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A discussion on mathematical models for option valuation

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Abstract. Options as a type of financial derivatives have become increasingly popular in recent years; therefore, understanding how to value options as a derivative is crucial. To analyse the problem, we can use mathematical stochastic analysis from the perspectives of risk management and return maximisation, and then develop an option valuation model. Using a literature restatement approach, the mathematical model of option valuation is examined in this study. Bachilier introduced the option valuation model in 1900 as a breakthrough in the study of financial mathematics. In the 1860s, the asset pricing model was introduced, followed by the famous B-S formula published by F. Black and M. Scholes, which is also used to research option valuation. The above-mentioned mathematical models have been refined and are now frequently utilised in the options market. They can also be used as a criterion for buyers to determine the value of options, thus it is important to investigate this subject, which will be the subject of this paper.

Keywords: option valuation, brownian motion, Ito's formula, capital asset pricing model, Black-Scholes model, volatility source model

1. Introduction

In the late 18th century on American and European markets, options gave the holder the right to buy or sell an asset at a preset price on a given day or at any time before that date. Options have become one of the most popular financial products in the world since the holder has rights but not liabilities and the underlying can be a range of commodity futures, equities, currencies, and bonds. As described in this book, quantitatively assessing options is essential. Bachelier pioneered stochastic processes in 1900 when he realised that continuous-time hazardous assets (stock indices, exchange rates, etc.) could be characterised by the Brown movement, which is the core of financial mathematics, but the stock price can be negative. But, everyone knows this is impossible. Samuelson rediscovered Bachelier's results in the 1850s, launching modern financial mathematics and two Wall Street revolutions. W.F. Sharpe, J. Lintner, and J. Mossin proposed the capital asset pricing model in 1964-1966 to assess how each company related to the market. The B.S. option pricing model started the second revolution. Black, Scholes, and Merton invented geometric Brownian motion in 1973 to illustrate risky price mechanisms [1]. Lognormal geometric Brownian motion solves negative values. Geometric Brownian motion, which cannot reach negative values, is used in option valuation models [2].

However, μ is a constant in the GBM, so many researchers (including Melton himself) have modified the model to some extent to obtain a more realistic model. This study will increase the stock price by employing the volatility source model. This research employs a literature review approach to comprehend the evolution of option valuation models. Although the above model was presented many years ago, it will help us develop a better option valuation model and provide us with a better understanding of how options are valued so that we may establish our own criteria for evaluating option purchases.

2. Definitions

2.1. Knowledge of options contracts

First, this study will define options [3]. Options are derivatives of underlying assets including stocks, currencies, bonds, commodity futures, and more. The buyer and seller sign a contract on the underlying asset and agree on the exercise price K and option expiration date. The option buyer has the right but not the responsibility to trade, but the option seller must fulfil the buyer's intentions.

Call and put options coexist. Options are a common financial derivative because if the price of an underlying asset rises, the buyer gains from a call option, and if it falls, the buyer can buy the asset and reverse the option contract for hedging. Due to these advantages, the buyer must pay an option premium when buying an option to avoid the arbitrage possibility, hence the pricing of the option premium is crucial and will be explored next in this paper.

2.2. Brownian motion, martingale and the Ito formula

First, we give the definition of Brownian motion [4]: for a stochastic process $\{W_t\}$, if it satisfies the following three properties:

1)$W_t$is a continuous orbit and starts from 0;

2)Incrementally independent, i.e., for any disjoint interval $(t_1, t_2]$ and $(s_1, s_2]$, there are mutually independent intervals with $B_{t_2} - B_{t_1}$ and $B_{s_2} - B_{s_1}$;

3)Obeying the normal distribution, $W_t - W_s \sim N(0, \sigma^2(t-s))$;

Then the stochastic process is said to be Brownian motion; when $W_0 = 0$, $\sigma^2 = 1$, it is standard Brownian motion.

Next, the concept of martingale is introduced: Suppose $\{F_t\}$ is an -algebra on the probability space (Ω, F, P), and for a cluster of productable random variables with finite mean $\{X_t\}(t \geq 0)$, there is $E(|x_t|) < \infty$, then $\{X_t\}(t \geq 0)$ is said to be a martingale; and for s $\leq$ t, there is $E(x_t|F_S)$=$X_s(\leq X_s, \geq X_s)$; and for a martingale $\{X_t\}$, there must be E(X_t)=E(X_0); such that for financial markets, the best way to predict future prices is to use present prices.

