Proceedings of the 2nd International Conference on Computing Innovation and Applied Physics (CONF-CIAP 2023)
DOI: 10.54254/2753-8818/5/20230301

An introduction to continuous functions, metric space, manifolds, topological spaces and its properties

Wangzhe He

Shenzhen College of International Education, Shenzhen, China

s21868.he@stu.scie.com.cn

Abstract. A n-dimensional topological manifold is defined as an n-dimensional local Euclidean space M that is also a second countable space and a Hausdorff space. If a topological space has a countable base, it is referred to as a second countable space. At the same time, continuous function and metric space can help people to understand topological space better. This article will express some basic ideas about continuous functions, metric space, topological space and its properties, and topological manifolds. Moreover, the paper shows some basic ideas of how topological manifolds, topological space and function could be recognized and proved. Through analysis, this paper demonstrates the connection between them, such as using properties of continuous function to prove the definition and properties of topological manifolds and spaces.

Keywords: topological space, manifolds, continuous function, metric space.

1. Introduction

Topology as a branch of mathematics, can usually be used to solve questions involving set and geometry, as well as to perform further analysis on them. Topology can also be an important field for some other subjects or fields, such as physics. In this article, we will go through some rudimental but crucial theorems and definition. Also, we are going to prove some of the theorem and do some simple calculation, such as proving the properties of manifolds. After reading this paper and some other references that appear in this paper on those topics we just mentioned in the abstract, it will be quite easy for people to get an understanding of some basic ideas in topology and be able to understand most of the basic points that other references or books may have mentioned.

2. Continuous function

2.1. Continuous function

In Euclidean metric, for a function, given $f: R \to R$, $x \in R$, f is called as a continuous function if for all $\varepsilon > 0$, there exist a $\delta > 0$, such that if $|y - x| < \delta$,
then $|f(y) - f(x)| < \varepsilon$.

Also, if $U \subseteq R$ is open, $f^{-1}(U)$ is open in R.

If f and g are continuous, then $g(f(x))$ is also continuous.

If a function f is surjective, then there exists $f: x \to Y$, where $f^{-1}: Y \to X$, such that: $f\left(f^{-1}(U)\right) = f^{-1}\left(f(U)\right) = f^{0}(U)$ [1].

DOI: 10.54254/2753-8818/5/20230301

2.2. *Function in topological space*

In topological space, giving a function $f: x \rightarrow Y$. There is an input $x \in X$ and obtaining an output of $f(x) \in Y$.

A function $f: x \rightarrow Y$ is called continuous if for all U open set in Y, we have that $f^{-1}(U)$ is open in X. We have $f^{-1}(U)$ as:

$$f^{-1}(U) = preimage\ of\ U = \{x \in X \mid f(x) \in U\}$$

Let X, Y be topological spaces, giving that there is a function $f: x \rightarrow Y$. Then this function is called homomorphism if f is continuous, f is invertible and its inverse f^{-1} is also continuous.

3. Metric space

A metric space is a set X such that there exists a distance d(x, y) that meets these four criteria for every x, y ∈ X.

1) $d(x, y) \geq 0$
2) $d(x, y) = 0$ if and only if when x=y
3) $d(x, y) = d(y, x)$
4) $d(x, y) + d(y, z) \geq d(x, z)$, where z is the third point. This property can be showed by using Pythagoras theorem.

3.1. *Open*

For a x ∈ X, when there is a ball at it, having a radius $r > 0$, then we have the set that $B(x, r) = \{y \in X \mid d(x, y) < r\}$.

we can say that $x \in B(x, r)$. This will be an important definition for us to define the term "open".

There is a subset U of X if it is a metric space, given $U \subseteq X$. Under this situation, if for all $x \in U$, there exist a r>0, such that $B(x, r) \subseteq U$, then X is known as open, the examples are showed by figure 1.

