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The colonel blotto game based on probability and statistics

Senhao Hu

Honour of Statistics, University of Waterloo, Waterloo 301003, Canada

s96hu@uwaterloo.ca

Abstract. The Colonel Blotto Game is famous as zero-sum game. The game asked players to get more passes(objects) than their opponents to win the game with limited regiments(resources). The one who put more regiments on the pass would get it, and player who has more passes would win the game. The Colonel Blotto Game could be used in athletics, business and competition in other forms: how the player uses specific amount of resource with strategies to gain more benefits than competitors. In this case, the Colonel Blotto Game could be seen transfer to a linear program problem, with constraints about limited resources to maximize what players get in the game. This article would analyze the strategy for the Colonel Blotto game in probability of winning and build the model by extending the Colonel Blotto Game with more regiments, more passes and weighted some passes to look for how these variables impact each other and find the general solution for this game. Then using linear program to check the final results. This article would focus on the resources, benefits and weighted of the benefits for the Colonel Blotto game to find out the directly relationship among these variables of the model with the strategy to win the game.

Keywords: colonel blotto game, zero-sum game, game theory.

1. Introduction

Zero-sum game is a kind of strategical games which has finite players with finite strategies. Given the value of winning as one, while losing be seeing as negative one, and drawing would be zero. It is clearly to see that the total sum of the benefits would be zero after the whole game which just like the name of zero-sum game, this is because that player's benefits from the game would equal to other's loss. The Colonel Blotto Game also known as Blotto Game. Which asked two players to use finite positive integer number of k force to obtain finite positive integer number of n battlefields. During the game, players do not know about their opponent's strategies. Suppose that player A focus on one of the battlefields with kA of force and player B use kB of force for it. If kA is not equal to kB, then the player who uses more force on the certain battlefields would get the battlefield. While if kA has same value with kB, then there will be a draw. In the end of game, the one who has more battlefields win the game [1]. In this paper, suppose two players with three battlefields which have same characteristics and have the same value, each players have five forces as the base case of Colonel Blotto Game, then the research will increase the force with the battle fields, and weighted some of battlefields based on it [2].

Game theory is useful in analyzing the mathematical model, especially in conflicts which is involved with multiplayer to make strategies. These strategies are interdependent which means one's strategy is not only influenced by his or her actions, but also influenced others' decisions [3]. The key is that players should consider who other's decision making would affect their own decisions. The most important

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thing for the game theory is the game. This is because without a reasonable and suitable model, players cannot make decisions which could help them to get more benefits from others. Game theory provides framework for the game, stable and rational outcomes for the player which would allow them to possibly to get more benefits from the game [4]. This game is not limited in the mathematics fields, it can also work in economics such as pricing strategy, market dynamic, political science, management and so on. The zero-sum game and game theory deeply related to humans' daily lives. They could be used in several different fields. Firstly, zero-sum Stackelberg game could be worked on the performance for network. The network performance would be improved by allocating energy and controlling the sequence of transmitting the BSs power in an efficient way with the strategies of zero-sum game and game theory [5]. Secondly, zero-sum game could be used in financial marketing manipulating and high frequency trading, which is called algorithmic trading, even though it is banned, people could still learn from it to analyze and monitor algorithmic trading which would influence the whole markets [6]. Furthermore, game theory also works on medical, such as using the zero-sum game to allocate healthcare resource to an emergency department for patients with different levels of needs and conditions [7]. In general, game theory and zero-sum game could be useful in the situation which needs to maximize the benefits/efficient or minimize the lose with limited resource.

The Colonel Blotto game can be tracked to 1921, French mathematician Emile Borel first mentioned the game in his paper and introduced it as Colonel´ Blotto game [8]. Borel initially proposed the Colonel Blotto game as a mathematical model which would be studied for decision making in the conflicts. Even though the model given by Borel was relatively simple, it paved the way to the advance zero-sum game and game theory. Recent years, more and more scholar connected Colonel Blotto game with Nash equilibrium. Nash equilibrium works in game theory which define the best strategy for a game is even if the player knows the opponents' strategy, he or she would keep his or her strategy as well, as there are no more benefits for the player to change strategy [9]. As the Nash equilibrium really exists in Colonel Blotto game, it is clearly to see that there exists the best strategy and the best strategy is pure. Otherwise, in some "stochastic asymmetric" version of the Blotto game which means that players have different number of resources, there exists mixed strategy [10].

