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The Boltzmann distribution in gravity field

Junyao Hao

Jilin University, Physics College, Changchun, China, 130000

13922851173@163.com

Abstract. The air is comprised of innumerable particles. In the atmosphere, the particles are mostly gas molecules, including nitrogen, oxygen, and hydrogen, extending in the air in a gravitational field caused by Earth. With thermal statistics, there are methods to calculate and observe some microscopic properties of this group of particles. This essay aims to disclose the density distribution of gas particles in the air from different altitudes and at different speeds. All the calculations have proceeded on the approximately ideal circumstance with some corrections. The calculation and observation method is based on ideal gas law and rudimentary Boltzmann distribution using knowledge of thermal dynamics and thermal statistics. At the end of the main body of this essay, the final formula is given with graphs created in Matlab providing more ocular illustrations. These parameters are mostly composed of physical fundamental constants and do not require information from a database.

Keywords: gravity, temperature, height, balance, ideal gas, adiabatic.

1. Introduction

As a branch of physics, thermal science is more like mathematical statistics which presents the traits of a huge group comprised of billions of particles from a sole unit.

In the atmosphere above the earth, there are innumerable particles existed constituting various gas. Due to the ideal gas law, sparsely, different gas particles obey the same law: ideal gas law, which presents the constraints between temperature, pressure, and volume of particles in a macroscopic way. The Boltzmann distribution, as a common way to present the secretive array of microcosmic particles, has been also used in many territories of science. This essay, which is aimed at proving and calculating the density of particles at different heights above the earth, can be divided into four sections, including macro and micro demonstrations of the equation only with the constant temperature, the velocity distribution of particles of different possibilities, and several formula corrections after introducing changes of temperature due to dissipative structure of the atmosphere.

The inferences and approximate calculations are based on the ground of the variety of Boltzmann distribution and equation of state of gas (ideal gas law). The latter correction will be an introduced adiabatic process of the inflation of ideal gas to unveil the general and simple regulation between temperature and altitude. Combined and transformed by corrections of temperature and gravity, a more accessible and more authentic formula came out and can be compared to the data collected. Performing this work can give a fundamental and mean method serving as a better reference value to some researchers when exploring more details about the distribution including the changes and the influence of temperature, gravity, humidness, and the source of particles.

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The conclusion can reveal more obvious and apparent connections between micro and macro observation, unveil more nature of the earth and present a more stable formula about the density of particles to process scientific research in different territories such as creating vaccum conditions, understanding more details about heat convection in atmosphere and manufacturing protection suits in extremely high and cold condition.

2. Macro calculation

Here is the ideal gas law, in which P stands for pressure, V presents volume, v is the total number of particles, and k, and T stand separately for Boltzmann constant and temperature.

$$PV = vkT \tag{1}$$

After transformation and introduction of the density of particles, we can get the pressure's expression: (n is the function of height: z)

$$n = \frac{v}{V} \tag{2}$$

$$P = \frac{vRT}{V} = n(z)kT \tag{3}$$

With the force balance formula:

$$\mathrm{F(upward) = F(downward)} \tag{4}$$

$$\mathrm{P(z + dz)\Delta S + nmg\Delta z = P(z + dz)\Delta S} \tag{5}$$

$$dP(z) = -n(z)mgdz \tag{6}$$

After a combination of two formulas replacing P by n(z), a differential equation with the only variable $n(z)$ can be reached.

$$dP(z) = \frac{kTdn(z)}{V} \tag{7}$$

$$\frac{dn(z)}{n(z)} = \frac{-mgdz}{kT} \tag{8}$$

After integration, the density can be figured out:

$$n(z) = n(0)e^{\frac{-mgz}{kT}} \tag{9}$$

$n(0)$ is the density of gas on the surface.

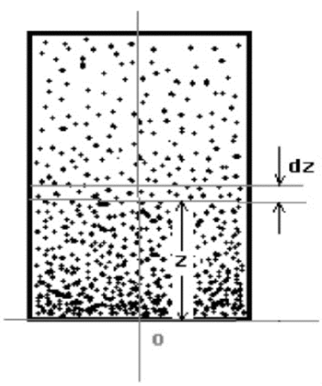

Figure 1. Ideal gas density distribution in an equilibrium state in a gravitational field [1].