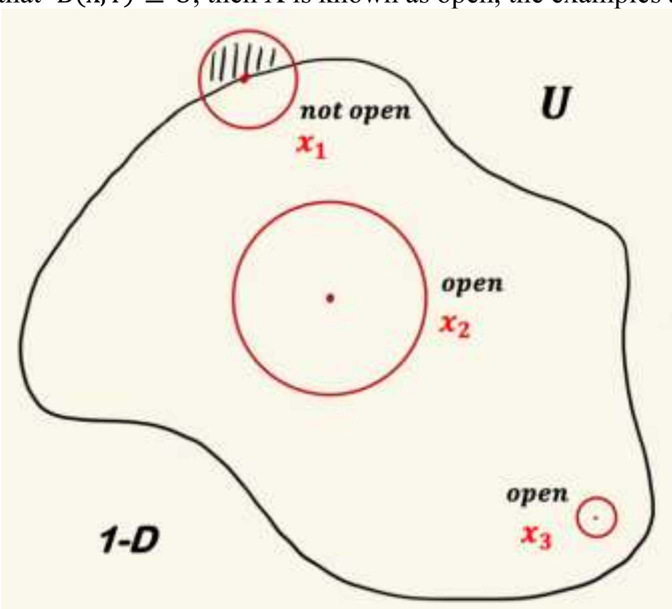

Figure 1. The example of balls that is open and closed in metric space.

Fundamental properties, if X is a metric space, then it satisfies:

1) X is open
2) ∅ is open
3) Any union of open subsets of X is also open.
4) Any finite set of open subsets of X has an intersection that is also open.

DOI: 10.54254/2753-8818/5/20230301

4. Topological space

Except from what we have mentioned in the abstract, topological space can also be considered as a generalization of metric space.

4.1. Definition of topological space

The following four characteristics define a set X with a collection A of subsets of x as a topological space.

1) $\emptyset \in A$
2) $X \in A$
3) $U_i \in A$, for every $i \in I$, then the union $\bigcup_{i \in I} U_i \in A$
4) $U_i, \dots, U_n \in A$, then $\bigcap_{i=1}^{n} U_i \in A$

In this case, any element of A is called an open subset of X. Automatically, any metric space will become a topological space due to its properties.

4.2. Example of some topological spaces

For X = R

1) Discrete topology: for $A = \{Y \subseteq X\}$, for all the subsets of X are exactly the same topology, which was define by the metric $d(x, y) = \{^{0 \text{ if } x=y}_{1 \text{ if } x \neq y}$

2) Trivial topology: for $A = \{\emptyset, R\}$, trivial topology is a topology under an extreme situation other than the discrete topology. If X is any non-empty collection, then the topology made up by X and an empty set is called trivial topology. [2]

3) Zariski topology: Zariski topology is usually defined by using closed subset in the space. The closed set of Zariski topology in an affine space A is a set of common zeros in a set of polynomials [3]. The Zariski topology for an algebraic variety can be guided from the Zariski topology in that space A. For an affine scheme Spec A, the Zariski closed set can be as: $V(I) = \{p \in \text{Spec A} \mid p \in I\}$.

Where I is ideal from Spec A [4].

Affine space is the collection of points and vectors, which have the definition:

1) Setting A as a point set, there are two ordered points P and Q correspond to a vector in n-dimensional vector space.

2) Set that P, Q and R are any three points in A, P and Q correspond to vector a, Q and R correspond to vector b, then P and R should correspond to vector a + b.

If the point set A satisfies both properties above, then A can be called an affine space [5].

Algebraic variety can be known as the common zero set defined by several multivariate polynomial equations. If this algebraic variety can be represented by only one function, then this is known as hypersurface. [6]

4.3. Hausdorff property

Giving that X is a topological space. X is known as Hausdorff if for all $x, y \in X$, where $x \neq y$, there exist $U_1, U_2 \in A$, such that $U_1 \cap U_2 = \emptyset$.

For a metric space, for X= metric space, d= distance, for $x, y \in X$, such that $x \neq y$. In this case, $d(x, y) \neq 0$.

