Proceedings of the 2023 International Conference on Mathematical Physics and Computational Simulation
DOI: 10.54254/2753-8818/11/20230417

On nonsingular bilinear maps over different fields

Ruoxuan Feng

WLSA Shanghai Academy, Shanghai, China, 200135

reinafeng2006@163.com

Abstract. The investigation of nonsingular bilinear forms originates from the classification of division algebras over the real number field. Building upon this foundation, researchers have delved into the study of nonsingular bilinear forms over real number fields, leading to significant results such as Hopf's theorem. However, the interest in understanding nonsingular bilinear forms extends beyond real number fields, prompting a desire to explore other fields as well. When it comes to algebraically closed fields, the theorem becomes well-understood, with essence captured by the Hopf-Smith theorem. Inspired by these established studies, we are motivated to further the comprehension of nonsingular bilinear forms over arbitrary fields. Given a field $\Bbbk$, we study in this article numerical constraints on r, s, n for the existence of nonsingular bilinear maps $\phi: \Bbbk^r \times \Bbbk^s \to \Bbbk^n$ for not only algebraically closed fields and the real number field but also the rational number field and finite fields. We reach the final conclusion mainly through algebro-geometric techniques and the use of determinantal varieties. We reprove a result of Hopf–Smith which states that the minimal possible value of n is $r+s-1$ when $\Bbbk$ is an algebraically closed field. When $\Bbbk$ is the real number field, we prove that under a combinatorial condition, the minimal possible value of n is still $r+s-1$. We also show that when $\Bbbk$ is the rational number field or a finite field, the minimal possible value of n is $\max\{r,s\}$.

Keywords: nonsingular bilinear forms, Hopf–Smith theorem, determinantal varieties, field extensions.

1. Introduction

Let $\Bbbk$ be a field. Let U, V, W be nonzero $\Bbbk$-vector spaces. We will be interested in the existence of bilinear maps satisfying certain *nonsingular* conditions. Following the notation of [1], we define

Definition 1.1. A bilinear map $\phi: U \times V \to W$ is called *nonsingular*, if

$$\phi(u, v) = 0 \text{ implies } u = 0 \text{ or } v = 0. \tag{1}$$

Given a nonsingular bilinear map $\phi: U \times V \to W$, it turns out that the dimension of W is related to that of U and V. For example, the non-singulariy of ϕ implies easily that

Lemma 1.2. $\max\{\dim U, \dim V\} \leq \dim W$.

This motivates us to define the following invariant $r\#_{\Bbbk}s$.

Definition 1.3. Let r, s, n be positive integers. We say the condition $\mathcal{H}_{\Bbbk}(r, s, n)$ holds, if there exist $\Bbbk$-vector spaces U, V, W of dimension r, s, n, respectively and there exists a *nonsingular* bilinear map $\phi: U \times V \to W$. Define $r\#_{\Bbbk}s$ to be the minimal integer such that $\mathcal{H}_{\Bbbk}(r, s, r\#_{\Bbbk}s)$ holds. In other words,

$$r\#_{\Bbbk}s := \min\{n \in \mathbb{N}: \mathcal{H}_{\Bbbk}(r, s, n) \text{ holds}\}. \tag{2}$$

We have evident bounds for $r\#_{\Bbbk}s$.

Proposition 1.4. $\max\{r,s\} \leq r\#_{\Bbbk}s \leq r+s-1$.

Let $n = r\#_{\Bbbk}s$. By the definition of $r\#_{\Bbbk}s$, there exists a nonsingular bilinear map $\phi: \Bbbk^r \times \Bbbk^s \to \Bbbk^n$. Fix $u \neq 0 \in \Bbbk^r$ and then we get a linear map $\phi_u: \Bbbk^s \to \Bbbk^n$ given by $v \mapsto \phi(u,v)$. The non-singularity of ϕ implies that ϕ_u is injective. Therefore, $n \geq s$. Similarly, $n \geq r$. Hence, we can derive that $r\#_{\Bbbk}s$ must be greater than or equal to $\max\{r,s\}$. For the upper bound of $r\#_{\Bbbk}s$, it suffices to construct the following map $\psi: \Bbbk^r \times \Bbbk^s \to \Bbbk^{r+s-1}$. Let $u \in \Bbbk^r$ be $(x_0, x_1, \ldots, x_{r-1})$, and $v \in \Bbbk^s$ be $(y_0, y_1, \ldots, y_{s-1})$. We define $w = \psi(u,v) \in \Bbbk^{r+s-1}$ as follows. Writing $w = (z_0, z_1, \ldots, z_{r+s-2})$, z_k is defined as $\sum_{i+j=k} x_i\, y_i$. It is obvious that this map is bilinear. It remains to check that ψ is nonsingular. Let $(u,v) \in \Bbbk^r \times \Bbbk^s$ satisfy $w = \psi(u,v) = 0$. We want to show that $u = 0$ or $v = 0$. Assume by contradiction that u, v are both nonzero. Writing $u = (x_0, \ldots, x_{r-1})$ (resp. $v = (y_0, y_1, \ldots, y_{s-1})$), we can take the minimal i (resp. j) such that x_i (resp. y_j) is not zero. Then z_{i+j} can be calculated as $\sum_{i+j}$ $x_i\, y_j = x_0 y_{i+j} + x_1 y_{i+j-1} + \cdots + x_i y_j + \cdots + x_{i+j} y_0 = x_i y_j$, which, obviously, cannot equal 0. This contradicts our assumption of $w = 0$. Therefore, ψ is nonsingular.

Note that the bounds in Proposition 1.4 do not depend on the base field $\Bbbk$. However, the subscript $\Bbbk$ in the definition of $r\#_{\Bbbk}s$ indicates that the precise value of this invariant is *dependent* of the base field $\Bbbk$, and it is indeed the case. In fact, the main purpose of this article is to try to understand this invariant with different base fields. Our main result is the following

Theorem 1.5 (Main Theorem). *(i) If $\Bbbk$ is an algebraically closed field, then* $r\#_{\Bbbk}s = r+s-1$.

(ii) In the real number cases, if the combinatoric number $\binom{r+s-2}{r-1}$ is an odd number, then $r\#_{\mathbb{R}}s = r+s-1$.

(iii) If $\Bbbk$ is the rational number field $\mathbb{Q}$ or a finite field $\mathbb{F}_q$, then $r\#_{\Bbbk}s = \max\{r,s\}$.

It turns out that in general, $r\#_{\Bbbk}s$ is quite hard to calculate. For example, determining the precise value of $11\#_{\mathbb{R}}14$ is still a well-known open problem in this domain according to Theorem 12.21 in [1]. This justifies our desire to ask for an additional condition in Theorem 1.5 (ii). Also note that in Lemma 5.8, we give some equivalent conditions for $\binom{r+s-2}{r-1}$ being an odd number.

Let us briefly describe here the history of this research domain and technicalities we use to prove the main theorem. The first statement is a direct consequence of the following theorem of Hopf and Smith stated and proved in [2, 3].

Theorem 1.6 (Hopf-Smith [2, 3]). *Let $\Bbbk$ be an* algebraically closed *field. Let U, V, W be nonzero finitely dimentional $\Bbbk$-vector spaces. Let*

$$\phi: U \times V \to W$$

be a nonsingular *bilinear map. Then* $\dim W \geq \dim U + \dim V - 1$.

Our statement here follows the notation as in Chapter 12 in [1]. An interesting application of this theorem is the Clifford's theorem in algebraic curves theory that relates the degree and the rank of a divisor on an algebraic curve (see Chapter III section 1 in [4] for detail). The classical proof of this theorem consists of using algebraic topology theory on projective spaces. We will *not* talk about this classical proof here. Instead, we give an algebro-geometric proof of the theorem of Hopf-Smith which, we believe, is more universal. Indeed, the proof of the second statement of Theorem 1.5 also uses this cycle of ideas, which will be discussed in Section 6. Return to the proof of the theorem of Hopf-Smith. We will use the construction of Grassmannian manifolds and determinantal varieties, whose basic theory will be recalled in Sections 2 and 3. The inequality in the theorem of Hopf-Smith then comes from the following fundamental theorem in algebraic geometry whose proof can be found in [5].

Theorem 1.7. Let $\Bbbk$ *be an algebraically closed field. Let $X, Y \subset \mathbb{P}^n$ be projective varieties over $\Bbbk$ such that* $\dim X + \dim Y \geq n$, *then* $X \cap Y \neq \emptyset$.