From this, we get that for a standard Brownian motion with a martingale. Then, the paper gives the introduction about the Ito formula. Suppose that Bt is a Brownian motion, Yt is a function for the variables t and Bt, it can get the Ito formula $d\,Y_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt$. Then, we define the stochastic differential equation. It has the form of $dx_t = \mu(t, x_t)\,d_t + \sigma(t, x_t)\,dB_t$ By giving the appropriate value of the function, we can get the Geometric Brownian Motion(GBM): $dx_t = \mu_0 x_t d_t + \sigma_0 x_t\,dB_t$ [5].

3. Early option valuation theories

In 1900, L. Bachlier presented one of the earliest models of option pricing in his doctoral thesis, The Theory of Speculative Trading. In his model, he assumed that the stock price obeyed an absolute Brownian motion model, let the stock price be S, the strike price be K and the time to expiry of the option contract be τ. Then, under risk-neutral conditions, he gave the formula for pricing options as $S_T = SF(\frac{S-K}{\sigma\sqrt{\tau}}) - KF(\frac{S-K}{\sigma\sqrt{\tau}}) + \sigma\sqrt{T}f(\frac{S-K}{\sigma\sqrt{\tau}})$, where F is the distribution function of the normal distribution and f is the density function of the normal distribution.

However, there are some problems with his assumptions. Firstly, in his assumptions, the price of the stock obeys an absolute Brownian motion model, and thus it may reach negative numbers, which is not in line with our reality; at the same time, the model does not place restrictions on the option price and the underlying price, which may result in the option price exceeding the underlying price, thus violating the no-arbitrage principle; Finally, the model does not take into account the time value of money, so the return is not the true return. If these problems could be solved, a general model of option valuation could be obtained, but unfortunately his paper went unnoticed for some time after publication, until Samuelson rediscovered his model 50 years later and aroused interest in its improvement, which eventually led to a more widely applicable model.

4. Capital asset pricing model (CAPM)

In the 1860s, W.F. Sharpe, J. Lintner, J. Mossin derived the capital asset pricing theorem based on the work of L. Bachlier and the Markowitz Mean-Variance Model, where they made the following assumptions.

1) The buyer wants as much wealth as possible;

2) Wealth is a function of the rate of return on investment, hence utility is a function of the rate of return;

3) The buyer understands that the return on investment probability distribution is the same;

4) Investment risk is the variation or standard deviation of the rate of return on investment;

5) The expected rate of return and risk are the major elements influencing investment decisions;

6) Investors follow the Dominance rule, choosing stocks with bigger returns and smaller risks.

With the above assumptions in mind, for an individual stock or portfolio, we use the market line (SML) and its relationship to expected return and systematic risk (β) to show how the market must price individual risky assets according to their security risk rating. The return-to-risk ratio for any risky asset in the market is equal to the market return-to-risk ratio, then there is: $\frac{E[r_s]-r_f}{\beta_s} = E[r_M] - r_f$, simplify it to obtain the CAPM formula: $E[r_s] = r_f + \beta_s(E[r_M] - r_f)$, where r_s is the return variable of portfolio S, r_M is the expected return of the market, r_f is the risk-free rate of the market, and β_s is the sensitivity to market risk, of which the calculation is the most meaningful part of it, and we can calculate it by $\beta_s = \frac{cov(r_s,r_M)}{var(r_M)}$.

At the same time, we know that there is systematic risk as well as unsystematic risk in the market, where systematic risk is the one that cannot be eliminated by diversification in the market, while unsystematic risk is the one that can be eliminated by investors changing their portfolio of stocks. And the premium that a risky investor needs to receive can be calculated through the CAPM, which also provides the possibility for investors to consider market risk, i.e. investors can choose the appropriate to make the proper asset allocation based on their judgement of the overall market trend and the need for risk control.

5. Black sholes model

Building on what has already been covered, Black, Scholes, and Merton give the most famous Black Sholes option valuation model. First, the following assumptions are given in their model [1]:

1) The short-term interest rate is fixed.

2) The stock price is a continual random walk with a variation proportional to its square. Hence, log-normal stock price distributions end limited intervals. Stock returns are constant.

3) No dividends are paid.

4) European options can only be exercised at maturity.

5) Stock and option transactions are free.

6) At the short-term interest rate, a security can be bought or held using a fraction of its price.