We can assume that U_1 is a ball contains x, U_2 is a ball contains y. Such that $U_1 = B(x, \frac{R}{2})$, $U_2 = B(y, \frac{R}{2})$. To show whether matric space is Hausdorff, we have to check that $U_1 \cap U_2 = \emptyset$.

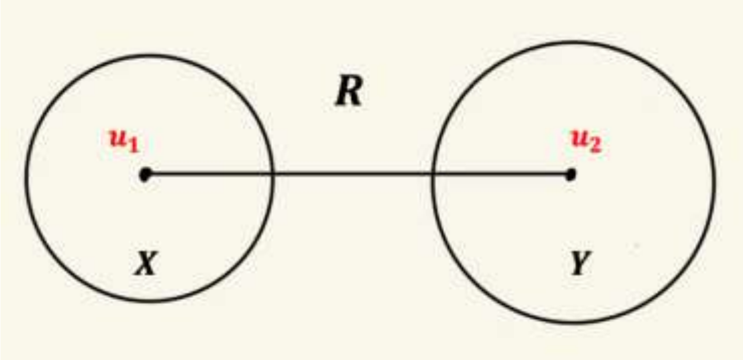

Figure 2. The example of showing Hausdorff of metric space.

We can first assume that $U_1 \cap U_2 \neq \emptyset$, meaning that two balls are intersecting. Given that $z \in U_1 \cap U_2$, so $z \in U_1$, $z \in U_2$.

In this case, $d(z,x) < \frac{R}{2}$, we can rewrite this into $z \in U_1 = B(x, \frac{R}{2})$. Also, we have $d(z,y) < \frac{R}{2}$, which can be turned into $z \in U_2 = B(y, \frac{R}{2})$. Hence, we have: $d(x,y) \leq d(x,z) + d(z,y) < \frac{R}{2} + \frac{R}{2} = R$

Hence, we get that $d(x,y) < R$, where there is a conflict with the assumption we state before. We can summarize by saying that all metric spaces are Hausdorff.

4.4. Product space

For there is a product between two spaces: $X \times Y = \{(x,y) | \ x \in X \quad y \in Y\}$

If X is a topological space, and Y is a topological space, then $X \times Y$ is also a topological space.

If X is a circle, giving $X = S^1 \subseteq R^2$, Y=R which is a Euclidean topology. Giving that $X \times Y \subseteq R^2 \times R = R^3$ which is a 3-dimensional space. The example of this product is showed by figure 3.

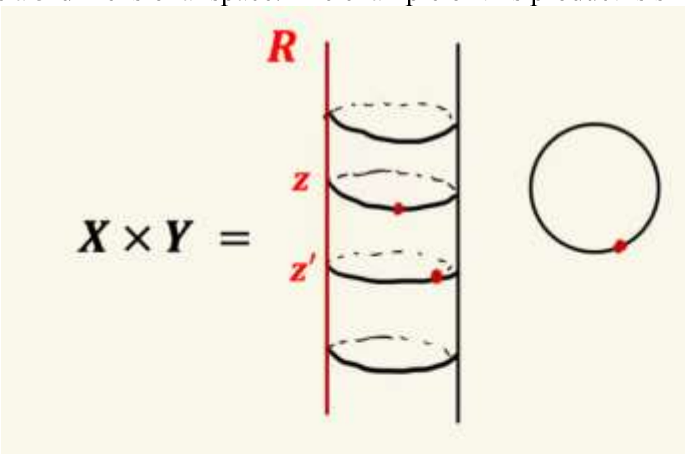

Figure 3. The example of product of topological space.

Since X is a topological space, there exists a collection of open subsets of X, also, since Y is a topological space, there exist B collection of open subsets of Y. The open subsets of $X \times Y$ are given by all the unions of $U \times V$, where $U \in A$, $V \in B$.