Remark. When $n = 2$, Theorem 1.7 is a special case of the Bézout's theorem: if X, Y are curves that do not have common components, then the number of $X \cap Y = \deg X \deg Y > 0$, counted with multiplicities.

Note that if $\Bbbk$ is *not* an algebraically closed field, then Theorem 7 becomes obviously wrong. For example, take $\Bbbk = \mathbb{R}$ and C and C' be real projective plane curves defined by $x_0 = 0$ and $x_0^2 - x_1^2 - x_2^2 = 0$, respectively, then C and C' have no intersections. This explains the reason why we need the condition that $\Bbbk$ is algebraically closed in Theorem 1.5 (i).

The proof of Theorem 1.5 (ii) is subtler. We shall give two proofs of quite different nature to this fact. The first proof relies on the following theorem due to Hopf.

Theorem 1.8 (Hopf [2]). *If there is a nonsingular bilinear map $\phi: \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$. Then whenever $n - s < k < r$, $\binom{n}{k}$ is even.*

Again the proof of this theorem uses homology group theory of real projective spaces and we refer those interested readers to Theorem 12.2 in [1] for the proof. Using this theorem and some elementary observations in combinatorial numbers that we will develop in Section 5, we finish the first proof the Theorem 1.5 (ii).

The second proof of Theorem 1.5 (ii) is more direct and does not utilize the combinatorics. Instead, it relies on the construction of Grassmannian manifolds and determinantal varieties we recall in Sections 2 and 3, and on the fact that under the condition stated in Theorem 1.5 (ii), the degree of the determinantal variety in the projective space is *odd* [6]. We write down the detail in Section 6 and terminate the second proof.

The proof of Theorem 1.5 (iii) is quite easy and algebraic. It is a direct consequence of the fact that the field $\Bbbk$ in question admits fields extensions of any degree. We give in Section 7 the proof of Theorem 1.5 (iii).

The organization of the article is as follows. In Section 2, we recall basic knowledge about Grassmannian manifolds and calculate their dimension. In Section 3, we present the determinantal varieties for later uses. In particular, we calculate the dimension of determinantal varieties. In Section 4, we prove Theorem 1.5 (i) using the algebro-geometric tools developed so far. In Section 5, we collect some elementary combinatorial observations and use them to give a first proof of Theorem 1.5 (ii) with the help of Hopf's theorem (Theorem 1.8). In Section 6, we present a result of the degree of the determinantal varieties and use algebro-geometric arguments to give another proof of Theorem 1.5 (ii). In Section 7, we recall some basic knowledge about the degree of field extensions and discuss its relation with the invariant $r\#_{\Bbbk}s$. We also use this discussion to prove Theorem 1.5 (iii).

2. Grassmannian manifolds

In this section, we present basic knowledge on Grassmannian manifolds. The main references of this part are [7, 8]. We give a local description of these manifolds and determine their dimensions.

Definition 2.1. The Grassmannian manifold $Gr(k, n)$ is defined as $\{k$-dimensional subspaces in $\Bbbk^n\}$.

Projective spaces are special cases of Grassmannian manifolds. By definition, projective manifolds parametrize 1-dimentional subspaces in a given vector space. Hence, $\mathbb{P}^{n-1} = Gr(1, n)$.

2.1. A canonical affine cover of the Grassmannian manifolds

As in the case of projective spaces, we can give a canonical affine cover of the Grassmannian manifolds. To do this, let us first give another description of elements in the Grassmannian manifolds. Let $Gr(k, n)$ be the Grassmannian manifold. We show that every element in $Gr(k, n)$ can be uniquely expressed by an equivalence class of $k \times n$ matrices, which plays the same role in the Grassmanian manifolds as the homogeneous coordinates in the projective spaces.

Definition 2.2. Let $W, W' \in M_{k\times n}(\Bbbk)$ by two $k \times n$ matrices of rank k. We say that W and W' are equivalent if there exists a $k \times k$ invertible matrix A such that $W' = WA$. For W a $k \times n$ matrix of rank k, denote $[W]$ the equivalence class of W.

Lemma 2.3. $Gr(k, n) = \{[W]: W$ is a $k \times n$ matrix of rank $k\}$.

Proof Every element in $Gr(k, n)$ is a k-dimensional subspace in $\Bbbk^n$, which can be determined by k linearly independent vectors, denoted by $(w_1, w_2, \ldots, w_k)$. Every vector in $\Bbbk^n$ can be written as a column vector of n tuples, so by doing so for each of the w_i, we can get a $k \times n$ matrix for each element in the Grassmannian manifold

$$\begin{pmatrix} w_{11} & \dots & w_{k1} \\ \dots & \dots & \dots \\ w_{1n} & \dots & w_{kn} \end{pmatrix}.$$

This matrix is of rank k since the k vectors $w_1, \dots, w_k$ are linearly independent. Now let us determine when two matrices give the same point in the Grassmannian manifold. Let $W := (w_1, w_2, \dots, w_k)$ and $W' := (w_1', w_2', \dots, w_k')$. Assume that W and W' determine the same subspace. Then every vector from W' can be expressed by a linear combination of the vectors from W, say

$$\begin{cases} w_1' = & a_{11}w_1 + a_{21}w_2 + \dots + a_{k1}w_k \\ \dots & \dots \quad \dots \\ w_k' = & a_{1k}w_1 + a_{2k}w_2 + \dots + a_{kk}w_k \end{cases}.$$

Let A be the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1k} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} \end{pmatrix},$$

So that we can get $W' = WA$. Therefore, every element in the Grassmannian manifold can be uniquely determined by the equivalence class of $k \times n$ matrices of rank k.

Definition 2.4. Let $I = \{i_1, \dots, i_k\} \subset \{1,2,\dots,n\}$ be a subset of k elements. Let U_I be defined as

$$\left\{ \begin{bmatrix} w_{11} & \dots & w_{k1} \\ \dots & \dots & \dots \\ w_{1n} & \dots & w_{kn} \end{bmatrix} \in Gr(k,n) : \det \begin{pmatrix} x_{1i_1} & \dots & w_{ki_1} \\ \dots & \dots & \dots \\ x_{1i_k} & \dots & x_{ki_k} \end{pmatrix} \neq 0 \right\}.$$

Lemma 2.5. $Gr(k,n) = \bigcup_{I \subset \{1,2,\dots,n\}, |I|=k} U_I$.

Proof Since the k vectors representing a point in $Gr(k,n)$ are linearly independent, some chosen $k \times k$ submatrix of the $k \times n$ matrix formed by these vectors should be of full rank, which meets the requirement of keeping the determinant nonzero, and thus is in U_I for some I. Therefore, the whole set of $Gr(k,n)$ can be covered by the union set of all U_I. ▫

Finally, let us show that each U_I can be identified as an affine space of dimension $k(n-k)$ via a natural map $\phi_I : U_I \to \mathbb{A}^{k(n-k)}$. Admitting this, we have

Corollary 2.6. The *dimension of the Grassmannian manifold* $Gr(k,n)$ *is* $k(n-k)$.

The construction of $\phi_I : U_I \to \mathbb{A}^{k(n-k)}$ as follows. Let $I = \{i_1, \dots, i_k\}$. Let $x \in U_I$ be a point. Then x is represented by a $k \times n$ matrix W whose submatrix formed by the $i_1, \dots, i_k$-th rows is invertible. But we can realize W by a matrix whose $i_1, \dots, i_k$-th rows form the identity matrix by doing WW_I^{-1}, where W_I is the submatrix formed by $i_1, \dots, i_k$-th rows of W. The matrix WW_I^{-1} represents x as well since W and WW_I^{-1} are equivalent in the sense of Definition 2.2. Apart from the $i_1, \dots, i_k$-th rows which form the identity matrix, the other entries of WW_I^{-1} are indetermined. There are $k(n-k)$ such indetermined numbers. The map $\phi_I : U_I \to \mathbb{A}^{k(n-k)}$ is defined by sending W to the $k(n-k)$ indetermined numbers in WW_I^{-1}.

Lemma 2.7. For *each I, the map* $\phi_I : U_I \to \mathbb{A}^{k(n-k)}$ *is well-defined and bijective.*

Proof To prove its well-definedness, let W and W' represent the same points in U_I. Then there is a $k \times k$ invertible matrix A such that $W' = WA$. In particular, we have $W_I' = W_I A$. Hence, $W'W_I'^{-1} = WA(W_I A)^{-1} = WAA^{-1}W_I^{-1} = WW_I^{-1}$, proving that the images of W and W' through ϕ_I operation are always equal.

