-x)(4 + x) = -x^2 - 4x \tag{7}$$

$$y_3 = (\tan\theta)(2 - y_4) = (-x)(x^2 + 4x + 2) = -x^3 - 4x^2 - 2x \tag{8}$$

$$y_2 = (\tan\theta)(3 + y_3) = (-x)(-x^3 - 4x^2 - 2x + 3) = x^4 + 4x^3 + 2x^2 - 3x \tag{9}$$

$$y_1 = (\tan\theta)(7 + y_2) = (-x)(x^4 + 4x^3 + 2x^2 - 3x + 7) = -x^5 - 4x^4 - 2x^3 + 3x^2 - 7x = 1 \tag{10}$$

Therefore,

$$-x^5 - 4x^4 - 2x^3 + 3x^2 - 7x - 1 = 0 \tag{11}$$

$$x^5 + 4x^4 + 2x^3 - 3x^2 + 7x + 1 = 0 \tag{12}$$

$$x^6 + 4x^5 + 2x^4 - 3x^3 + 7x^2 + x = 0 \tag{13}$$

So, $x = -\tan\theta$ is a root of $x^6 + 4x^5 + 2x^4 - 3x^3 + 7x^2 + x = 0$.

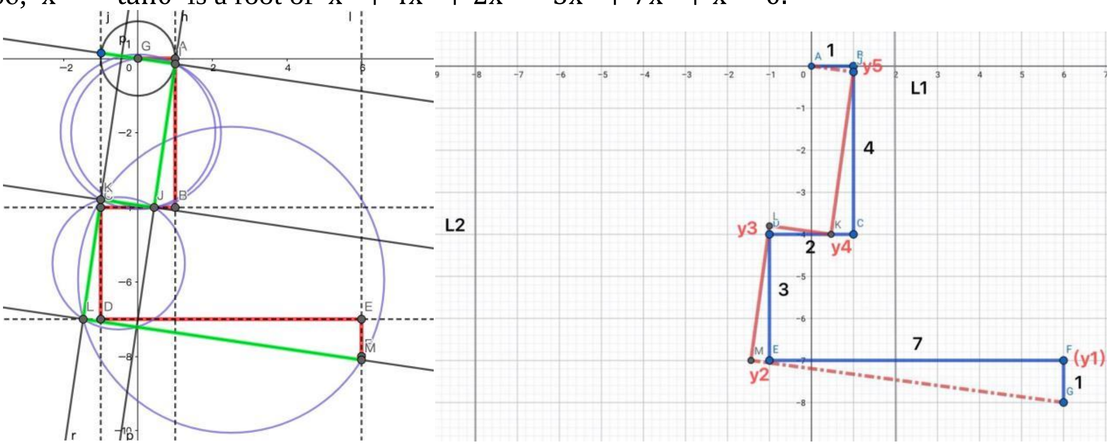

Figure 5. solving the six order equation using three folds [9].

3.2. Comparison between traditional methods and origami
As a way of graphing in mathematics, origami has always been compared to compass-and-straightedge constructions, and origami is better and more powerful than compass-and-straightedge in the following aspects:

Firstly, origami constructions can solve a broader range of equations. Since compass-and-straightedge can only draw line (order 1) and circles (order 2), it is limited to solving quadratic or liner functions [10]. However, origami is able to construct all of the constructions possible with compass-and-straightedge. With the help of O6, moreover, origami can solve the problems that cannot be solved with compass-and-straightedge, including angle trisection, cube doubling, and many

DOI: 10.54254/2753-8818/5/20230282

regular polygons. This is because origami is capable of constructing cubic and thereby quartic equations.

Secondly, origami is more accessible. The compass-and-straightedge constructions cannot be operated without rules and compass, but origami is possible only with a sheet of paper.

Thirdly, origami constructions are more accurate. With the modern technique, it is easy to make the angle of too large or too small when using the compass, leading to inaccurate results. On the other hand, origami only involves aligning points and lines, which can be made in great accuracy.

4. Conclusion

In this paper, one and two folds origami axioms have been analyzed. The specific geometric method of solving equations——Lill's method is identified. An equation $x^6 + 4x^5 + 2x^4 - 3x^3 + 7x^2 + x = 0$has be proven solvable with only three simultaneous folds using Lill's method. Therefore, there exist some six order equations to be solved by three simultaneous folds. This may shed some lights on the problem regarding how many simultaneous folds are sufficient to solve equations of order n and encourage further research in that area. Also, origami is proven to be much more powerful than the traditional compass-and-straightedge constructions in terms of the field of equations solvable, the accuracy, as well as the accessibility, which encourage people to take use of origami constructions more frequently.

The limitations of this research include the three-fold axioms are not identified and the paper does not prove that three simultaneous folds can solve general six order equations, but give an example instead. Future researches will work on three-fold alignments as well as axioms and try to find a more general solution (it maybe specified to a specific type) to six order equations.

Acknowledgment

I am particular grateful for professor Shun Tang of The Capital Normal University for providing me academic guidance in this research. His insights and creative methods to solve complex problems have inspired me greatly. Also, I am extremely grateful to Beijing Teenager Science and Technology Club for offering me the opportunity to conduct in-depth research. Finally, I would like to express my gratitude to all of the supports and assistance my family, friends, and teachers gave me.

Reference

[1] M. Nadinne. History of Origami, https://plymouthlibrary.org/wp-content/uploads/History-of-Origami.pdf
[2] Y. Huang, B. Li. *Origami and Mathematics,* 2012.
[3] Alperin, R. C., & Lang, R. J. (2009). One-, two-, and multi-fold origami axioms. *Origami*, *4*, 371-393.
[4] Hull, T. (1996). A note on" impossible" paper folding. *American Mathematical Monthly*, *103*(3), 240-241.
[5] Lucero, J. C. (2018). Geometric solution of a quintic equation by two-fold origami. *arXiv preprint arXiv:1801.07460*.
[6] Bertschinger, T. H., Slote, J., Spencer, O. C., & Vinitsky, S. The Mathematics of Origami.
[7] Hull, T. C. (2011). Solving cubics with creases: the work of Beloch and Lill. *The American Mathematical Monthly*, *118*(4), 307-315.
[8] Tabachnikov, S. (2017). Polynomials as polygons. *The Mathematical Intelligencer*, *39*(1), 41-43.
[9] TivnanR. Lill's Method. https://www.geogebra.org/m/jreyczgq
[10] S. Gu. Origami, *The Fun of Thinking*, 2012.