As mentioned in the previous paragraphs, zero-sum game works to achieve maximizing benefits and minimizing loss with limited resources. This could be expressed as the linear program. The allocated resources on each battlefield could be seen as individual variables in the linear program. The sets which contain these variables would be seen as a strategy for the game. And the sum of winning probability of all different sets would be 1. For the linear program, it is asked to maximize the number of battlefields, it has two kinds of constraints, the first one is allocation constraints, which means the sum of resources that is allocated on each battlefield for each player must not over the play's total resource. The other one is strategy constraints; it limits that the sum of probability for the opponent's allocation on a certain battlefield should be equal to one.

After talking about the basic condition of Colonel Blotto Game, this paper would increase resource and battlefields to find some general solution for Colonel Blotto Game. To make Colonel Blotto Game strategic, a proper proportion for the resource and battlefields would be essential. There are two extreme situations. Firstly, the number of battlefields is far greater than the number of resources. Take the example of three force with one thousand battlefields. Players just need to allocate three battlefields with one resource, then they will not lose the game. The second one is the number of resources is far greater than the number of battlefields. Take the example of three battlefields with one thousand resources. In this case, players would have same strategies with three battlefields and five resource. Players just stack meaningless hundreds of resources on these three battlefields which would not influence strategy. It may just make calculation and programming hard to work. This research will discuss the proportion for battlefields and regiments.

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2. Method

2.1. Base case

The base case for the Colonel Blotto Game is that, for each of two players, they must work on three battlefields with five regiments. The one who pays more resources on a certain battlefield would get it. If a battlefield received same numbers of regiments from two players, t Colonel Blotto Game here would be a draw and none of these two players would get this pass. In the end of the whole game, the player who has more battle fields would win the game.

2.2. Strategies

To show strategies for each player, the research would present strategies as $a, b, c \dots, k$,where $a, b, c, \dots, k \in \mathbb{Z}$, and $a, b, c, \dots, k \geq 0$, k is the number of battlefields. $\sum_{i=1}^{k} k \leq n$, where $n \in \mathbb{Z}$ is the number of resources for each of players. $a, b, c, \dots, k$presents for each of battlefields, the number of resources is used by one of players. To remove ambiguity, in this research, all n! kinds of strategies in different order would be sorted in ascending orders, which means $a \leq b \leq c \leq \cdots \leq k$. It is true that different order strategies would lead to a different result, the researcher would calculate all $(n!)^2$ kinds of results for certain selecting strategies from two players.

2.3. Payoff Matrix

This research would use Payoff Matrix to present results from two players' strategies. Payoff Matrix shows all available strategies from two players and calculate all net winning (the difference in winning between two players) for each pair of strategies and presents the winning probability.

3. Results and discussion

Table 1 shows the Payoff Matrix for the base case of Colonel Blotto Game. The rows correspond to the strategies the player use in the game and the columns correspond to the strategies of the opponent. After add all the possibilities of winning for each rows, it is clear to see that P ((0,0,5)) < P ((0,1,4)) < P((0,2,3)) < P((1,1,3)) < P((1,2,2)). So (1,2,2) are the best selecting strategies.

Table 1. Payoff Matrix for Colonel Blotto Game base case.

	(0,0,5)	(0,1,4)	(0,2,3)	(1,1,3)	(1,2,2)
(0,0,5)	0	-1/3	-1/3	-1	-1
(0,1,4)	1/3	0	0	-1/3	-2/3
(0,2,3)	1/3	0	0	0	1/3
(1,1,3)	1	1/3	0	0	-1/3
(1,2,2)	1	2/3	-1/3	1/3	0

Different numbers of regiments may lead to a different selecting strategy for a certain number of passes. Just as mentioned above, too few resources lead to a lack of strategy; and too many resources will lead to a meaningless increase in calculation.

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Table 2. Payoff Matrix for 3 passes with 6 regiments.

