# Baryonic impacts on dark matter at different scales

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Abstract. Contemporarily, dark matter remains a mysterious part of the universe where no exact model has been confirmed experimentally. Nevertheless, various studies have discussed possible impacts on dark matter based on theoretical framework and numerical simulations in recent years. In this study, it aims to find baryonic impacts on dark matter by comparing dark matter distribution in dark matter only simulations and hydrodynamic simulations on different scales (cosmic web and dark matter halos). To be specific, some previous results are compared between several studies analysing baryonic impacts, mainly focusing on the mass fraction and volume fraction of cosmic web, halo shape, halos' orbital properties, phase space density, some physical processes related to baryons, etc. Additionally, the state-of-art hydrodynamic simulations and dark matter only simulations are introduced briefly as a basic knowledge. Overall, these results shed light on guiding further exploration of dark matter behaviours on different scales under the impact of baryons.

Keywords: cosmological simulation, baryonic impacts, dark matter.

## 1. Introduction

Cosmological simulations, different from observational astronomy, break the limits of time or space, allowing us to test our physical understanding in different areas and epochs, thus provide us much more information about universe [1]. Hydrodynamic simulations which include baryonic physics are demanding because of the much more complex physical processes. However, baryons, consisting of our visible galaxies, cannot be ignored even it only takes 5% of the universe for their various properties. Fortunately, the rise of computational power has led to hydrodynamic simulations of larger volume. Thus, one can study baryonic impacts on dark matter by comparing dark matter distribution in hydrodynamic simulations with those in dark matter only simulations.

The various physical processes in hydrodynamical simulations make the simulation expensive. By changing dark matter properties into those derived from the tasks above and adding baryons inferred from the dark matter distribution by finding the relationship between baryon distribution and dark matter distribution into dark matter only simulations, one can develop a semi-analytical model to generate hydrodynamic simulations from dark matter only (DMO) simulations to reduce costs [2]. Additionally, one can employ those empirical principles in machine learning to deduct baryon distribution simply from DM distribution. This study will be organized into four parts. First, hydrodynamic simulations and dark matter only simulations will be introduced as a basic knowledge. Subsequently, baryonic impacts on dark matter distribution at different scales, cosmic web, halos, will be introduced in sequence.

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## 2. Simulating methods

## 2.1. Introduction of cosmological simulations

Different from observational astronomy, cosmological computer simulations, based on cosmological models and initial conditions, can construct a man-made universe and therefore make detailed predictions for different components of universe, e.g., dark matter and baryons. By altering different models and initial conditions, cosmological computer simulations can output different results, which can be used to compare with nowadays' observational facts. With the consistency of the results from observational facts and simulating outputs, one can better confirm the more reliable physical models therefore find the physics inside them. Moreover, cosmological computer simulations can provide predictions for certain astronomical facilities to narrow their range of observations. THESAN, a suite of large volume hydrodynamic simulations with self-consistent radiation, can provide predictions for James Webb Space Telescope in various observations for high redshift galaxies [3].

## 2.2. Dark matter only (DMO) simulation

The first appearance of dark matter can be traced back to 1937 when Fritz Zwicky found the luminous mass in the Coma Berenice galaxy cluster was much smaller than the total mass needed to hold these galaxies. His assumption was supported by Vera Cooper Rubin in her work of comparing measured radial velocity of stars located at a distance from the galaxy centre with their expected velocity calculated from Newton's law. Additionally, dark matter can influence the observations of Cosmic Microwave Background which can be used to study dark matter distribution [4]. Dark matter, taking nearly 25% of the total universe and reacting through gravitational force, is considered to be the backbone for matter clustering. Halos are areas of dark matter over-densities with galaxies forming at the centres of them. The collisionless Boltzmann equation and Poisson's equation as follows are used to describe the continuum limit of non-interacting dark matter particles where f represents space-phase density. The differentiation of f with respect to f implies mass f0 remains the same in each trajectory. And the equations should be solved in the Friedmann equation [1].

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \Phi}{\partial \mathbf{r}} \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{1}$$

$$\nabla^2 \Phi = 4\pi G \int f d\mathbf{v} \tag{2}$$

There are many numerical methods developed for DMO simulation to overcome the obstacles raised by multidimensional collisionless Boltzmann equation, among which the N-Body Method is the most famous. The phase-space density is sampled with an ensemble of N phase-space points with mass  $m_i$  to describe the dynamics of collisionless dark matter. Monte Carlo technique is employed, and the gravitational field is softened on small scales to avoid unphysical two body scatterings. Usually, Newton's gravitational force is employed as a good approximation. So, calculating gravitational force and replace  $\frac{\partial \Phi}{\partial \mathbf{r}}$  in Eq. (2) with it become fundamental tasks. Methods to calculate gravitational forces can roughly be divided in two groups: accelerating calculations through approximations, mesh-based methods [1].