3. Micro calculation

Using thermal statistics, the same formula can be reached. Firstly, some definitions should be clarified, ω_l is the degeneracy of possible states in the same energy level. Under the uncertainty principle of

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quantum mechanics, $\omega_l = \frac{\Delta x \Delta p}{h}$. Here, h is the Planck constant. After integration in three-dimension (x-y-z coordinate) space (V is fixed).

$$\omega_l = \frac{VV_p}{h^3} \quad V_p = \int\int\int dpx \ \text{dpy dpz} \tag{10}$$

Two constraints separately introduce two factors α and β [2]

$$\sum a_l = N \tag{11}$$

$$\sum_l a_l \varepsilon_l = E \tag{12}$$

N is the symbol of the total number of particles, and E is the total energy of the whole group of particles, both of which can be indicated by the sum of the number and energy in each energy level. After functional analysis, two factors (α and β) are fixed due to the fixed N and E, comprising the formula under Boltzmann distribution connecting the number of particles and energy in each level. Among them, $\beta = \frac{1}{kT}$

$$a_l = \omega_l e^{-\alpha-\beta\varepsilon_l} \tag{13}$$

Use integration in x-y-z coordinate to present fixed N:

$$N = \frac{V}{h^3} e^{-\alpha-\beta\varepsilon_l} \int\int\int dpx \ \text{dpy dpz} \tag{14}$$

At different heights, particles possess different potential energy due to gravity. Set the surface of the earth as zero potential l surface. Potential energy ε_P is equivalent to mgh. The energy expression involves particles' pot involves energy and kinetic energy in three-dimensional field d [3] and can be presented in the following equation:

$$\varepsilon_l = \varepsilon_K + \varepsilon_P = \frac{P^2}{2m} + \varepsilon_P = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgh \tag{15}$$

Put this ε_l into the expression of N, we can compare the density of particles at different heights. *Iatz=0*, the heights just equivalent to kinetic equivalent ch will be divided into three dir-divided (x-y-z). The equation goes to this:

$$\mathrm{N(0)} = \frac{\mathrm{V}}{\mathrm{h}^3} \mathrm{e}^{-\alpha} \int\int\int \mathrm{e}^{-\frac{\mathrm{p_x^2}}{\mathrm{2mkT}}} \mathrm{dpx}\, \mathrm{e}^{-\frac{\mathrm{p_y^2}}{\mathrm{2mkT}}} \mathrm{dpy}\, \mathrm{e}^{-\frac{\mathrm{p_z^2}}{\mathrm{2mkT}}} \mathrm{dpz} \tag{16}$$

Set $z=h$ (a constant) in the equation whose energy involves different kinds of energy (kinetic energy changes with different positions according to quantum mechanics, and potential energy is fixed for every particle). This equation can be presented:

$$\varepsilon_l = \varepsilon_K + mgz \tag{17}$$

$$N(h) = \frac{V}{h^3} e^{-\alpha-\frac{mgh}{kT}} \int\int\int e^{-\frac{p_x^2}{2mkT}} dpx\, e^{-\frac{p_y^2}{2mkT}} dpy\, e^{-\frac{p_z^2}{2mkT}} dpz \tag{18}$$

Comparing the two expressions, the same conclusion can be proved in a micro way:

$$N(h) = N(0) e^{\frac{-mgh}{kT}} \tag{19}$$

$$n(h) = n(0) e^{\frac{-mgh}{kT}} \tag{20}$$

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4. Particulate distribution and correction

4.1. The velocity distribution of particles

Solving the triple integral with fixed N, the factor $e^{-\alpha}$ can be got. Its value is $\frac{N}{V\left(\frac{2kT\pi m}{h^2}\right)^{3/2}}$

$$\frac{V}{h^3}\int\int\int e^{-\frac{p_x^2}{2mkT}}dpx\, e^{-\frac{p_y^2}{2mkT}}dpy\, e^{-\frac{p_z^2}{2mkT}}dpz = V\left(\frac{2kT\pi m}{h^2}\right)^{3/2} \tag{21}$$

$$N = e^{-\alpha}\frac{V}{h^3}\int\int\int e^{-\frac{p_x^2}{2mkT}}dpx\, e^{-\frac{p_y^2}{2mkT}}dpy\, e^{-\frac{p_z^2}{2mkT}}dpz \tag{22}$$

$$e^{-\alpha} = \frac{N}{V\left(\frac{2kT\pi m}{h^2}\right)^{3/2}} \quad (N\ is\ fixed) \tag{23}$$

We transform the momentum signal into the velocity expression $p = mv$, changing the x-y-z rectangular coordinates to infinite extendable sphere coordinates.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2 sin\theta dr d\theta d\varphi = 4\pi\int_0^{\infty} r^2 dr \tag{24}$$

t $p=mv$ into the sphere coordinates, the equations can be this: (with extra m^3), combining the N expression with speed, we can have the relationship between the possibility (ratio) of particles and velocity, which means velocity distribution.

$$\int\int\int dpx\ \text{dpy dpz} = 4\pi\int p^2 dp = 4\pi m^3\int v^2 dv \tag{25}$$

$$\frac{dn}{n} = D(v) = e^{-\frac{mgh}{kT}-\frac{mv^2}{2kT}} * 4\pi v^2 * \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \tag{26}$$

The simulated graph can be made by coding in Matlab, setting some fixed authentic parameters (mean particle mass is $4.81 \times 10^{-26} kg$, gravity constant is $9.8\ N/kg$, Planck constant $1.38 \times 10^{-23} N \cdot m \cdot s$, the stable temperature is about 25℃ (300 K)).