To prove the injectivity, let W and W' satisfy $\phi_I(W) = \phi_I(W')$. Since the I-th rows of WW_I^{-1} and $W'W_I'^{-1}$ are both the identity matrix, $\phi_I(W) = \phi_I(W')$ implies that $WW_I^{-1} = W'W_I'^{-1}$. Hence, $W' =$

$W(W_I^{-1}W_I') = WA$, where $A = W_I^{-1}W_I'$ is a $k \times k$ invertible matrix, which means that $[W'] = [W]$. This proves its injectivity.

To prove the surjectivity, we construct a $k \times n$ matrix to represent an element in $\mathbb{A}^{k(n-k)}$. In this matrix, the $i_1, \dots, i_k$-th rows form the identity matrix, while the rest $n-k$ rows are filled by the given element in $\mathbb{A}^{k(n-k)}$. Since any element in $\mathbb{A}^{k(n-k)}$ can be represented by this sort of matrix, the surjectivity of this map proves to be true. ▫

2.2. Tangent spaces of the Grassmannian manifolds

In this section, we exhibit an explicit expression of the tangent space of the Grassmannian manifold at a given point. This description gives us another way to calculate the dimension of $Gr(k,n)$ by the following

Theorem 2.8. Let *X be an irreducible algebraic variety and let $x \in X$ be a regular point. Then* $\dim X = \dim T_{X,x}$.

We refer the reader to §6 in for a proof of this result.

Now let us give an explicit description of the tangent space of $Gr(k,n)$ at the point $x \in Gr(k,n)$. Let $V = k^n$ be the n-dimensional vector space. The point x corresponds to a k-dimensional subspace $W \subset V$. To describe a tangent vector v of $Gr(k,n)$ at x, let γ be a path in $Gr(k,n)$ passing through the point x whose tangent vector is exactly $v \in T_{Gr(k,n),x}$. This path gives us a family of k-dimensional subspaces $\{W_t\}$ such that $W_0 = W$. We claim that we can get a canonical linear map $g_v: W \to V/W$ by this family of k-dimensional subspaces. To do this, let $w \in W = W_0$. Let $\{w_t\}$ be a smooth family of vectors such that $w_t \in W_t$ for any t. This family of vectors gives us a path in $\mathbb{k}^n$ passing through the $w \in \mathbb{k}^n$, thus induces a tangent vector $u \in \mathbb{k}^n$ at the point w. This vector $u \in V$ depends not only on $w \in W$ but also on the choice of families $\{w_t\}$. Now let $u' \in \mathbb{k}^n$ be another vector constructed as above but with another family of vectors $\{w_t'\}$ with $w_t' \in W_t$. Then $u' - u = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(w_t' - w_t) \in W$, since $w_t' - w_t \in W_t$ for each t and $\lim_{t\to 0} w_t = \lim_{t\to 0} w_t'$. Hence, modulo W, the vector $u \in V$ is uniquely determined by w. Thus, we have conducted a linear map $g_v: W \to V/W$ by sending $w \in W$ onto \$\Bar{u}\in V/W\$. This linear map g_v depends on the tangent vector $v \in T_{Gr(k,n),x}$ and thus we acquire a map

$$g: T_{Gr(k,n),x} \to \mathrm{Hom}(W, V/W)$$

$$v \mapsto g_v.$$

By a local calculation, we can check that this g is an isomorphism between vector spaces. Since the dimension of $\mathrm{Hom}(W, V/W)$ is clearly $k(n-k)$, we get

Corollary 2.9. *The dimension of the Grassmannian manifold $Gr(k,n)$ is $k(n-k)$.*

3. Determinantal varieties

We present in this part basic knowledge of determinantal varieties with reference to Chapter II in [6].

Definition 3.1. Let $M_{m\times n}(\mathbb{k})$ be the set of $m \times n$ matrices with entries in $\mathbb{k}$. Then the *generic determinantal variety* $M_r(m,n)$ is defined as $\{m \times n \text{ matrices of rank} \le r\} \subset M_{m\times n}(\mathbb{k})$, in which $r \le \min\{m,n\}$.

Lemma 3.2. *$M_r(m,n)$ is a cone.*

Proof Let $\lambda \in \mathbb{k}$ and $A = (w_1, w_2, \dots, w_n) \in M_r(m,n)$ with $\mathrm{rank}(A) = p \le r$, in which $w_1, w_2, \dots, w_p$ are linearly independent. We can get that if $\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_p w_p = 0$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_p = 0$. Hence, when $\lambda \ne 0$, if $\lambda_1 \lambda w_1 + \lambda_2 \lambda w_2 + \cdots + \lambda_p \lambda w_p = 0$, $\lambda_1 = \lambda_2 = \cdots = \lambda_p = 0$. Therefore, $\lambda w_1, \lambda w_2, \dots, \lambda w_p$ are linearly independent when $\lambda \ne 0$. Since $\lambda A = (\lambda w_1, \lambda w_2, \dots, \lambda w_n)$, $\mathrm{rank}(\lambda A) = p = \mathrm{rank}(A) \le r$. When $\lambda = 0$, all the entries of λA are zero, so $\mathrm{rank}(A) = 0 \ge r$. Therefore, no matter whether λ is zero or not, the rank of λA is always less than or equal to r, which means that $\lambda A \in M_r(m,n)$. Therefore, $M_r(m,n)$ is a cone. ▫

The main goal of the rest of this section is to calculate the dimension of the generic determinantal variety $M_r(m,n)$. We give two methods to calculate this dimension. Both methods need us to introduce, following[4, Chapter II], the variety $\widetilde{M}_r(m,n)$.

Definition 3.2. $\widetilde{M}_r(m,n)$ is defined as $\{(A,W) \in M_{m\times n}(\mathbb{k}) \times Gr(r,n): \mathrm{Im}(A) \subset W\}$.

Our first method consists in studying the two projections from $\widetilde{M}_r(m,n)$. The second method consists in understanding the tangent space of the generic determinantal variety $M_r(m,n)$ at a smooth point.

3.1. Study of the two projections from $\widetilde{M}_r(m,n)$

Proposition 3.4. *Let $pr_1: \widetilde{M}_{m\times n}(\mathbb{k}) \to M_{m\times n}(\mathbb{k})$ be defined as $(A,W) \mapsto A$. Then,*

(i) $\mathrm{Im}(pr_1) = M_r(m,n)$*;*

(ii) For A of rank r, there is only one element in its preimage;

(iii) $\{A \in M_r(m,n): rank(A) = r\} \subset M_r(m,n)$ *is a nonempty open subset.*

Proof (i) If $A \in \mathrm{Im}(pr_1)$, then there exists $W \in Gr(r,n)$ such that $\mathrm{Im}(A) \subset W$. As $\mathrm{rank}(A) = \mathrm{dim}\mathrm{Im}(A) \le \mathrm{dim}W = r$, we can get that $A \in M_r(m,n)$. Conversely, if $A \in M_r(m,n)$, then $Im(A)$ is of dimension $\le r$. Hence, there exists W of dimension r such that $\mathrm{Im}(A) \subset W$. Therefore, $(A,W) \in \widetilde{M}_r(m,n)$, and $A = pr_1(A,W) \in W$.

(ii) $(A,W) \in M_{m\times n}(\mathbb{k}) \times Gr(r,n)$ is the preimage of A in $\widetilde{M}_r(m,n)$ if and only if $\mathrm{Im}(A) \subset W$. Since $\mathrm{rank}(A) = r = \mathrm{dim}W$, we have $W = \mathrm{Im}(A)$, so the only preimage of A is $(A, \mathrm{Im}(A))$.

(iii) $\{A \in M_r(m,n): \mathrm{rank}(A) = r\}$ is nonempty. Since $\mathrm{rank}(A) \le r-1$, any $r\times r$ submatrix of A has determinant 0. Hence,

$$\{A \in M_r(m,n): \mathrm{rank}(A) \le r-1\} = \bigcap_{\substack{\{i_1,\dots,i_r\}\subset\{1,\dots,m\},\\ \{j_1,\dots,j_r\}\subset\{1,\dots,n\}}} \left\{A \in M_r(m,n): \mathrm{det}A_{\substack{i_1,\dots,i_r,\\ j_1,\dots,j_r}} = 0\right\}.$$

Because $\mathrm{det}: M_{r\times r}(\mathbb{k}) \to \mathbb{k}$ is a polynomial function, $\left\{A \in M_r(m,n): \mathrm{det}A_{\substack{i_1,\dots,i_r,\\ j_1,\dots,j_r}} = 0\right\}$ is a closed subset. Therefore, as a finite intersection of closed subsets, $\{A \in M_r(m,n): \mathrm{rank}(A) \le r-1\}$ is still a closed subset.