## 2.3. Hydrodynamic simulations

Baryons react through electro-magnetic force additional to gravitational force, which allows baryons behave so differently as radiations are considered. Hydrodynamic equations can roughly be divided into three groups: Lagrangian, Eulerian or arbitrary Lagrange-Eulerian techniques. The numerical approaches can be divided in two groups: mesh-free and mesh-based algorithms [1]. Due to the limit of computational power, sub-resolution models and sub-grid models sometimes are employed to implement various physical processes induced by baryons to complement the hydrodynamical equations.

## 3. Baryonic impacts on different scales

#### 3.1. On cosmic web

Cosmic web is consisted of halos, filament, walls, and voids. Sunseri applied a modified version of NEXUS to identify the types of cosmic structures to TNG-300 and TNG-300-Dark simulations [5]. They analysed the mass fractions of structures, the volume fractions, the matter power spectrum, and the probability density function of density field in both simulations. Halos are found to lose over 10% of their mass while the volume fractions remain nearly the same without significant shift of boundaries of structures as a result of matter expelled by feedback from halos to the surrounding environment. Besides, baryonic impacts suppress matter clustering on the dependence of structure types and structure mass. The redistribution of matter with halos possibly related to processes of radiative cooling, stellar and AGN feedback. The suppression or boost can act quite differently in space with different density. However, since they only studied in IllustrisTNG, different sets of hydrodynamic simulations should be run to validate their results and they should further study the time revolution of each simulating box to break the degeneracy between baryonic effects and cosmological parameters. Furthermore, physical processes leading to the differences need to be explained detailed.

## 3.2. On dark matter halo shape

A comparison of dark matter halo shape between the hydrodynamic and DMO simulation of the Aq-C halo was made by Zhu [6]. They studied baryonic effects on dark matter by analysing a Milky Way-sized halo Aq-C-4 in both hydrodynamical simulation and DMO simulation. Applying an automatic orbit classification code called smile, they achieved to classify the DM particle orbits in the halo. A clear result that halos in hydrodynamic simulations tend to be more spherical than their DMO counterparts which have a clear box shape were reached. Apart from simply looking at the projected DM density maps of Aq-C halos in x-z plane, they also calculated halo shape parameters (a >b >c), axial ratios ( $q = \frac{b}{a}$  and  $s = \frac{c}{a}$ ), triaxial parameter ( $T = \frac{a^2 - b^2}{a^2 - c^2}$ ), as functions of galactocentric distance r. For r<100 Kpc, q remains above 0.8 in hydrodynamic simulations but a bit lower ( $\approx$  0.6) in the counterparts. In the same range of r, s remains above 0.6 in hydrodynamic simulations but around 0.4 in the counterparts. The triaxiality parameter T in hydrodynamic simulations gets close to 0 when r gets around 6 Kpc, which implies that the inner regions of Aq-C halos with baryonic effects are nearly spherical. As a comparison, the triaxiality parameter T in DMO simulations is above 0.6 for r<100 Kpc, which shows a prolate inner shape of Aq-C in DMO simulations. To be noticed, as these three parameters vary as functions of r, the global shapes of DM halos are not exact ellipsoids.

Kun Ting Eddie Chua also reached a conclusion that galaxy formation results in rounder halos compare to the DMO simulation. They studied the halo shapes from TNG50 and TNG100 together with their counterparts in DMO simulations. Additionally, they used a friends-of-friends (FOF) group finder algorithm to identify and to match halos [7]. They visualize some representative halos of TNG100 and TNG50 in plots showing the dark matter density in a 30Kpc-thick slice with half of the virial radius [7]. Similar to Zhu's result, halos in hydrodynamic simulations are more spherical than their DMO counterparts. Moreover, they reached a result that the inclination of halos can be changed due to baryonic physics. Plots of q, s, T as functions of  $\frac{r}{R_{200}}$  contain information in full physics runs of TNG50, TNG100 and Illustris as comparisons of their DMO counterparts. They found that the amount of sphericalization depends on radius. By comparing the difference of shape parameters between the full physics run and its DMO counterparts, they found baryons play a stronger role near the halo centre and the impacts reduce towards virial radius. They also noticed that halos tend to be more spherical in TNG50 than TNG100 halos with same mass. They attributed it to both a resolution dependency of models in TNG50 or TNG100 and the relation between galaxy properties and halo shape. To examine the relation between halo shape and halo mass, they focus on the shape at two specific radii: the virial radius  $R_{200}$ representing the outer halo and  $r_{15} = 0.15 R_{200}$  representing the inner halo. Furthermore, they showed the difference in median shapes in different simulations by calculating  $\Delta q = q_{fp} - q_{DMO}$ . As mentioned

