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Evaluation, analysis, and the state-of-applications of Taylor expansion

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Abstract. Contemporarily, Taylor expansion is a pretty powerful mathematical tool widely used in various fields for approximating complex functions with simpler polynomial approximations. It has applications in fields inducing physics, engineering, and computer science, ranging from numerical methods for solving differential equations to optimization algorithms and machine learning scenarios. Despite its extensively use, there are still open questions and challenges related to its application, including accuracy, convergence, computational cost, and numerical stability, etc. This study will provide a comprehensive summary as well as analysis of the existing literature on the application of Taylor expansion, identifies the current state-of-the-art, and proposes future research directions. The paper aims to address the limitations and challenges of Taylor expansion, improve its accuracy and efficiency, and explore new applications. The findings of this research can benefit researchers, practitioners, and engineers interested in utilizing Taylor expansion in their fields. Overall, these results contribute to the advancement of the field of Taylor expansion-based approximations.

Keywords: Taylor expansion, simpler polynomial approximations, applications.

1. Introduction

The Taylor expansion was first proposed by the English mathematician Brook Taylor in the early 18th century. “Given Maclaurin’s striking results in geometry, it is ironic that today his name is recalled almost exclusively in connection with the so-called Maclaurin series, which appeared in his Treatise of Fluxions of 1742 but is only a special case of the more general Taylor series, published by Brook Taylor (1685-1731) in 1715 in his Methodus Incrementorum Directa et Inversa [1]. Brooke Taylor's main contribution was proposing a general mathematical method for expanding any differentiable function into a power series of infinite terms at some point, known as Taylor's series or Taylor's expansion. The formulation of Taylor's developments significantly advanced the discipline of calculus and mathematical analysis and became an important tool in many scientific fields. Later, based on Taylor, some mathematicians improved and extended Taylor expansions and proposed some new series expansions which can be better adapted to different mathematical and physical problems. In 1770 Lagrange derived a generalization of Taylor's expansion in which the independent variable x is defined by an implicit equation $x=a+\epsilon\phi(x)$. He showed that another function f(x) can, within limits, be expanded in powers of epsilon [2]:

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$$f(x) = f(a) + \sum_{1}^{\infty} \frac{\epsilon^r d^{r-1}}{r!\, d\alpha^{r-1}} \left[\frac{\phi^r(\alpha) df(\alpha)}{d\alpha} \right] \quad (1)$$

A Lagrangian expansion is a type of series expansion that represents a function as an infinite number of terms, similar to the Taylor expansion. The difference is that the Taylor expansion expands the derivative of a function at a point into a series of terms, while the Lagrange expansion approximates a function over an interval. In addition, the Lagrangian expansion can be used to gradually improve the approximation accuracy by increasing the number of terms, which is suitable for certain functions where Taylor expansion cannot be performed at a certain point.

Moreover, there is the use of Fourier series expansions."Fourier series, named after its originator, the French mathematician and physicist Jacques Fourier (1768-1830), is a key technique in signal processing. Jacques Fourier introduced this concept in his memorable work Théorie Analytique de la Chaleur (The Analytical Theory of Heat) published in 1822, in which he developed the theory of heat conduction [3]. The Fourier series can be used to expand a periodic function into an infinite-term triangular series. This expansion can be used to compute periodic functions and oscillatory phenomena in fields such as signal processing, physics, electrical engineering, and cybernetics.

Taylor expansion is a widely used mathematical tool in various fields, including physics, engineering, and computer science. It allows for approximate representations of complex functions using simpler polynomial approximations, which can greatly simplify calculations and enable efficient numerical computations. The applications of Taylor expansion are diverse, ranging from numerical methods for solving ordinary and partial differential equations, to optimization algorithms for system identification and control, and to machine learning algorithms for function approximation and prediction.

Despite its widespread use, there are still many open questions and challenges related to the application of Taylor expansion in different domains. For example, the accuracy and convergence properties of Taylor approximations may vary depending on the specific function being approximated, the order of the approximation, and the domain of interest. Moreover, the computational cost and numerical stability of Taylor expansion-based methods may also be a concern, especially for high-dimensional problems or in the presence of noise or uncertainties.

Therefore, there is a need for further research and investigation into the application of Taylor expansion in various fields to understand its limitations, improve its accuracy and efficiency, and explore new avenues for its application. This research paper aims to address these challenges by comprehensively reviewing the existing literature on the application of Taylor expansion, identifying the current state-of-the-art, and proposing future research directions to advance the field of Taylor expansion-based approximations.

By conducting in-depth research on the applications of Taylor expansion and addressing the limitations and challenges associated with its use, this research paper seeks to contribute to the advancement of the field and provide valuable insights for researchers, practitioners, and engineers who are interested in utilizing Taylor expansion in their respective fields of study or applications.