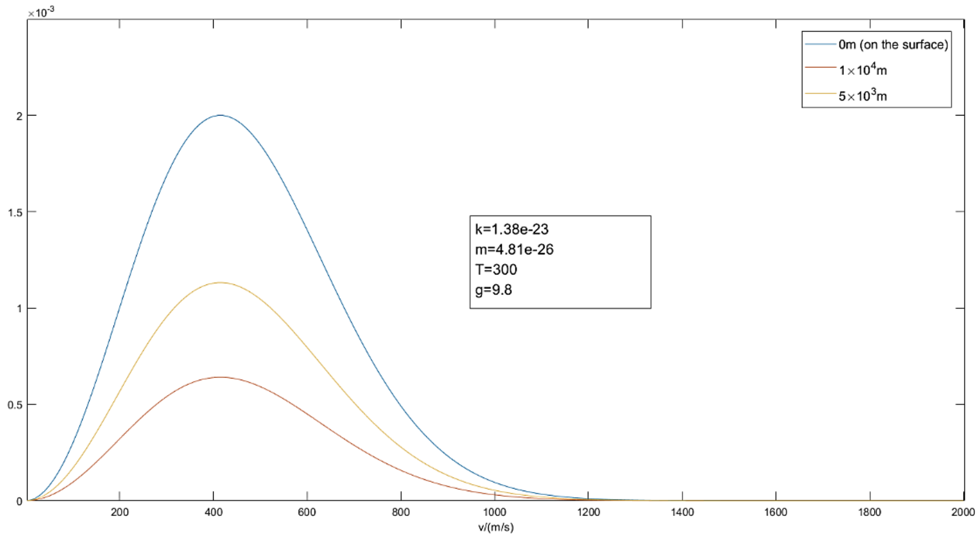

Figure 2. The proportion of particulate number in different speed.

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The graph compares the density of particles in three separate altitudes (on the surface, 10 thousand meters, 5 thousand meters). The vertical coordinates stand for the proportion of the number of particles in determined speed and the horizontal coordinates express velocities with m/s as its unit. The most probable velocity is about 1800 m/s and remains the same without consideration of height. The curves of possibilities get blunt with the increase of altitude, which means that there is a bound at a certain high-level space where the ratio of numbers is almost the same reaching zero and the gas particles hardly exist. This is a so-called vacuum circumstance without atmosphere. We can also divide more easily different speeds of particles in the upper air because of the convergence of possibilities.

4.2. Potential energy correction

In the previous assumption, the gravitational constant remains the same without the differences in height. To conform to the real situation in the atmosphere, this constant must comply with the calculation of the law of universal gravitation to get the data more precise. For instance, the common limitation of the existence of the atmosphere is at over 100 kilometers, while the earth possesses a ratius of about *6400km*. Combining them in the formula, the gravitational constant varies greatly. Here is Newton's law of universal gravitation. M stands for the mass of the earth.

$$F = \frac{GMm}{r^2} = mg \tag{27}$$

To be more specific, we put this change to the calculation of potential energy to make corrections. There is an integral of the gravity force a particle overcomes from the surface to the upper air with its lower limit and upper limit of integration. $r0$ refers to the radius of the Earth, which means a particle's initial position is on the surface. $r0 + h$ represents the distance from the core of the Earth to a certain altitude of the atmosphere, h is the height.