Remark. This proposition shows that $\mathrm{dim}M_r(m,n) = \mathrm{dim}\widetilde{M}_r(m,n)$.

Proposition 3.5. *Let $pr_2: \widetilde{M}_r(m,n) \to Gr(r,n)$ be defined as $(A,W) \mapsto W$. For any $W_0 \in Gr(r,n)$,* $\mathrm{dim}pr_2^{-1}(W_0) = mr$.

Proof By the definition of pr_2 and W_0, we can get $pr_2^{-1}(W_0) = \{(A,W) \in \widetilde{M}_r(m,n): pr_2(A,W) = W_0\}$, which is equal to $\{(A,W_0) \in \widetilde{M}_r(m,n)\}$. By integrating the definition of $\widetilde{M}_r(m,n)$, the expression is further transformed into $\{A \in M_{m\times n}(\mathbb{k}): \mathrm{Im}(A) \subset W_0\}$, which refers to a set of linear maps denoted by $\{A: \mathbb{k}^m \to W_0\}$–that is, $\mathrm{Hom}(k^m, W_0)$. Hence, $\mathrm{dim}pr_2^{-1}(W_0) = \mathrm{dim}\mathrm{Hom}(k^m, W_0) = mr$.

Corollary 3.6. $\mathrm{dim}\widetilde{M}_r(m,n) = mr + nr - r^2$.

Consider the map $pr_2: \widetilde{M}_r(m,n) \to Gr(r,n)$ defined as in Proposition 3.5. The fibers of the map pr_2 is of dimension mr according to Proposition 3.5. We also know that $\mathrm{dim}Gr(r,n) = r(n-r)$. Hence, by Corollary 2.6 and Theorem 3.7 below, $\mathrm{dim}\widetilde{M}_r(m,n) = \mathrm{dim}pr_2^{-1} + \mathrm{dim}Gr(r,n) = mr + r(n-r) = mr + nr - r^2$. ▫

Remark. This corollary, together with the remark under the Proposition 3.4, implies that the dimension of the generic determinantal variety $M_r(m,n)$ is $mr + nr - r^2$.

Theorem 3.7. *Let $f: X \to Y$ be a morphism between algebraic varieties. Assume*

(i) $\mathrm{dim}Y = n$*, and*

(ii) $\mathrm{dim}f^{-1}(y) = m$ for any $y \in Y$,

then $\mathrm{dim}X = m + n$.

We refer to [5] for a proof of this theorem.

Example. Let $f: \mathbb{A}^3(\Bbbk) \to \mathbb{A}^1(\Bbbk)$ be a morphism defined as $(x_1, x_2, x_3) \mapsto x_1$. For any $t \in \mathbb{A}^1(\Bbbk)$, $f^{-1}(t) = \{(t, x_2, x_3) \in \mathbb{A}^3(\Bbbk): x_2, x_3 \in \Bbbk\} \simeq \mathbb{A}^2(\Bbbk)$. Therefore, $\dim\mathbb{A}^1(\Bbbk) = 1$, $\dim f^{-1}(t) = 2$. Hence, $\dim\mathbb{A}^3(\Bbbk) = 3 = \dim\mathbb{A}^1(\Bbbk) + \dim f^{-1}(t)$.

3.2. Tangent space of $\boldsymbol{M_r(m,n)}$ *at a smooth point*

Proposition 3.8. Let $(A, W) \in \widetilde{M}_r(m,n)$. *Then the tangent space of* $\widetilde{M}_r(m,n)$ *at* (A, W) *is*

$$T_{\widetilde{M}_r(m,n),(A,W)} = \{(B, \phi) \in \mathrm{Hom}(\mathbb{C}^m, V) \times \mathrm{Hom}(W, V/W): B \equiv \phi \circ A(\bmod W)\}.$$

Proof Let $\{(A_t, W_t)\}$ be a smooth family of elements in $\widetilde{M}_r(m,n)$. By the definition, $\mathrm{Im}(A_t) \subset W_t$. Let $(A_0, W_0) = (A, W)$. Let $(B, \phi) \in \mathrm{Hom}(\mathbb{C}^m, V) \times \mathrm{Hom}(W, V/W)$ be the tangent vector of the family $\{(A_t, W_t)\}$ at $t = 0$. Then for any $u \in \mathbb{C}^m$,

$$\frac{d}{d_t}|_{t=0} A_t u = Bu. \tag{3}$$

For any $w \in W$, let $\{w_t\}$ be a family of vectors such that $w_t \in W_t$, $w_0 = w$. Then $\frac{d}{d_t}|_{t=0} w_t \equiv \phi(w)(\bmod W)$, combining with our description of the tangent space of $Gr(r,n)$ at W. In particular, by (3), $\phi(Au) \equiv Bu(\bmod W)$. But $u \in \mathbb{C}^m$ in arbitrary, we get $B \equiv \phi \circ A(\bmod W)$, as desired. ▫

Proposition 3.9. Let $A \in M_r(m,n)$ *be of rank* r. *Then* $T_{M_r(m,n),A} = \{B \in M_{m\times n}(\Bbbk)$: there exists $\phi \in \mathrm{Hom}(\mathrm{Im}A, V/\mathrm{Im}A)$ such that $B \equiv \phi \circ A(\bmod \mathrm{Im}A)\}$

Proof Let $pr_1: \widetilde{M}_r(m,n) \to M_r(m,n)$. The preimage of A under pr_1 has only one element $(A, \mathrm{Im}A)$. Therefore, $T_{Mr(m,r),A} = pr_{A*}T_{\widetilde{M}_r(m,n),(A,\mathrm{Im}A)}$. By the previous proposition, $T_{M_r(m,n),A}$ refers to a set of $B \in M_{m\times n}(\Bbbk)$ where there exists $\phi \in \mathrm{Hom}(\mathrm{Im}A, V/\mathrm{Im}A)$ such that $B \equiv \phi \circ A(\bmod \mathrm{Im}A)$.

Corollary 3.10. $\dim M_r(m,n) = r(n-r) + mr$.

Proof Let $B \in T_{M_R(m,n),A}$. Then there exists $\phi \in \mathrm{Hom}(\mathrm{Im}A, V/\mathrm{Im}A)$ such that $B \equiv \phi \circ A(\bmod \mathrm{Im}A)$. Therefore, there exists a linear map $\psi \in \mathrm{Hom}(\mathbb{C}^m, \mathrm{Im}A)$ such that $B = \phi \circ A + \psi$. Then B uniquely determines ϕ and ψ, and vice versa. Hence, $\dim T_{M_r(m,n),A} = \dim\mathrm{Hom}(\mathrm{Im}A, V/\mathrm{Im}A) + \dim\mathrm{Hom}(\mathbb{C}^m, \mathrm{Im}A) = r(n-r) + mr$, as desired.

4. Proof of the theorem of Hopf and Smith

In this section, we give an algebro-geometric proof of the theorem of Hopf and Smith with the tools developed so far.

Theorem 4.1 (Hopf-Smith [2, 3]). *Let* $\Bbbk$ *be an algebraically closed field. Let* U, V, W *be nonzero finitely dimensional* $\Bbbk$-vector *spaces. Let*

$$\phi: U \times V \to W$$

be a nonsingular bilinear *map. Then* $\dim W \geq \dim U + \dim V - 1$.

Lemma 4.2. *Let* V *be* $\Bbbk$*-vector space. Let* $C \subset V$ *be a cone of* $\dim n$. *Let* $\mathbb{P}(C)$ *be the set of* 1-*dimensional subspaces of* V *lying in* C. *Then,* $\dim\mathbb{P}(C) = n - 1$.

Proof Let

$$\begin{array}{rccc} f: & C - \{0\} & \to & \mathbb{P}(C) \\ & u & \mapsto & [u]. \end{array}$$

Let $l \in \mathbb{P}(C)$. Then $f^{-1}(l) = \{v \in C - \{0\}: [v] = l\} = l - 0$. Since $\dim(C - \{0\}) = \dim C = n$, $\dim f^{-1}(l) = \dim(l - \{0\}) = \dim l = 1$, by Theorem 3.7, we get $\dim\mathbb{P}(C) = n - 1$. ▫

Let $\phi: U \times V \to W$ be a bilinear map. One defines a linear map

$$\begin{array}{rccc} \psi: & U & \to & \mathrm{Hom}(V, W) \\ & u & \mapsto & \big(v \mapsto \phi(u, v)\big). \end{array}$$

Proposition 4.3. *Let $\phi: U \times V \to W$ be a nonsingular bilinear map as in Theorem* 4.1. *Let ψ be the associated linear map defined as above. Let $R := \mathrm{Im}\psi \subset \mathrm{Hom}(V, W)$. For $r \leq 0$ a nonnegative integer, let* $\mathrm{Hom}_r(V, W)$ *be the set of linear maps from V to W whose rank is less than or equal to r. Then*

(i) $\dim R = \dim U$, *and*

(ii) $R \cap \mathrm{Hom}_{\dim V-1}(V, W) = \{0\}$.