2. Basic descriptions

Taylor's expansion is a type of series expansion that represents a function as an infinite number of terms and is used to approximate the value of a function near a point. The essence of the Taylor series is to use the derivatives of each order of a function at a point to approximate the performance of the function in the vicinity of that point and apply it to the approximate calculation of various functions in the local range. "Taylor's series as follows [4]:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \cdots \quad (2)$$

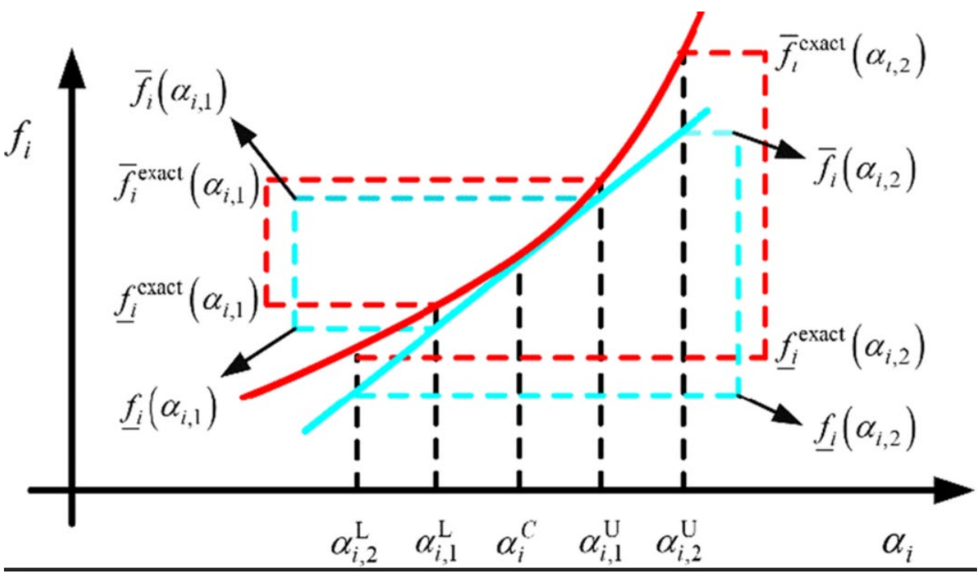

Figure 1. A sketch of implementation of Taylor expression in Economy.

In addition, Taylor expansions have several important properties. The first property is accuracy. The accuracy of the Taylor expansion increases as the number of terms increases. In other words, as the number of terms increases, the Taylor expansion gradually approximates the original function until the highest accuracy at infinity. The second property is convergence. The Taylor expansion converges in some neighborhoods of the expansion point. The size of the radius of convergence depends on the derivability of the function and the choice of a particular point. The third property is uniqueness. It means that given a function and an expansion point, its Taylor expansion is unique. The fourth property is derivative. The terms of the Taylor expansion can be derived term by term to obtain the derivative of each order of the function at the point of expansion.

3. Application in economics

In economics, the Taylor expansion is a widely used mathematical tool that can be used to solve a variety of economic problems. For example, in optimization problems, we usually need to find the maximum or minimum value of a function. Taylor expansions can be used to approximate the derivative of a function at a point and thus help us find the optimal value of the function. Specifically, we can use Taylor expansions to expand a function into an infinitely differentiable polynomial at a point to approximate the derivative and higher-order derivatives of the function at that point. Sakakibara argued that we shall use the optimal investment-GNP ratio and capital stock generated in the first step. If we substitute them into the function U and expand it by a Taylor series around the point ucu =0, the term U(C) in the social welfare function can be treated as a linear function of ucu [5]. Eisuke Sakakibara Used the Taylor expansion to derive the equation for ucu. The picture below is a graph of the equation he made. A typical example is given in Fig. 1.

Moreover, Taylor expansions are also widely used in economic models. For example, Taylor expansions can be used to predict future trends of economic variables. For trend analysis of an economic variable, Taylor expansions can be used to approximate it as a polynomial function and use that function for future forecasting. Other scholars stated that a method for integrating optimization and control during on-line process operation is known as economic model predictive control (EMPC). One formulation of EMPC which can maintain closed-loop stability in the presence of sufficiently small disturbances is Lyapunov-based EMPC (LEMPC). In this work, we make precise connections between closed-loop stability considerations under LEMPC and numerical approximations (via Taylor series) of the solution of the nonlinear dynamic model of the process used in the controller [6]. In their model, they found truncated Taylor series approximation solutions for the nonlinear process model, so they used the Taylor series to help them optimize and predict the model.

4. Application in computer science

In the field of computer science, Taylor expansions are used in several subfields, including numerical analysis, computer graphics, optimization, and machine learning. One use of Taylor expansions in numerical analysis is that they can be used to approximate complex functions, thereby improving computational speed and accuracy. For example, in numerical integration and differentiation, Taylor expansions can be used to compute approximations to function values; in solving ordinary differential equations (ODEs) and partial differential equations (PDEs), Taylor expansions help to construct numerical solution methods, such as finite difference methods and finite element methods. The magnitude of this error can be determined by using the Taylor expansion [7]. Weideman used Taylor expansions to compute error bounds which obtained the results given in Table. 1.

Table 1. Calculation results about standard error estimation and error bound.