before, baryonic impacts are nearly negligible at virial radius, which is reasonable since baryons distributes much more densely near the halo centre. This is also indicated by the slight difference between hydrodynamic and DMO runs at virial radius. From the fact that *s* and *q* show their negative correlations with halo mass while *T* shows a positive correlation, they concluded that halos with more mass tend to be less spherical and more oblate on average in both hydrodynamic simulations and DMO simulations. Shape parameters' dependence on mass are no longer monotonic in hydrodynamic simulations. In fact, *s* and *q* in baryonic simulations are greater than those in DMO simulations, showing halos in Full-Physics runs are more spherical. The change of *T* from DMO simulations to hydrodynamic simulations can also quantify this. In addition to their TNG results, they also compared other simulations from NIHAO and EAGLE simulations. By turning on/off nine different feedback models of the galaxy formation model, they analysed the halo properties and their dependence on galaxy formation efficiency or stellar mass fractions to evaluate the effects of different physical processes.

In Chua's another article published in 2019, they analysed halos from the full physics (FP) runs of Illustris suite at three different resolutions with their DMO counterparts. Additionally, two non-radiative (NR) simulations with different number of elements were performed. They calculated the median shape parameters as functions of radius for Illustris-Dark and Illustris-NR for their two lower resolutions [8]. Simply from the diagram they draw, the Illustris-NR results are identical to the Illustris-Dark results when r is over  $0.1\ R_{200}$ . This interesting phenomenon suggests without radiative processes gas evolves similarly to the DM. Additionally (a >b >c),  $q=\frac{b}{a}$  and  $s=\frac{c}{a}$  of both Illustris-Dark and Illustris-NR are generally lower with smaller  $\frac{r}{R_{200}}$  than those with larger  $\frac{r}{R_{200}}$  and the triaxiality decreases towards the virial radius. Hence, halos are prolate near halo center but become more oblate as radius increases. Similar plots were also drawn to compare the halo parameters in Illustris and Illustris-Dark with respect to halo-centric distance. The decreases of  $q=\frac{b}{a}$  and  $s=\frac{c}{a}$  from Illustris to Illustris-Dark indicate halos in hydrodynamic simulations have rounder shapes. The shapes of DM halos vary as functions of radius in Illustris and Illustris-Dark, but the relation between shapes and radius of Illustris-Dark is much weaker than that in Illustris. As a complement, the shape-radius relation depends on the halo mass. Halo behaves differently in shapes as its mass falls in different range. It's noteworthy that their results contrast with some of the previous papers maybe due to different baryonic physics employed in simulations.

In general, it is found that halos tend to be more spherical in hydrodynamic simulations than those in DMO simulations by analysing radius profiles which usually contain three parameters,  $q = \frac{b}{a}$ ,  $s = \frac{c}{a}$  and  $T = \frac{a^2 - b^2}{a^2 - c^2}$ , of halos in both simulations. The amounts of sphericalization can depend on various factors, physical processes added in hydrodynamic simulations, the galactocentric radius, the halo mass, etc.

# 3.3. On halos' orbital properties

Orbital properties of halo constituents are closely related to intrinsic shape of DM halos. Zhu used spectral methods to analyse the orbital properties of halos, applying an automatic orbit classification called SMILE to a Milky Way-sized halo named Aq-C-4 [6]. By following Hernquist & Ostriker [9], Binney& Spergel [10] and Carpintero & Aguilar [11], they divided orbits into six class: SAT, LAT (inner), LAT (outer), resonant orbit, thin orbit, box orbit based on three fundamental frequencies,  $\Omega_x$ ,  $\Omega_y$ ,  $\Omega_z$ . The frequencies of all other spectral lines can be expressed as follows where l, m, n are integer triplets.