Proof To prove (i), first we claim that ψ is injective. In fact, when $\psi(u) = 0$, for any nonzero $v \in V$, $\phi(u, v) = \psi(u)(v) = 0$, which implies $u = 0$ by the property of non-singularity. Hence, the dimension of $R = \mathrm{Im}\psi$ is equal to the dimension of U. Now let us prove (ii). Let $\psi(u) \in R \cap \mathrm{Hom}_{\dim V-1}(V, W)$. We must show that $\psi(u) = 0$. Assume by contradiction that $\psi(u) \neq 0$. We claim that $\psi(u): V \to W$ is injective. Indeed, if $\phi(u, v) = 0$, v must be zero in case that $u \neq 0$ due to its non-singularity. Hence, $\mathrm{rank}\, \psi(u) = \dim V$, which contradicts to the fact that $\psi(u) \in \mathrm{Hom}_{\dim V-1}(V, W)$.

Proof of Theorem 4.1. Let us first calculate the dimensions of $\mathbb{P}\mathrm{Hom}_{\dim V-1}(V, W)$ and $\mathbb{P}\mathrm{Hom}(V, W)$. Let $m = \dim V$, $n = \dim W$. After a choice of bases of V and W, respectively, we can get $\mathrm{Hom}_{m-1}(V, W) \cong M_{m-1}(m, n)$. Since $\mathrm{Hom}_{m-1}(V, W) \subset \mathrm{Hom}(V, W)$ is a cone, by Lemma 3.2, we have

$$\begin{aligned}
\dim \mathbb{P}\mathrm{Hom}_{m-1}(V, W) &= \dim \mathrm{Hom}_{m-1}(V, W) - 1 \quad \text{by Lemma 4.2} \\
&= \dim M_{m-1}(m, n) - 1 \\
&= m(m-1) + n(m-1) - (m-1)^2 - 1 \quad \text{by Corollary 3.6} \\
&= mn + m - n - 2 \\
&= \dim V \dim W + \dim V - \dim W - 2
\end{aligned}$$

According to Lemma 4.2, the dimension of $\mathbb{P}\mathrm{Hom}(V, W)$ can be calculated as $\dim \mathrm{Hom}(V, W) - 1$, which is $\dim V \dim W - 1$. After deriving those required dimensions, let us use the notation mentioned in Proposition 4.3, which is to let $R := \mathrm{Im}\psi \subset \mathrm{Hom}(V, W)$. Regard $\mathbb{P}R$, $\mathbb{R}\mathrm{Hom}_{\dim V-1}(V, W)$ as subvarieties of $\mathbb{P}\mathrm{Hom}(V, W)$, which satisfies an inequality such that $\dim \mathbb{P}\mathrm{Hom}(V, W) \geq \dim \mathbb{P}\mathrm{R} + \dim \mathbb{P}\mathrm{Hom}_{\dim V-1}(V, W)$. Through a process of substitution and simplification, we can get the desired conclusion such that $\dim W \geq \dim U + \dim V - 1$.

5. Application of Hopf's theorem

In this section, we are going to use the Hopf's theorem (Theorem 1.8) to prove Theorem 1.5 (ii). Most of the results in this section can also be found in Chapter 12 of [1] but Lemma 5.8 and the proof of Theorem 1.5 (ii) thereafter are to our knowledge new. Let us first recall the statement of the Hopf's theorem.

Theorem 5.1 (Hopf [2]). *If there is a nonsingular bilinear map $\phi: \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$. Then whenever $n - s < k < r$, $\binom{n}{k}$ is even.*

For a proof of this theorem, the readers are invited to see Theorem 12.2 in [1]. Motivated by this theorem, let us define some combinatorial invariants following [1].

Definition 5.2. Let r, s, n be positive integers. We say that $\mathcal{H}(r, s, n)$ holds if $\binom{n}{k}$ is even for any $n - s < k < r$.

Definition 5.3. Following the definition above, $r \circ s$ is defined as the minimum value of n for $\mathcal{H}(r, s, n)$ to hold.

Remark . Hopf's theorem says that $r \circ s \leq r \#_{\mathbb{R}} s$ for any positive integers r, s.

Lemma 5.4. *Let $m \leq n$. If $\mathcal{H}(r, s, m)$ holds, then $\mathcal{H}(r, s, n)$ holds.*

Proof It suffices to show that if $\mathcal{H}(r, s, m)$ holds, then $\mathcal{H}(r, s, n)$ holds. Since $\mathcal{H}(r, s, m)$ holds, for any $m - r < k < s$, we have $\binom{m}{k}$ even. Now let k satisfy $m + 1 - r < k < s$. Then both k and $k - 1$ is greater than $m - r$ and less than s. Hence, by assumption, $\binom{m}{k}$ and $\binom{m}{k-1}$ are even. Hence, $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$ is even. Hence, $\mathcal{H}(r, s, m+1)$ holds. ▫

Proposition 5.5. $\max\{r, s\} \leq r \circ s \leq r + s - 1$.

Proof Let $n = r \circ s$. By definition of $r \circ s$, $\mathcal{H}(r,s,n)$ holds and $\mathcal{H}(r,s,n-1)$ does not hold. Let us first confirm the lower bound of $r \circ s$. Assume by contradiction that $n < \max\{r,s\}$. In this case, the value of k can be taken to be zero, which incapacitates the construction of $\mathcal{H}(r,s,n)$ since $\binom{n}{0} = 1$. Hence, it is clear that $n \geq \max\{r,s\}$. For the upper bound, we again assume by contradiction that $r \circ s \leq r + s - 2$. Since $\mathcal{H}(r,s,m+1)$ holds, by Lemma 5.4, $\mathcal{H}(r,s,r+s-2)$ holds. Therefore, after a combination of lower and upper bounds, we can get that $\max\{r,s\} \leq r \circ s \leq r + s - 1$.

Proposition 5.6. *If a is even and b is odd, then $\binom{a}{b}$ is even.*

Proof Assume $a = 2m$ and $b = 2n+1$. If $2n+1 > 2m$, then $\binom{2m}{2n+1} = 0$, which is even. If $2n+1 < 2m$ ($n \leq m-1$), we can get the exponent of 2 in $\binom{2m}{2n+1}$ by combining the exponents of 2 in $(2m)!$, $(2n+1)!$, and $(2m-2n-1)!$ together. $(2m)!$ can be divided by $2^{m+\left[\frac{m}{2}\right]+\left[\frac{m}{4}\right]+\cdots+1}$; $(2n+1)!$ can be divided by $2^{n+\left[\frac{n}{2}\right]+\left[\frac{n}{4}\right]+\cdots+1}$; $(2m-2n-1)!$ can be divided by $2^{(m-n-1)+\left[\frac{m-n-1}{2}\right]+\cdots+1}$. Hence, the exponent of 2 in $\binom{2m}{2n+1} = \frac{(2m)!}{(2n+1)!(2m-2n-1)!}$ is equal to $\left(m + \left[\frac{m}{2}\right] + \left[\frac{m}{4}\right] + \cdots + 1\right) - \left(n + [\frac{n}{2} + \left[\frac{n}{4}\right] + \cdots + 1\right) - \left((m-n-1) + \left[\frac{m-n-1}{2}\right] + \cdots + 1\right) > 0$. This proves that $\binom{2m}{2n+1}$ can be divided by 2, and thus must be even.

Proposition 5.7. *$r \circ s$ is* odd *if and only if*

(i) r and s are both odd;

(ii) $r \circ s = r + s - 1$.