	(0,0,6)	(0,1,5)	(0,2,4)	(1,1,4)	(0,3,3)	(1,2,3)	(2,2,2)
(0,0,6)	0	-1/3	-1/3	-1	-1/3	-1	-1
(0,1,5)	1/3	0	0	-1/3	0	-2/3	-1
(0,2,4)	1/3	0	0	0	0	0	0
(1,1,4)	1	1/3	0	0	1/3	-1/3	-1
(0,3,3)	1/3	0	0	-1/3	0	1/3	1
(1,2,3)	1	2/3	0	1/3	-1/3	0	0
(2,2,2)	1	1	0	1	-1	0	0

After calculating all selecting strategies from table 2 in the form as the sum of payoff value. It is clearly to see (2,2,2) is the best selecting strategies.

To find a suitable number of regiments for a certain number of passes. This research will increase regiments gradually until the best strategy for a player tends to be stable.

Table 3. Payoff Matrix for 3 passes with 7 regiments.

	(0,0,7)	(0,1,6)	(0,2,5)	(1,1,5)	(0,3,4)	(1,2,4)	(1,3,3)	(2,2,3)
(0,0,7)	0	-1/3	-1/3	-1	-1/3	-1	-1	-1
(0,1,6)	1/3	0	0	-1/3	0	-2/3	-2/3	-1
(0,2,5)	1/3	0	0	0	0	0	-1/3	-1/3
(1,1,5)	1	1/3	0	0	1/3	-1/3	-1/3	-1
(0,3,4)	1/3	0	0	-1/3	0	0	1/3	2/3
(1,2,4)	1	2/3	0	1/3	0	0	0	1/3
(1,3,3)	1	2/3	1/3	1/3	-1/3	0	0	-1/3
(2,2,3)	1	1	1/3	1	-2/3	1/3	-1/3	0

Table 4. Payoff Matrix for 3 passes with 8 regiments.

	(0,0,8)	(0,1,7)	(0,2,6)	(1,1,6)	(0,3,5)	(1,2,5)	(0,4,4)	(1,3,4)	(2,2,4)	(2,3,3)
(0,0,8)	0	-1/3	-1/3	-1	-1/3	-1	-1/3	-1	-1	-1
(0,1,7)	1/3	0	0	-1/3	0	-2/3	0	-2/3	-1	-1
(0,2,6)	1/3	0	0	0	0	0	0	-1/3	-1/3	-2/3
(1,1,6)	1	1/3	0	0	1/3	-1/3	1/3	-1/3	-1	-1
(0,3,5)	1/3	0	0	-1/3	0	0	0	0	1/3	1/3
(1,2,5)	1	2/3	0	1/3	0	0	1/3	0	1/3	-2/3
(0,4,4)	1/3	0	0	-1/3	0	-1/3	0	1/3	-1/3	1
(1,3,4)	1	2/3	1/3	1/3	0	0	-1/3	0	0	1/3
(2,2,4)	1	1	1/3	1	-1/3	1/3	-1/3	0	0	-1/3
(2,3,3)	1	1	2/3	1	-1/3	2/3	-1	-1/3	1/3	0

Table 5. Payoff Matrix for 3 passes with 9 regiments (part 1).

	(0,0,9)	(0,1,8)	(0,2,7)	(1,1,7)	(0,3,6)	(1,2,6)
(0,0,9)	0	-1/3	-1/3	-1	-1/3	-1
(0,1,8)	1/3	0	0	-1/3	0	-2/3
(0,2,7)	1/3	0	0	0	0	0
(1,1,7)	1	1/3	0	0	1/3	-1/3
(0,3,6)	1/3	0	0	-1/3	0	0
(1,2,6)	1	2/3	0	1/3	0	0
(0,4,5)	1/3	0	0	-1/3	0	-1/3
(1,3,5)	1	2/3	1/3	1/3	0	0
(2,2,5)	1	1	1/3	1	-1/3	0
(2,3,4)	1	1	2/3	1	0	2/3
(3,3,3)	1	1	1	1	0	1

Table 6. Payoff Matrix for 3 passes with 9 regiments (part 2).