4.5. Closed subset

In topological space, a subset Y is known as a closed subset if X – Y is open. Or we can say that Y is a closed subset when its complementary set is open.

Giving an example, for trivial topology $A = \{R, \emptyset\}$, each subset in it is both open and closed because R and $\emptyset$ are complement.

DOI: 10.54254/2753-8818/5/20230301

In a metric space, if all the limit points in the collection are all in this set, then we can consider this as a closed set. [7]

5. Topological manifolds

5.1. Basic properties

Given that if both X and Y are homomorphic, then they will have the same properties. For example, If X is Hausdorff and X and Y are homomorphic, then Y should definitely be Hausdorff.

5.2. manifolds

X topological space is called second countable if there exist $\{U_i\}$ is a countable set of open sets such that if V is open in X, then there exists a $U_i \subseteq V$.

X is called a topological manifold when X is a topological space that satisfies these three following properties:

1) X is Hausdorff
2) X is second countable
3) For all $x \in X$, there exist a $U \subseteq X$ which is open, such that $x \in U$, and there also exist $f: u \to u^{\sim}$ (homomorphism), where $u^{\sim}$ is an open subset of R^n where n represents for the dimension of X. [8]

Showing that S^1 is a topological manifold of dimension 1, given that R is a topological manifold of dimension one. For $x \in R$, $U \in R$

Now we got a circle: $X = S^1 = \left\{(x, y) \in R^2 \,\middle|\, x^2 + y^2 = 1\right\}$

We are showing that for any $(x, y) \in S^1$, there exists a U open $\in (x, y)$ such that U is homomorphic to an open subset of R.

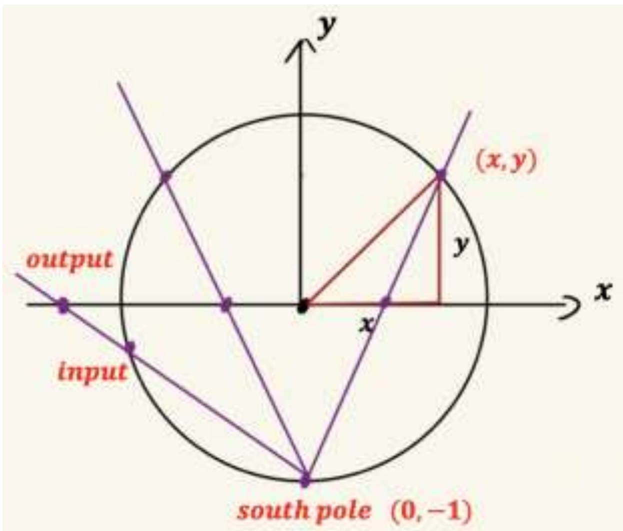

Figure 4. Method to prove the circle that it is a topological manifold.

It is important that $(x, y) \neq (0, -1)$ which is known as a south pole, means that $U = S^1 - (0, -1)$. We first draw a line, connecting south pole (0, -1) with point (x, y). This line will then intersect with x-axis at a point $(x_0, 0)$. We obtain a function $f: u \to R$ after connecting points, the function f is known as stereographic projection. This function has the equation: $\frac{Y - Y_0}{y - Y_0} = \frac{X - X_0}{x - X_0} = 0.$

For this circle, the equation can be written as: $\frac{Y+1}{y+1} = \frac{X}{x}$.

After we plug in south pole (0, -1) and (x, y), we can further evaluate it to $Y = \frac{(y+1)}{x}X - 1$, also we can find out the intersection $x_0 = \frac{x}{y+1}$. Due to the case that $(0, -1) \notin U$, so the function $f(x, y) = \frac{x}{y+1}$

is continuous. Hence, $f^{-1}: R \to U: f(x) = (\frac{2x}{1+x^2}, \frac{1-x^2}{1+x^2}))$ is continuous due to the properties we mentioned before. $u' = S^1 - \{(0,1)\}$ which is the north pole, then we have $f' = u' \to R, (x,y) \to \frac{x}{1-y}$, and hence we got $(f')^{-1}: R \to u',\ x \to (\frac{2x}{1+x^2}, \frac{1-x^2}{1+x^2})$.