Proof If r and s are both odd, then $r+s-1$ must also be odd, which means, obviously, that $r \circ s$ is odd. To prove the uniqueness of this condition, let $r \circ s = 2m-1$, so that (1) $\mathcal{H}(r,s,2m)$ holds (which means that $\binom{2m+1}{k}$ is even for any $2m+1-s < k < r$), while (2) $\mathcal{H}(r,s,2m)$ does not hold (which means that there exists $2m-s < k < r$ that makes $\binom{2m}{k}$ odd). Let k_0 be the minimal of k such that $2m-s < k < r$ and $\binom{2m}{k}$ is odd. If $k_0 < r-1$, then $k_0+1 < r$–that is, $2m-s+1 < k_0+1 < r$. By (i), $\binom{2m+1}{k_0+1}$ is even. But $\binom{2m+1}{k_0+1} = \binom{2m}{k_0+1} + \binom{2m}{k_0}$, so $\binom{2m}{k_0+1}$ is odd. Hence, for any $k_0 < k < r$, $\binom{2m}{k_0+1}$ is odd. In particular, $\binom{2m}{r-1}$ is odd. Then, $\binom{2m+1}{r-1} = \binom{2m}{r-1} + \binom{2m}{r-2}$. If $2m+1 \neq r+s-1$, then $2m+1 < r+s-1$, which means that $2m+1-s < r-1 < r$. Since $\binom{2m+1}{r-1}$ is even by (i), $\binom{2m}{r-2}$ is odd. Through a series of induction, we can get that $\binom{2m}{k}$ is odd for every $2m-s < k < r$. One cannot have two consecutive combinatorial numbers that are odd since $2m$ is even, which is contradictory to what we have just calculated. Hence, $2m+1 = r+s-1$, which proves (ii). To show (i), assume by contradiction that r, s are even and $r \circ s \leq r+s-2$. It suffices to show that $\mathcal{H}(r,s,r+s-2)$ holds–that is, for any $r-2 < k < r$, we have $\binom{r+s-2}{k}$, or $\binom{r+s-2}{r-1}$, even. However, by proposition 5.6, if a is even and b is odd, then $\binom{a}{b}$ is even. Hence, $\binom{r+s-2}{r-1}$ is even, as desired.

Lemma 5.8. *The* following *statements are equivalent:*

(i) $\binom{r+s-2}{r-1}$ is odd.

(ii) $r \circ s = r + s - 1$.

(iii) In writing $r - 1 = a_n 2^n + a_{n-1} 2^{n-1} + \cdots + a_0$, $s - 1 = b_m 2^m + b_{m-1} 2^{m-1} + \cdots + b_0$, where $a_i, b_j \in \{0,1\}$, there does not exist i such that $a_i = b_i = 1$.

Proof Let us first prove that (i) is equivalent to (ii). Assume that $\binom{r+s-2}{r-1}$ is odd, let us prove $r \circ s = r + s - 1$. Suppose by contradiction that $r \circ s \leq r+s-2$, so that $\binom{r+s-2}{k}$ is even for any $n-s < k < r$. However, $\binom{r+s-2}{r-1}$ is odd, which is contradictory with the statement above, so $r \circ s \geq r+s-2$. According to Proposition 5.5, $r \circ s \leq r+s-1$, leading to the conclusion that $r \circ s = r+s-1$. Now let us prove the opposite direction. Since $r \circ s = r+s-1$, there exists $r-2 < k < r$ such that $\binom{r+s-2}{k}$ is odd. Obviously, k can only be $r-1$, which indicates that $\binom{r+s-2}{r-1}$ is odd. The next step is to prove the equivalence between (i) and (iii). To prove that (i) implies (iii), let us determine the requirements for $\binom{r+s-2}{r-1} = \frac{(r+s-2)!}{(r-1)!(s-1)!}$ to be odd. The power of 2 in $(r+s-2)!$ is $\lfloor\frac{r+s-2}{2}\rfloor + \lfloor\frac{r+s-2}{2^2}\rfloor + \cdots$; the power of 2 in $(r-1)!$ is $\lfloor\frac{r-1}{2}\rfloor + \lfloor\frac{r-1}{2^2}\rfloor + \cdots$; the power of 2 in $(s-1)!$ is $\lfloor\frac{s-1}{2}\rfloor + \lfloor\frac{s-1}{2^2}\rfloor + \cdots$. According to (iii), $r-1$ can be written as $a_n 2^n + a_{n-1}2^{n-1} + \cdots + a_{02}^0$, so that $\lfloor\frac{r-1}{2^i}\rfloor = \lfloor a_n 2^{n-i} + a_{n-1}2^{n-i-1} + \cdots + a_{02}^{-i}\rfloor = a_n 2^{n-i} + a_{n-1}2^{n-i-1} + \cdots + a_i$. By the same token, $\lfloor\frac{s-1}{2^i}\rfloor = b_m 2^{m-i} + b_{m-1}2^{m-i-1} + \cdots + b_i$. $\binom{r+s-2}{r-1}$ is odd when 2 cannot divide $\frac{(r+s-2)!}{(r-1)!(s-1)!}$–that is, $\lfloor\frac{r-1}{2^i}\rfloor + \lfloor\frac{s-1}{2^i}\rfloor = \lfloor\frac{r+s-2}{2^i}\rfloor$, for the reason that $\lfloor\frac{r-1}{2^i}\rfloor + \lfloor\frac{s-1}{2^i}\rfloor \le \lfloor\frac{r+s-2}{2^i}\rfloor$ always establishes by the nature of rounding down. Then we can get $a_{i-1}2^{i-1} + \cdots + a_0 + b_{i-1}2^{i-1} + \cdots + b_0 < 2^i$. It is obvious that a_{i-1}, b_{i-1} cannot both be 1, which implies that a_i, b_i are not both 1. For the opposite direction, assume by contradiction that $\binom{r+s-2}{r-1}$ is even, so that 2 can divide $\frac{(r+s-2)!}{(r-1)!(s-1)!}$. Based on the expressions we have already derived above, there must exist such i that $\lfloor\frac{r-1}{2^i}\rfloor + \lfloor\frac{s-1}{2^i}\rfloor < \lfloor\frac{r+s-2}{2^i}\rfloor$. This implies that $a_{i-1}2^{i-1} + \cdots + a_0 + b_{i-1}2^{i-1} + \cdots + b_0 \ge 2^i$, which means that there exists $k \in \{0, \ldots, i-1\}$ such that $a_k = b_k = 1$.

*Proof of Theorem 1.5 (*ii*).* Since $\binom{r+s-2}{r-1}$ is odd, by Lemma 5.8, we can get that $r \circ s = r+s-1$. According to Theorem 5.1, $r\#_{\mathbb{R}}s \ge r \circ s = r+s-1$. According to Proposition 1.4, $r\#_{\mathbb{R}}s \le r+s-1$. Therefore, $r\#_{\mathbb{R}}s = r+s-1$.

6. Degrees of the determinantal varieties

In this section, we discuss the degrees of the projective determinantal varieties in the projective space of all matrices of a given size. The main result is Theorem 6.3 which is proven in [6, Proposition 12(a)]. We will use this result to give another proof of Theorem 1.5 (ii).

6.1. Degree of a projective variety

We start with some elementary observations about the degree of a projective variety.

Definition 6.1. Let $\Bbbk$ be an algebraically closed field. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension n. Then the degree of X in $\mathbb{P}^N$ is defined by the number of the intersection points of $H \cap X$ in which H is a general $(N-n)$-dimensional linear subspace.

Lemma 6.2. *Let $X \subset \mathbb{P}^N(\mathbb{R})$ be a real projective variety of dimension n. If the degree of X is odd, then for any linear subspace of dimension $\ge N-n$, $H \cap X \ne \phi$.*

Proof Let $H \subset \mathbb{P}^N(\mathbb{R})$ be a linear subspace of dimension $\ge N-n$. Let $H' \subset H$ be a subspace of dimension $N-n$. Since the degree of $X \subset \mathbb{P}^N$ is odd, by definition, $X(\mathbb{C}) \cap H_{\mathbb{C}}$ has an odd number of points, counted with multiplicity. Let $x \in X(\mathbb{C}) \cap H_{\mathbb{C}}$ be a complex point. Then the complex conjugate $\Bar{x}$ of x is also in $X(\mathbb{C}) \cap H_{\mathbb{C}}$, counted with multiplicity. Hence, the set of non-real points in $X(\mathbb{C}) \cap H_{\mathbb{C}}$ is even. Hence, there is a real point in $X(\mathbb{C}) \cap H_{\mathbb{C}}$. In other words, $X \cap H \ne \phi$. ▫

The degrees of the projective determinantal varieties are calculated in Proposition 12 (a) in [6].

Theorem 6.3 (Harris–Tu [6])). *The degree of* $\mathbb{P}(M_r(m,n)) \subset \mathbb{P}M_{m\times n}(\Bbbk)$ *is*

$$\prod_{\alpha=0}^{m-r-1} \frac{\binom{n+\alpha}{m-1-\alpha}}{\binom{n-r+\alpha}{m-r-1-\alpha}}.$$

6.2. Application to the calculation of $\boldsymbol{r\#_{\mathbb{R}}s}$

In this part, we give another proof of Theorem 1.5 (ii) using the result of the previous part.