N	$\lvert I(f_3)-T_N(f_3)\rvert$	Error bound (3)
10	3.3×10^{-4}	1.6×10^{-1}
20	2.0×10^{-5}	3.9×10^{-2}
40	1.3×10^{-6}	9.7×10^{-3}

In addition, in the article Numerical Simulation of the Potential Flow around a Submerged Hydrofoil with Fully Nonlinear Free-Surface Conditions Jian Hu, Lei Guo, Shili Sun wrote that the latter is in fact the first-order Taylor expansion with respect to y = 0 on thebasis of the former [8]. Thus, they also used Taylor expansions to derive their linear model.

5. Application in Localization algorithm

In the field of localization algorithm, the Taylor series expansion method is applied to improve the accuracy and robustness of positioning algorithms. Specifically in the article a study of indoor positioning algorithms based on RSSI [9], the Taylor series expansion method is used to approximate the nonlinear equations involved in the positioning algorithm by a linear equation, which can significantly reduce the error caused by nonlinear effects by the writer. The resulting linear equation can then be solved using various optimization techniques to obtain more accurate and robust position estimates. Overall, the Taylor series expansion method is a powerful tool for improving the performance of indoor positioning algorithms.

In the article improved Three-time Correction DV-Hop Positioning Algorithm [10], the writer used Taylor's mean value theorem to expand the unknown node's coordinates around the first correction coordinates. By using this expansion, an optimization algorithm is applied to obtain the second correction coordinates of the unknown node. The third correction coordinates are obtained by correcting the coordinate values that do not meet the actual value based on scene restriction conditions. The experiments have proved that this algorithm has excellent positioning capability under three test conditions.

In addition, Taylor expansion can also be used improve accuracy of the algorithm. For example, the Taylor series expansion method is proposed for improving the accuracy of Time of Arrival (TOA) based location estimation using Ultra-Wideband (UWB) technology [11]. The method involves utilizing total least square to estimate the initial reference point and implementing nonlinear equations Taylor series expansion with the initial reference point. This approach is used to solve nonlinear equations and improve the robustness and accuracy of location estimation.

6. Limitations & prospects

Taylor expansion as a widely used mathematical method has many advantages, but at the same time there are some limitations. The first limitation problem is the convergence problem. Taylor expansions may not converge in some cases. For some functions, especially those that do not converge in certain intervals, Taylor expansions may not provide a valid approximation. In such cases, other methods need to be found to approximate the function. The second limitation is the approximation error. Taylor expansions usually only provide an approximation of the function. When truncating the Taylor series, a certain approximation error is introduced. For applications requiring high accuracy, Taylor expansion may not meet the requirements and other more accurate methods need to be used. The third limitation is localization. Taylor expansions usually only provide a valid approximation in a local area. When the behaviour of a function over a larger region needs to be considered, Taylor expansion may not provide satisfactory results. In such cases, it may be necessary to consider other global approximation methods such as Fourier series or Chebyshev polynomials. A fourth limitation is the difficulty of calculating higher order derivatives. The Taylor expansion requires the calculation of higher order derivatives of the function. For some complex functions, it may be very difficult to calculate the higher order derivatives, thus limiting the application of Taylor expansions.

Despite these limitations, Taylor expansions have great potential in many fields. The first major potential is numerical method improvement. To overcome the limitations of Taylor expansion, Taylor expansion can investigate improved numerical methods, such as adaptive truncation and convergence acceleration techniques, to improve the accuracy and speed of the approximation. The second major potential is to combine other approximation methods. Taylor expansions can be combined with other approximation methods to overcome the limitations. For example, Taylor expansions can be combined with methods such as Fourier series or Chebyshev polynomials to achieve a more comprehensive approximation of functions. The third major potential is high performance computing. With the development of high-performance computing technology, Taylor expansion can use parallel computing and high-speed hardware to accelerate the computational process of Taylor expansion, thus expanding its application in complex problems. The fourth major potential is machine learning and artificial intelligence. Taylor expansion has many applications in the field of machine learning and artificial intelligence. By investigating more advanced Taylor expansion methods, the training of models can be improved.

7. Conclusion

In summary, this study provides an in-depth study of Taylor expansions and their applications in various fields. First, we review the historical background of Taylor expansions and the improvement of level expansions. Next, through literature review and research progress, we analyse the applications of Taylor expansions in different scenarios. For three specific domains, we discuss in detail the practical applications of Taylor expansions and the results they produce. Then, we discuss the limitations of the current use of Taylor expansions and provide an outlook on future research. To sum up, Taylor expansions are a powerful mathematical tool with a wide range of applications in several fields. Although there are still limitations in practical applications, this paper demonstrates successful cases of Taylor expansions in several fields. In the future, researchers can continue to focus on improved methods of Taylor expansion to overcome the existing challenges and limitations. In addition, the exploration of new application scenarios will help to expand the application scope of Taylor expansion. Overall, these results provide researchers, practitioners, and engineers with insight into Taylor expansion and its applications, thus promoting the further development of Taylor expansion in various fields.

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