	(0,4,5)	(1,3,5)	(2,2,5)	(2,3,4)	(3,3,3)
(0,0,9)	-1/3	-1	-1	-1	-1
(0,1,8)	0	-2/3	-1	-1	-1
(0,2,7)	0	-1/3	-1/3	-2/3	-1
(1,1,7)	1/3	-1/3	-1	-1	-1
(0,3,6)	0	0	1/3	0	0
(1,2,6)	1/3	0	-1/3	-2/3	-1
(0,4,5)	0	0	0	2/3	1
(1,3,5)	0	0	0	0	0
(2,2,5)	0	0	0	-1/3	-1
(2,3,4)	-2/3	0	1/3	0	0
(3,3,3)	-1	0	1	0	0

From Table 3, 4, 5 and 6 above, it is obviously to see that the best strategies which have the highest payoff value for each number of regiments are that separating all regiments to three battlefields evenly.

From previous calculations, the research can draw the preliminary conclusions, since amounts of regiments and passes influence each other, to find a properly proportion between the numbers of regiments and passes, this research would then find best strategies as increasing the number of regiments with battlefields. In the table above, it will show best strategies for different number of regiments and battlefields, the column would present the number of regiments, while the row would show the number of battlefields. Since if the number of battlefields is even, there would be too much draw for two players, in this case, this research will just discuss battlefields in odd numbers, as Table 7 and 8 show.

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Table 7. Best Strategies for Colonel Blotto Game with different number of regiments and battlefields (part 1).

	5	6	7	8
3	(1,2,2)	(2,2,2)	(2,2,3)	(2,3,3)
5	(1,1,1,1,1)	(1,1,1,1,2)	(1,1,1,2,2)	(1,1,2,2,2)
7	(0,0,1,1,1,1,1)	(0,1,1,1,1,1,1)	(1,1,1,1,1,1,1)	(1,1,1,1,1,1,2)
9	(0,0,0,0,1,1,1,1,1)	(0,0,0,1,1,1,1,1,1)	(0,0,1,1,1,1,1,1,1)	(0,1,1,1,1,1,1,1,1)

Table 8. Bes tstrategies for Colonel Blotto Game with different number of regiments and battlefields (part 2).

	9	10	11
3	(3,3,3)	(3,3,4)	(3,4,4)
5	(1,2,2,2,2)	(2,2,2,2,2	(2,2,2,2,3))
7	(1,1,1,1,1,2,2)	(1,1,1,1,2,2,2)	(1,1,1,2,2,2,2)
9	(1,1,1,1,1,1,1,1,1)	(1,1,1,1,1,1,1,1,2)	(1,1,1,1,1,1,1,2,2)

4. Conclusion

From the previous calculation, it is obviously to see that the best strategy for Colonel Blotto Game is dividing resources evenly to all of battlefields. Given n resources and k battlefields, where$n, k \in \mathbb{N}$, the best strategy for the Colonel Blotto Game is putting $\frac{n}{k} - 1$regiments in each of m battlefields, where $m \in \mathbb{N}$, m is the remainder of $\frac{n}{k}$, and putting $\frac{n}{k}$, regiments to each of the rest of battlefields$(k - m)$.

It is true that one of strategy also has good performance in payoff matrix and winning probability. This strategy is just focusing on r battlefields, where $r \in \mathbb{N}, r = \frac{k+1}{2}$. Putting all of resources evenly in r battlefields. This kind of strategy has a positive payoff value as the opponent using the strategy which is talked before. But this does not mean that this strategy is better than the previous strategy. This strategy has the problem that it has uncertainty: how the player selects r battlefields from n battlefields. Different selection would lead to a different result. Also, this research just looks for the strategy which has the highest payoff value (the highest winning probability). Back to these two strategies, the previous one has higher winning probability and payoff value than this strategy for all kinds of the opponent's strategies. Thus, this research would see the previous one as the best strategy.

Back to the question this paper talked in the introduction, too many regiments would lead to meaningless calculation; and too few regiments would cause no general results to find best strategy. From the calculation above, the research shows that as the number of regiments is less than the number of battlefields, there would be too few strategies to use. Even two players probably put their regiments into different battlefields. Thus, the number of regiments should equal or greater than the number of battlefields. From table 1 to table 5, it is clearly to see that all of them have a part of similar payoff values. As the increase of regiments, it would just add some new rows and columns, the whole table would appear in the next table. Based on the facts that too many regiments would increase the calculation, the payoff matrix larger number of regiments is a matrix based on the previous matrix from smaller amount of regiments with some new rows and columns. As a result, the good proportion for the number of regiments and battlefields should be that resources should greater or equal to the battlefields and less or equal to the two times of battlefields.

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