Theorem 42. *If* $\binom{r+s-2}{r-1}$ *is odd, then* $r\#_{\mathbb{R}}s = r+s-1$.

Proof Since $r\#_{\mathbb{R}}s \le r+s-1$ by Proposition 1.4, it suffices to show that $r\#_{\mathbb{R}}s \ge r+s-1$. Let $\phi: \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ be a nonsingular bilinear map. We must show that $n \ge r+s-1$. The map ϕ

induces the linear map $\psi: \mathbb{R}^r \to \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$. The non-singularity of ϕ implies the following (c.f. Proposition 4.3):

(i) ψ is injective;

(ii) $\mathrm{Im}\psi \cap \mathrm{Hom}_{s-1}(\mathbb{R}^s, \mathbb{R}^n) = 0$,

where $\mathrm{Hom}_{s-1}(\mathbb{R}^s, \mathbb{R}^n)$ signifies the linear maps from $\mathbb{R}^s$ to $\mathbb{R}^n$ with rank $\leq s-1$. By Theorem 6.3, the degree of $\mathbb{P}\mathrm{Hom}_{s-1}(\mathbb{R}^s, \mathbb{R}^n)$ in $\mathbb{P}\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$ is $\binom{n}{s-1}$. Assume by contradiction that $n < r + s - 1$. We may assume $n = r + s - 2$, since the existence of a nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ with $n < r + s - 1$ implies the existence of a nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^{r+s-2}$. The degree of $\mathbb{P}\mathrm{Hom}_{s-1}(\mathbb{R}^s, \mathbb{R}^{r+s-2})$ in $\mathbb{P}\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^{r+s-2})$ is $\binom{r+s-2}{s-1}$, which is odd by assumption. By Corollary 3.6 and Lemma 4.2, the dimension of $\mathbb{P}\mathrm{Hom}_{s-1}(\mathbb{R}^s, \mathbb{R}^{r+s-2})$ is $s(s-1) + (r+s-2)(s-1) - (s-1)^2 - 1 = (s+r-1)(s-1) - 1$, whereas the dimension of $\mathbb{P}\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^{r+s-2})$ is $s(r+s-2) - 1$. Since $\psi: \mathbb{R}^r \to \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^{r+s-2})$ is injective, the dimension of $\mathbb{P}\mathrm{Im}\psi$ is $r-1$. Hence, $\dim\mathbb{P}\mathrm{Im}\psi + \dim\mathbb{P}\mathrm{Hom}_{s-1}(\mathbb{R}^s, \mathbb{R}^{r+s-2}) = r - 1 + (s+r-1)(s-1) + 1 = s(r+s-2) - 1 = \dim\mathbb{P}\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^{r+s-2})$. Therefore, by Lemma 6.2, $\mathbb{P}\mathrm{Im}\psi \cap \mathbb{P}\mathrm{Hom}_{s-1}(\mathbb{R}^s, \mathbb{R}^{r+s-2}) \neq \phi$, which contradicts (ii). This proves that $n \geq r + s - 1$. Therefore, it can be finally concluded that $r\#_{\mathbb{R}}s = r + s - 1$, as desired.

7. Field extensions

In this section, we recall some basic knowledge on field extension theory. The principal reference is [9]. We study the relation between degree of field extensions and the invariant $r\#_{\mathbb{k}}s$ defined in the Introduction. Finally, we concentrate on the rational number field and finite fields and complete the proof of Theorem 1.5 (iii).

7.1. Degrees of field extensions

Definition 7.1. L is called a field extension of $\mathbb{k}$ if L is a field containing $\mathbb{k}$ as a subfield.

Remark. If L is a field extension of $\mathbb{k}$, then L has a canonical $\mathbb{k}$-vector space structure.

Definition 7.2. The degree of the field extension $L/\mathbb{k}$ is defined as the dimension of L as a $\mathbb{k}$-vector space. It is denoted by $[L:\mathbb{k}]$.

Example. The field $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2}: a, b \in \mathbb{Q}\}$, viewed as a $\mathbb{Q}$-vector space, has a basis $\{1, \sqrt{2}\}$. The extension degree is thus $[\mathbb{Q}[\sqrt{2}]:\mathbb{Q}] = 2$.

Definition 7.3. Let L be a field extension of $\mathbb{k}$. An element $\alpha \in L$ is called an *algebraic element* over $\mathbb{k}$, if there exists a monic $\mathbb{k}$-coefficient polynomial $f(X) = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$ where $c_i \in \mathbb{k}$ such that $f(\alpha) = 0$.

Proposition 7.4. The *following statements are equivalent:*

(i) $x \in L$ *is algebraic over* $\mathbb{k}$.

(ii) $\mathbb{k}[x]$ *is a finitely dimensional* $\mathbb{k}$*-vector space.*

(iii) $\mathbb{k}[x]$ *is a field.*

Proof Let us first prove that (i) implies (ii)–that is, if $x \in L$ is algebraic over $\mathbb{k}$, then $\mathbb{k}[x]$ is a finitely dimensional $\mathbb{k}$-vector space. Since $x \in L$ is algebraic over $\mathbb{k}$, there exists an expression $x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0$, where $c_i \in \mathbb{k}$. Therefore, any expression $a_0 + a_1x + \cdots + a_gx^g$ can be expressed in terms of $1, x, \ldots, x^{n-1}$. Hence, $\mathbb{k}[x]$ is generated by $1, x, \ldots, x^{n-1}$ over $\mathbb{k}$, which proves that $\mathbb{k}[x]$ is a finitely dimensional $\mathbb{k}$-vector space. Then let us prove that (ii) implies (iii)–that is, if $\mathbb{k}[x]$ is a finitely dimensional $\mathbb{k}$-vector space, then $\mathbb{k}[x]$ is a field. It suffices to show that for $x \in \mathbb{k}[x]$ nonzero, x has an inverse in $\mathbb{k}[x]$. Since $\mathbb{k}[x]$ is finitely dimensional, $1, x, x^2, \ldots, x^n, \ldots$ become linearly dependent for $n \geq \dim\mathbb{k}[x]$. Hence, there exists n such that $x^n = c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ where $c_i \in \mathbb{k}$. We may assume $c_0 \neq 0$. Then $\frac{1}{c_0}x^{n-1} - \frac{c_{n-1}}{c_0}x^{n-2} - \cdots - \frac{c_1}{c_0}$ is the inverse of x, as desired. The final step is to prove that (iii) implies (i). If $x = 0$, then x is obviously an algebraic element. If $x \neq 0$, since $\mathbb{k}[x]$ is a field, x has an inverse in $\mathbb{k}[x]$. Suppose $a_0 + a_1x + \cdots + a_gx^g$ is the inverse of x–that is, $x\left(a_0 + a_1x + \cdots + a_gx^g\right) = 1$. May assume $a_g \neq 0$, then $x^{g+1} + \frac{a_{g-1}}{a_g}x^g + \cdots + \frac{a_0}{a_g}x - \frac{1}{a_g} = 0$. Hence, x is an algebraic element over $\mathbb{k}$.

Definition 7.5. Let $x \in L$ be an algebraic element over $\mathbb{k}$. A minimal polynomial $\mu_{x,\mathbb{k}}(X)$ of x over $\mathbb{k}$ is defined as a polynomial with $\mathbb{k}$-coefficients satisfying:

(i) $\mu_{x,\mathbb{k}}(x) = 0$.

(ii) If $f(X) \in \mathbb{k}[X]$ is nonzero such that $f(x) = 0$, then $\deg\mu_{x,\mathbb{k}} \leq \deg f$.

(iii) $\mu_{x,\mathbb{k}}(X)$ is monic.

Lemma 7.6. *Let $x \in L$ be an algebraic element over $\mathbb{k}$. Let $f(X) \in \mathbb{k}[X]$ be monic polynomial such that $f(x) = 0$. Then the following statements are equivalent:*

(i) f is the minimal polynomial of x over $\mathbb{k}$.

(ii) f is an irreducible polynomial in $\mathbb{k}[X]$.