Hence the circle S^1 is a topological manifold of dimension one.

Further proving shows that a sphere is a two-dimensional topological manifold, having the equation of:

$$S^2 = \{(x_1, x_2, x_3) \in R^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

For all $(x_1, x_2, x_3) \in R^2$, we are showing that there exist $U = \text{open} \ni (x_1, x_2, x_3)$, such that $\exists f: u \to R^2$ which is homomorphism. Like the circle we mentioned just before, we can take a south pole from the sphere, which is (0, 0, 1), hence we have

$$u = S^2 - \{(0,0,1)\}$$

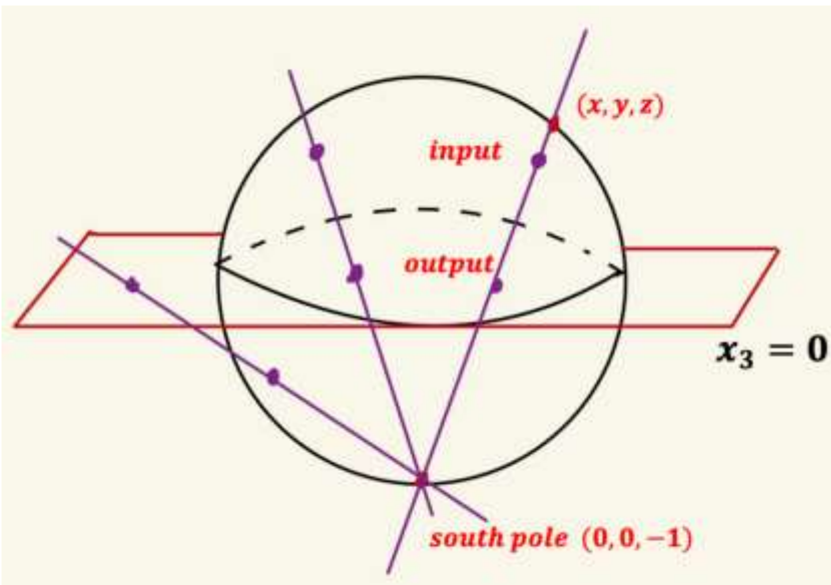

Figure 5. Method to prove the sphere that it is a topological manifold.

For $f: u \to R^2$, the line between (x_1, x_2, x_3) and (0, 0, 1) is $\frac{z+1}{x_3+1} = \frac{y}{x_2} = \frac{x}{x_1}$. Using the similar method, we can obtain the intersection with the plane $x_3 = 0$ is $(\frac{x_1}{x_3+1}, \frac{x_2}{x_3+1}, 0)$. Hence there is: $f: u \to R^2,\ (x_1, x_2, x_3) \to (\frac{x_1}{x_3+1}, \frac{x_2}{x_3+1}, 0)$, so there is $f^{-1}: R^2 \to u$. Hence, we obtain the line between $(x_1, x_2, 0)$ and $(0, 0, -1)$ is

$$\frac{x}{x_1} = \frac{y}{x_2} = z + 1$$

The intersection with S^2 is

$$(\frac{2x_1}{1 + x_1^2 + x_2^2}, \frac{2x_2}{1 + x_1^2 + x_2^2}, \frac{1 - x_1^2 - x_2^2}{1 + x_1^2 + x_2^2})$$

DOI: 10.54254/2753-8818/5/20230301

Hence $f^{-1}: R^3 \to u,\ (x_1, x_2, 0) \to (\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2}, \frac{1-x_1^2-x_2^2}{1+x_1^2+x_2^2})$. Similarly, as what we did just now, take the north pole (0, 0, 1), getting $u' = S^2 - \{(0,0,1)\}$, so we obtain:

$$f': u' \to R^2, (x_1, x_2, x_3) \to (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0)$$

Finally getting:

$$(f')^{-1}: R^2 \to u', \qquad (x_1, x_2, 0) \to (\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2}, \frac{1-x_1^2-x_2^2}{1+x_1^2+x_2^2})$$

Hence we prove that a sphere is a two-dimensional topological manifold.