$$\omega_k = l\Omega_x + m\Omega_y + n\Omega_z \tag{3}$$

If l, m, n are not integer triplets, it's a box orbit. If equation [4] is satisfied, one obtains a thin orbit. Furthermore, if a thin orbit meets the condition that one of l, m, n is 0, one derives a resonant orbit.

$$l\Omega_x + m\Omega_y + n\Omega_z = 0 (4)$$

If two of these three parameters are equal, it is classified as tube orbit among which orbits with  $\Omega_x \approx \Omega_y$  are called short axis tubes (SAT or Z-tubes) and orbits with  $\Omega_y \approx \Omega_z$  are called long axis tubes (LAT

or X-tubes). Additionally, Long tubes are further classified as outer and inner. There are also some unstable orbits which could not be expressed by equation (3). 'Chaos' is recognized as an additional property since SMILE doesn't have a separate class of them. To be noticed, several assumptions including the frozen potential which could have a profound impact on the results were made in their analysis. Frequency maps were drawn, showing SAT and LAT take great portions in both simulations. But classes with  $\Omega_x/\Omega_y \sim 0.6$  or  $\Omega_x/\Omega_z \sim 0.8$  take great proportions in DMO simulations rather than in hydrodynamic simulations. They found different orbit habitats in their frequency maps where some dominant lines can be seen when  $0.55 < \Omega_{\chi}/\Omega_{z} < 0.70$  and  $0.75 < \Omega_{\gamma}/\Omega_{z} < 0.85$  in DMO simulations instead of their hydrodynamic counterparts. Gaps corresponding to chaotic orbits are also found along the various resonant relations. As orbital families are sorted out, it is found that the fraction of SAT orbits dominates the other fractions in Hydrodynamic simulations reaching a maximum of nearly 0.65 and a minimum of 0.45 at different radius. The fraction of both LAT (outer) and box orbits are subdominant in hydrodynamic simulations, with LAT (outer) fraction's rising from 0.05 in the inner halo to a maximum of 0.3 and box orbit fraction's remaining stable slightly below 0.2. The fraction of LAT (outer) is not monotonic with respect to radius to the halo centre. For the DMO simulation, the fraction of box orbit dominants with r < 15kpc approximately and the fraction of SAT orbits becomes the dominant with r > 15kpc. The total fraction of SAT and box orbit is much lower in C-4 DMO estimated as 0.5 as average to radius than that estimated as 0.6 in C-4 simulation. Additionally, LAT (inner) takes a fraction of 0.15 with r < 10kpc in C-4-DMO while it barely appears in C-4. The fraction of both resonant orbit and thin orbit is slightly higher in C-4-DMO. IAT identified as  $\Omega_x/\Omega_z \approx 1$  appears in C-4 simulation at radius of nearly 200kpc. As a conclusion, triaxial halos in DMO simulation are dominant by box, resonant and thin orbits. They also studied the chaotic orbits by the frequency diffusion rate, since chaotic orbits have no definite frequencies. The definition of the frequency rate  $\Delta\Omega$  can be expressed in equation (5) which quantifies the relative change of frequencies between the first and second halves of the interval  $(\Omega_{k,1}, \Omega_{k,2})$ . Again, they draw the distribution of  $log(\Delta\Omega)$  for different orbits in both simulations.

$$\Delta\Omega = \frac{1}{3} \sum_{k} \frac{|\Omega_{k,1} - \Omega_{k,2}|}{(\Omega_{k,1} + \Omega_{k,1})/2}$$
 (5)

Here,  $\Delta\Omega=0.01$  is used as a boundary for regular orbits and chaotic orbits because of an intrinsic scatter of ~0.5 dex considered. Most of the tube-like orbits are regular orbits while most of the box-like orbits are chaotic orbits. It seems that halos in DMO simulations tend to have chaotic orbits. The fraction of chaotic orbits reduces from DMO simulations to hydrodynamic simulations. Moreover, some peaks appeared in figures of both simulations can be characterized in detail to further study the reasons behind them. Additionally, a study focusing on the relation between r (galactocentric distance) and the dominant type of orbits could be done in the future. However, they used triaxial symmetry to the potential expansion and the assumption of frozen potential which may lead to changes. They also assessed the impact and found that most differences occur in the outer region where SAT orbits are replaced by box orbits without the assumption of triaxial symmetry.