This stands for the ultimate position of a particle.

$$\varepsilon_P = \int_{r0}^{r0+h} \frac{GMm}{r^2} = \frac{GMmh}{r_0(r_0 + h)} \tag{28}$$

After the gravity correction, the first equation of the density formula has been changed in its index number. The formula becomes intricate with more physical constants and more calculations. G is the universal gravitational constant, whose value is about 6.67×10^{-11}.

$$n(h) = n0 * e^{\frac{-GMm}{kT(r0(r0+h))}} \tag{29}$$

4.3. The correction of temperature

Previous calculations and inferences are based on the assumption that the temperature in the atmosphere always remains the same because of the balance of energy, at about 25℃ as the author supposed. This also means that there is a stationary field above earth where the heat convection and energy transduction seldom happen. However, practically, this is not the real situation in the authentic atmosphere. The global mean state of the atmosphere is maintained by a statistical balance between sources and sinks of energy and momentum in the atmospheric circulation [4], which needs successive energy import to maintain. The energy resource has its origin from the heat radiation of the sun by daylight, while the outer space absorbs the heat from the gas particles with its temperature reaching approximately absolute zero. As a result, there is heat convection in the air between particles and particles. So the temperature varies according to the height. The main cause of the drop in temperature with the increase of height in the troposphere is that the air pressure falls with the altitudes. The expansion occurs when the air rises and the atmosphere shrinks when descending [5]. Because of the low heat conductivity coefficient, the process of expansion and contraction can be considered adiabatic [6], which means the energy of gas only changes through work according to the first law of thermo ($\Delta U = \Delta Q + \Delta W$). The formula can be expressed more specifically.

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$$dU = dQ - pdV \tag{30}$$

The heat changes can be considered zero. Combining this with the equation of state in a differential way, we could get the transformation of the formula.

$$dU = -pdV \tag{31}$$

$$Vdp + pdV = nkdT \tag{32}$$

$$Vdp - dU = nkdT \tag{33}$$

For gas particles without consideration of intermolecular attractive force and repulsive force, we can assume that all their energy comes from kinetic movement. If the volume is fixed, the energy changes are only related to the changes in temperature, multiplied by a constant named heat capacity. This signal with fixed volume can be expressed like this:

$$dU = C_V dT \tag{34}$$

$$Vdp - C_V dT = nkdT \tag{35}$$

For fixed pressure and moveable volume:

$$dU = C_p dT \tag{36}$$

$$C_p = C_V + nk \tag{37}$$

Using ideal gas law to transform *T*:

$$dT = \frac{Vdp + pdV}{nk} \tag{38}$$

$$Vdp - C_V\left(\frac{Vdp + pdV}{nk}\right) = Vdp + pdV \tag{39}$$

$$C_V\left(\frac{Vdp + pdV}{nk}\right) + pdV = 0 \tag{40}$$

$$\frac{C_V Vdp}{nk} = -\left(\frac{C_V p}{nk} + p\right) dV \tag{41}$$

Introducing $C_p = C_V + nk$:

$$\frac{1}{nk} C_V Vdp = -\frac{1}{nk} C_p pdV \tag{42}$$

$$C_V Vdp = -C_p pdV \tag{43}$$

Let γ represent the ratio of C_p over C_V (for diatomic molecule, set mean ideal γ is 1.4 in the air)

$$\frac{dp}{p} + \gamma \frac{dV}{V} = 0 \tag{44}$$

$$lnp + \gamma lnV = constant \tag{45}$$

$$pV^{\gamma} = constant \tag{46}$$

Transforming V into T:

$$\frac{T^{\gamma}}{p^{\gamma-1}} = constant \tag{47}$$

The gradient of temperature from pressure in the diabatic process goes to:

$$\left(\frac{\partial T}{\partial p}\right)_S = \frac{\gamma - 1}{\gamma}\frac{T}{p} \tag{48}$$

As the explanation given above, the temperature is the function of altitude:

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$$\frac{d}{dz}T(z)=\left(\frac{\partial T}{\partial p}\right)_S\frac{d}{dz}p(z) \tag{49}$$

$$\frac{d}{dz}T(z)=-\frac{\gamma-1}{\gamma}\frac{mg}{k}=-\frac{\gamma-1}{\gamma}\frac{Mg}{R} \tag{50}$$

The relation of temperature and height is linear, we can construct a formula of T:

$$T=T_0+az \tag{51}$$

$$a=\frac{\gamma-1}{\gamma}\frac{mg}{k} \tag{52}$$

Put it into the expression of particulate density to make the correction:

$$\frac{\mathrm{dn}}{dz}=-\frac{n(z)mg}{kT(z)}=-\frac{n(z)mg}{k[\![(T]\!]_0+az)} \tag{53}$$

After integration, here comes the conclusion of density after corrections by gravity and temperature. After temperature correction:

$$n=n_0\left(1+\frac{az}{T_0}\right)^{-\frac{mg}{aR}} \tag{54}$$

After gravity correction:

$$n=n_0\left(1+\frac{az}{T_0}\right)^{-\frac{mGM}{ak(r_0(r_0+h))}} \tag{55}$$

Making illustration with the fixed parameters, we can get the graph in Matlab comparing the present formula to the former one in different altitude (assuming n_0 is a random constant)