7) Short selling is unpunished. A seller who does not own a security will accept the buyer's price and promise to settle with him at a later date by paying him the security's price.

Based on the above assumptions, we first calculate the price S_t of the stock in the option, which we base on the GBM model obtained above $\frac{dS_t}{S_t} = \mu d_t + \sigma\, dB_t$, where μ is the expected return on the stock and σ is the volatility of the stock.

Next, we consider $\ln x_t$, which can be presented with the GBM formula:
$d\ln x_t = \frac{\partial f}{\partial t}\mu_0 X_t d_t + \frac{\partial f}{\partial x}\sigma_0 x_t\, dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\sigma_0 x_t)^2\, dt$.

Calculating that with Ito formula, we finally get the formula $d\ln x_t$=$(\mu_0 - \frac{1}{2}\sigma_0^2)\, d_t + \sigma_0\, dB_t$.

In this way, we can get that the logarithmic rate of the stock by $\ln\frac{x_t+\Delta t}{x_t} \sim N((\mu_0 - \frac{1}{2}\sigma_0^2)\Delta t, \sigma_0^2\Delta t)$, which is the accurate value of the logarithmic rate of return. Then $Y_t = \ln S_t$, we can get $S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t}$. This is also the stock price model given by the Black Scholes model.

Next, we calculate the price of the call option. Let the discounted price of the stock be S_t', then we have $S_t' = S_t e^{-rt}$. With the Ito formula, we get: $\frac{dS_t'}{S_t'} = (\mu - r)d_t + \sigma\, dB_t$.

We define $\theta = \frac{\mu - r}{\sigma}$, and $W_t = B_t + \theta t$, then, $dS_t' = S_t'\sigma\, dW_t$. Thus, $\frac{dS_t}{S_t} = rd_t + \sigma\, dW_t$.

By the first theorem of asset pricing, we know that there exists an equivalent probability measure P; under measure Q, W_t is a standard Brownian motion, and thus S_t' is a martingale under probability measure Q. Thus we get $S_t' = S_0' e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$. From $S_t' = S_t e^{-rt}$, we get $S_t = S_0 e^{rt - \frac{1}{2}\sigma^2 t + \sigma W_t}$.

The resulting formula does not contain the expected rate of return μ but the risk-free rate r. Therefore, we obtain the same expected rate of return for the risky asset S_t as for the risk-free asset, and thus the probability measure Q is the risk-neutral measure.

Under measure Q, the option price is: $U_t = E^Q\left[(S_T - K)^+ e^{-r(T-t)}\right]$. Next, we consider the separation of the main part, when $y = \frac{\ln\frac{K}{x} - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma}$, that is the division point taken by the main part. Then we calculate $\int_{y_0}^{\infty} e^{-rT} S_t e^{\sigma y + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}\, dy$, which can be simplified to obtain it as $N(d_1)$. Among it, $d_1 = \frac{\ln\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ and $\int_{y_0}^{\infty} e^{-r(T-t)} K e^{-\frac{y^2}{2(T-t)}}\, dy$, which can be simplified to $Ke^{-r(T-t)}N(d_2)$. Among it, $d_2 = \frac{\ln\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$. As such, we obtain the call option price as: $u(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$, where d_1 and d_2 is the formula given above.

Correspondingly, the put option price is $v(t, S_t) = -S_t N(d_1) + Ke^{-r(T-t)}N(-d_2)$. This gives us the most famous B-S formula, which is one of the most important and widely circulated models for option valuation [6-7].

6. Conclusion

Starting with L. Bachlier's option valuation model, this study traces the mathematical models' evolution. Each researcher enhanced the mathematical model, offering the capital asset pricing theorem in the 1860s, the model, and a formula for stock prices based on geometric Brownian motion in 1973. Black, Scholes, and Merton have recently produced a pricing model for options. This model also earned the Nobel Prize in Economics. After improving the stock pricing model with the volatility source model, the option prices were the same as in the Black-Scholes model, proving that equity option prices are independent of expected rate of return. This study does not address the issue where the risk-free interest rate is a known function of time or a random variable, which requires improving the Black-Scholes model. Merton improved the mathematical model for this problem, and later scholars updated the Black-Scholes model [8]. He investigates the example where the interest rate is a continuous stochastic process and provides an option valuation model. After examining the essential facts, the author will pursue further study in this field.

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