# 3.4. On halos' angular momentum

Zhu draw a figure of the mean total angular momentum  $|\mathbf{L}|$  and the mean absolute value of angular momentum in the z direction  $|L_z|$  within concentric spherical shells of the Aq-C-4 halo in both simulations and compared them. [6] A clear net rotation appeared in the central region of DM halo as  $|L_z|$  rises going from DMO simulations where  $|L_z| \approx 0$  to hydrodynamic simulations. The sign of  $|L_z|$  represents the rotating orientations of halos. The positive sign represents halos' co-rotating with respect to the stellar disc while the negative sign represents halos' counterrotating.  $|L_z|$  is negative in hydrodynamic simulations with r < 10 kpc and becomes positive as r increases.  $|L_z|$  in DMO simulations is much lower than that in hydrodynamic simulations, remaining close to 0 with r < 20 kpc and becoming negative in the range of 20 kpc < r < 40 kpc. It's consistent with the result that the fraction of SAT orbits dominates the other fractions in C-4 simulation. Additionally, the fraction of counterrotating SAT orbits dominates the fraction of co-rotating SAT orbits in hydrodynamic

simulations within r = 10kpc.

## 3.5. On halo's phase space density

In Zhu's work [6], they calculated a numerical estimate of phase space density using the ENBID code and an SPH-like kernel smoothing. A figure of estimate of phase-space density of DM particles with respect to galactocentric distance r shows subhalos and newly formed streams indicated by high phase space density locate at large r (r > 10 kpc). The value f of phase space density can be regarded as the combination of mass density and local velocity dispersion. The curve of phase space density is smooth near the halo center while it shakes to form some sharp peaks in the outer area. A slight increase of log(f) from C-4 to C-4-DMO is found and 90th precentile f of C-4 halo is close to the median values of f in C-4-DMO halo in the very central region. They both lead to their assumption that an acceleration of mixing processes leads to a smooth component of DM halos [6]. Additionally, combining others' progresses [12-14], they reached a conclusion: the reversal that DM halos turn to their original shape before baryons are added cannot be achieved in their simulations by removing the central mass distribution slowly and artificially. They considered the angular momentum exchange by the 'collisions' between baryons and DM particles and the shorter dynamical time scale in hydrodynamic simulations caused by the lower phase-space density in C-4 simulation lying in the combination of higher velocity dispersion and higher density as two of reasons for the irreversibility. Their result consists with some of Valluri's predictions while contrasts with others [15].

# 3.6. On relevant physical processes

Zhu also studied the baryonic effects as his other work and identified three physical processes, adiabatic contraction, tidal disruption, and reionization. These physical processes, listed in Table 1, strongly relate to mass of the subhalos.

Objects	Types of Processes	Results
Massive Subhalos, V <sub>max</sub> <35km/s	Adiabatic Contraction	DM Concentration increased
Low Massive Subhalos, V <sub>max</sub> >35km/s	Reionization	Mass and V <sub>max</sub> reduced
Intermediate Subhalos, 20Km/s <v<sub>max&lt;35km/s</v<sub>	Strong Tidal Truncation	$V_{\text{max}}$ Reduced

#### 4. Conclusion

In summary, baryonic impacts on two different scales (cosmic web and dark matter halos) are studied by comparing dark matter distribution in different simulations, dark matter only simulations and hydrodynamic simulations. Some results in common are reached from several studies. Generally, dark matter halos behave more spherical in hydrodynamic simulations according to their radius profiles. Their shapes relate to the physical processes implemented, halo mass, galactocentric radius, etc. For orbital properties and angular momentum, the fraction of SAT orbits dominates the others in hydrodynamic simulation and DMO simulation at large galactocentric radius while the fraction of box orbits dominates in DMO simulation at small galactocentric radius. Halos in DMO simulations tend to have chaotic orbits. |L<sub>z</sub>| in DMO simulations is much lower than that in hydrodynamic simulations, which can directly reflect the change of orbital families from hydrodynamic simulation to DMO simulation. The result that 90th precentile phase space density f of C-4 halo is close to the median values off in C-4-DMO halo in the very central region suggests halos in hydrodynamic simulation tend to have a smooth density distribution. Physical processes causing the variances are briefly introduced in this article by showing certain halos' behaviour due to three physical processes, adiabatic contraction, reionization and strong tidal truncation. This article only covers several papers, and the parts of orbital properties and angular momentum are mainly from Zhu's work as resources. It would be better if different work of different simulations can be compared. Furthermore, this article neglects the scale of dark matter subhalos and only give a rough introduction on various processes. Different results of the same physical processes due to the evolution history of halos can be further explained in the future. Also, the balance reached by different physical mechanisms can be discussed.

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