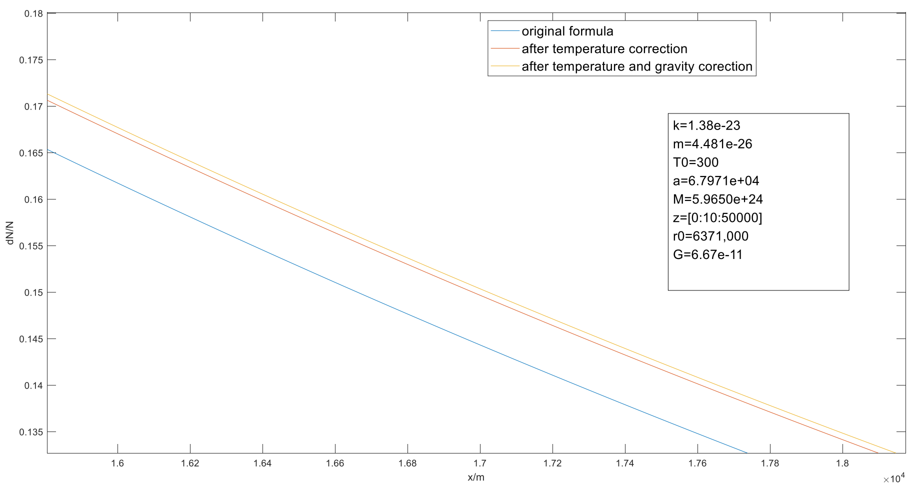

Figure 3. The ratio(y) of number of particles in different height(*x(m)*).

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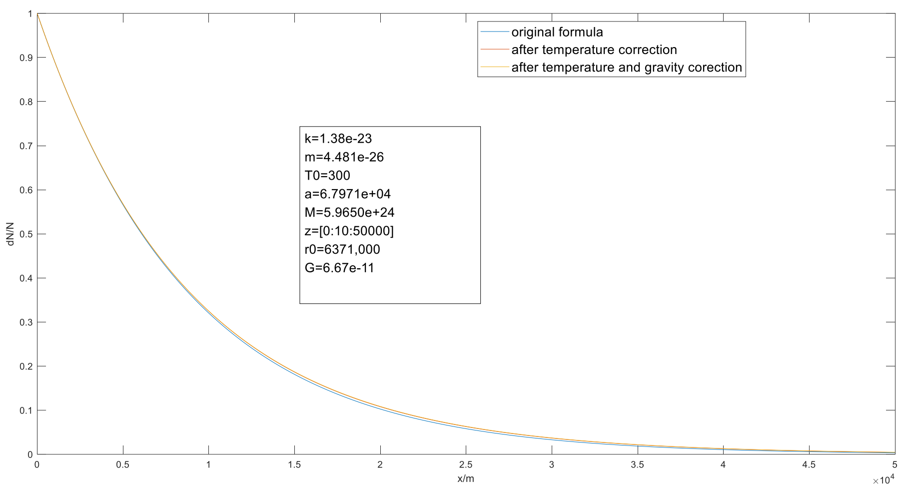

Figure 4. The ratio(y) of number of particles in different height(x(m)).

Observing the graph, the temperature amendment has a nonnegligible impact on the results, especially at high levels of space, while the gravity correction influences little the conclusion. Both of them can be ignored in the low-level atmosphere.

5. Conclusion

This essay proves theoretically the Boltzmann distribution and illustrates the particulate density from different speeds. Two corrections separate from gravity and temperature are made throughout the essay with a graph comparing them in different heights. However, even with the two corrections, the calculations are based on the ideal circumstance. In different levels of the atmosphere exist clouds that can absorb the heat radiation from the sun causing different heat resources, while our assumption considers that the original heat resource only exists on the surface of the earth, making air expansion occur through all layers of the atmosphere. In fact, due to the intricate meteorology, the condition conceived in the essay only happens in the troposphere (starts at the Earth's surface and extends to 8 to 14.5 kilometers high) [7] where the convection in the air is strong. When it comes to the stratosphere, generally speaking, ozone contents in the stratosphere are higher than that in the troposphere [8] which can easily absorb the ultraviolet increasing the temperature, the ideal circumstance is not suitable for this layer. Apart from the error made by the real atmosphere, the formula adopted an approximate solution when combining temperature and gravity corrections altogether because little change can be made in gravity correction. The graph set the limit up to 100 kilometers while the convenient upper limit is at 10 kilometers, which means the corrections affects little the changes to the ratio of particulate numbers at different height. In the future, the research can focus more on the variety of particles that can affect the temperature and potential energy discontinuously from different layers.

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