A torus is a surface generated by revolving a circle in R^3 about an axis that is coplanar with the circle $\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2$, when $R > r$, that is the ring torus [9].

6. Connectedness and compactness

6.1. Compactness

Let X be a topological space. We can say that X is compact if for any set of open subsets u_i of X, such that $X = \bigcup_{i \in I} u_i$, that exist $i_1, i_2, i_3, \ldots, i_n$ such that $X = \bigcup_{j=1}^{n} u_{i_j}$ which means that it has finite subcovers.

Tychonoff's Theorem: In topology, the arbitrary product of compact space is also compact space. [10]

Proving that for $X = R$, the trivial topology $A = \{\emptyset, R\}$ is compact.

Given that $X \cup \emptyset = X = R$, hence $\bigcup U_k = R$. In this case, given that $U_k = X$ or R, hence there will be the existence of a value of k, leading to $U_k = R$. Under this condition, there will be a finite subcover which satisfies the definition of compactness. Hence, the trivial topology is compact.

Three more basic theorems for compactness:

1) Stating a function $f: x \to Y$, it is continuous between topological space. Assume that X is compact. Then $\mathrm{Im} f = f(x)$ is also compact.

2) When X is a compact topological space, and $C \subseteq X$ closed, then C is also compact.

3) R^n with Euclidean topology, and $C \subseteq X$ closed, then C is also compact.

6.2. Connectedness

Given that X is a topological space, we can say that X is connected if we cannot find open subsets A and B that satisfy following properties:

1) $X = A \cup B$

2) $A \cap B = \emptyset$

3) $A, B \neq \emptyset$

A is both open and closed because $X - A = B$ is open

If $f: x \to Y$ is continuous and X is connected, then $f(x)$ is connected.

7. Conclusion

This paper introduces and explains some fundamental concepts in topological space and its properties, as well as topological manifolds and their properties, as well as continuous function and metric spaces. To make a conclusion, the paper has introduced the connection between those chapters. For example: learning the concept of metric space will help further understanding in topological space, or how continuous function performs in topological space and using continuous function to prove an n-dimensional topological manifolds. However, this article only introduces very rudimental points, so it will be unhelpful for those who are seeking for a deeper understanding and explanation of those chapters.

DOI: 10.54254/2753-8818/5/20230301

In the future, the author plans to conduct additional research in these chapters in order to write an article that will deliver senior concepts while also making it much easier to understand.

References

[1] Surjective function. https://www.cuemath.com/algebra/surjective-function/
[2] "Mathematic Dictionary Volume II" China Western Education press, Southeast University Press, China Science and Technology Press, 2002.
[3] "Basic Algebraic Geometry2, 3rd edition", Igor R. Shafarevich, Springer, 2013
[4] "Some studies on the regular monoids of affine algebras and the profiles of affine regular monoids" Fang Xiao, University of Zhejiang, 2009
[5] "College Graduate Textbook, Tensor Analysis[M]" Zongruo Tian, Northwestern Polytechnical University Press, 2005
[6] "Some studies on piecewise algebraic curves and piecewise algebraic variety[D]" Yisheng Lai, Dalian University of Technology, 2002
[7] "A Hausdorff Topology for the Closed Subsets of a Locally Compact Non-Hausdorff Space" J. M. G. Fell. American Mathematical Society, 1962
[8] "Introduction to Topological Manifolds, Second Edition", John M. Lee, Springer, 2011
[9] "Notes on Basic 3- Manifold Topology", Allen Hatcher, math.unl.edu, 2007
[10] "Tychonoff's Theorem" David G. Wright, American Mathematical Society, 1994