Proof Let us first prove that (i) implies (ii). Assume that f is reducible. There exists monic polynomials $g, h \in \mathbb{k}[X]$ with strictly smaller degrees than that of f such that $f = gh$. Since $0 = f(x) = g(x)h(x)$, $g(x) = 0$ or $h(x) = 0$, which is contradictory to the minimality of the degree of f. Then let us prove that (ii) implies (i). Let $\mu_{\alpha,\mathbb{k}}$ be the minimal polynomial. By Euclidean division, $f = g\mu_{\alpha,\mathbb{k}} + r$ with $\deg r < \deg\mu_{\alpha,\mathbb{k}}$. Then, $0 = f(x) = g(x)\mu_{\alpha,\mathbb{k}}(x) + r(x) = r(x)$, so that $r \equiv 0$, which means that $f = g\mu_{\alpha,\mathbb{k}}$. But f is irreducible, implying that $f = \mu_{\alpha,\mathbb{k}}$, as desired.

Proposition 7.7. *Let $x \in L$ be an algebraic element over $\mathbb{k}$. Let $\mu_{x,\mathbb{k}}(X) \in \mathbb{k}[X]$ be the minimal polynomial of x over $\mathbb{k}$.* Then $\deg\mu_{x,\mathbb{k}} = [\mathbb{k}[x]{:}\,\mathbb{k}]$

Proof Let $\mu_{x,\mathbb{k}}(X) = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$. Since every x^k can be expressed in terms of $1, x, \ldots, x^{n-1}$ since $\mu_{x,\mathbb{k}}(x) = 0$, every polynomial expression of x can be expressed via $1, x, \ldots, x^{n-1}$. This proves that $\{1, x, x^2, \ldots, x^{n-1}\}$ spans $\mathbb{k}[x]$. Assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are not all zero when $\lambda_{11} + \lambda_2X + \cdots + \lambda_nx^{n-1} = 0$. Then there is a nonzero polynomial of degree $\leq n-1$ annihilating x, which contradicts the minimality of $\mu_{x,\mathbb{k}}$. This proves that $\{1, x, x^2, \ldots, x^{n-1}\}$ is linearly independent. Hence, it can be concluded that $\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis of $\mathbb{k}[x]$ as a $\mathbb{k}$-vector space, indicating that $\deg\mu_{x,\mathbb{k}} = [\mathbb{k}[x]{:}\,\mathbb{k}]$.

7.2. Relation between the extension degree and $\boldsymbol{r\#_{\mathbb{k}}s}$

Proposition 7.8. *Let $\mathbb{k}$ be a field that admits a field extension of degree d. Then for any $r \leq d$, $r\#_{\mathbb{k}}d = d$.*

Proof It suffices to construct a nonsingular bilinear form $\phi: \mathbb{k}^r \times \mathbb{k}^d \to \mathbb{k}^d$. Since $\mathbb{k}$ admits a field extension of degree d, say L, then the multiplication of L gives a symmetric $\mathbb{k}$-bilinear map

$$\begin{array}{cccc} \psi: & L \times L & \to & L \\ & (x, y) & \mapsto & xy. \end{array}$$

On the other hand, L is a $\mathbb{k}$-vector space of dimension d, so by a choice of basis, $L \cong \mathbb{k}^d$ as $\mathbb{k}$-vector spaces, which makes ψ_L to be a $\mathbb{k}$-bilinear map $\phi_L: \mathbb{k}^d \times \mathbb{k}^d \to \mathbb{k}^d$. Since $r \leq d$, we can choose a subspace of dimension r in $\mathbb{k}^d$. The restriction of ϕ_L gives a $\mathbb{k}$-bilinear map $\phi: \mathbb{k}^r \times \mathbb{k}^d \to \mathbb{k}^d$. We claim that ϕ is nonsingular. Let $u \in \mathbb{k}^r, v \in \mathbb{k}^d$ be nonzero elements. Regard u, v as elements in L. By the definition of ϕ_L and ϕ, since L is a field, $uv = 0$ implies either $u = 0$ or $v = 0$. Hence, $\phi: \mathbb{k}^r \times \mathbb{k}^d \to \mathbb{k}^d$ is nonsingular. ▫

Corollary 7.9. *Let $\mathbb{k}$ be a field that admits field extensions of any degree. Then for any $r, s \in \mathbb{N}$, $r\#_{\mathbb{k}}s = \max\{r, s\}$.*

7.3. Finite field theory

In this section, we recall following [9] some basic facts about finite fields that will be useful in the next section to the proof of Theorem 1.5 (iii), namely Proposition 7.16 . It is a well-known elementary fact that for any prime number p, the congruence system of integers modulo p forms a field, which we denote by $\mathbb{F}_p$. This is the starting point of the whole theory.

Proposition 7.10. *If F is a* finite *field, then the cardinal of F is a power of a prime number.*

Proof For any natural number n, we can regard n as an element in F defined to be the sum of n copies of 1's. Consider the following subset $\{0,1,2,\ldots,n,\ldots\} \subset F$. Since F is a finite field, this sequence terminates, say at $n-1$. Since F is a field, every element in $\{0,1,\ldots,n-1\}$ is invertible in F. In particular, the product of nonzero elements is nonzero. This implies that n is a prime number, say p. Hence, $\{0,1,2,\ldots,n-1\} \cong \mathbb{F}_p \subset F$. In other words, F is a field extension of $\mathbb{F}_p$, so it can be viewed as a $\mathbb{F}_p$-vector space. Since F is a finite field, it is a finitely dimensional $\mathbb{F}_p$-vector space. By a choice of a basis of F as an $\mathbb{F}_p$-vector space, $F \cong \mathbb{F}_p^n$ as $\mathbb{F}_p$-vector spaces. Hence, $|F| = |\mathbb{F}_p^n| = p^n$, as desired.

To construct other finite fields, we need the following fundamental result on algebraically closed fields.

Theorem 7.11. *Let $\Bbbk$ be a field. Then there exists a field extension Ω of $\Bbbk$ which is algebraically closed. Furthermore, if L is a field extension over $\Bbbk$ such that every element in L is algebraic over $\Bbbk$, then L can be embedded into Ω.*

Proposition 7.12. *Let p be a prime number and let n be a positive integer. Then there exists a unique field F with p^n elements, up to field isomorphism.*

Proof First, let us show the existence of such a field by an explicit construction. By Theorem 7.11, there is an algebraically closed extension Ω_p of $\mathbb{F}_p$. We define a subset $\mathbb{F}_{p^n} \subset \Omega_p$ as follows:

$$\mathbb{F}_{p^n} := \{x \in \Omega_p : x^{p^n} = x\}.$$

We are going to show that $\mathbb{F}_{p^n}$ is a field with p^n elements. To show that $\mathbb{F}_{p^n}$ has p^n elements, we prove that $X^{p^n} - X = 0$ has no multiple roots. Let $f(X) = X^{p^n} - X$, so that $f'(X) = p^n X^{p^n-1} - 1 = -1$. Suppose there exist multiple roots. It means that there exists some $x \in \Omega_p$ that satisfies

$$\begin{cases} f(x) = 0 \\ f'(x) = 0' \end{cases}$$

which is impossible since $f(x) = -1 \neq 0$. Hence, $X^{p^n} - X = 0$ has no multiple roots, implying that there are p^n distinct roots. Therefore, there are p^n elements in $\mathbb{F}_{p^n}$. Now let us show that $\mathbb{F}_{p^n}$ is a field. Assume $x, y \in \mathbb{F}_{p^n}$, so we can get that $x^{p^n} = x, y^{p^n} = y$. Since $x + y = x^{p^n} + y^{p^n} = (x+y)^{p^n} \in \mathbb{F}_{p^n}$, it meets the addition rule of a field. Assume $x \in \mathbb{F}_{\mathbb{p}^{\mathbb{m}}}, m \in \mathbb{N}$. Then mx can be expressed as $x^{p^n} + x^{p^n} + \cdots + x^{p^n}$ with m terms, which equals $(x + x + \cdots + x)^{p^n} = (mx)^{p^n}$. This proves that $mx \in \mathbb{F}_{p^n}$, so it also meets the multiplication rule of a field. Therefore, it can be concluded that $\mathbb{F}_{p^n}$ is a field.

Next, let us prove the uniqueness. Let F be a field with p^n elements. We must show that F is isomorphic to $\mathbb{F}_{p^n}$. Let F be a finite field with p^n. Similarly to Proposition 7.10, F is an extension of $\mathbb{F}_p$. By the Proposition 7.11, F can be embedded into Ω_p. TO show that $F \cong \mathbb{F}_{p^n}$, it suffices to check that for any $x \in F$, $x^{p^n} - x = 0$. If $x = 0$, then x clearly satisfies this equation. Now let us assume that $x \neq 0$. Let us consider $F^\times := F - \{0\}$ for which $|F^\times| = p^n - 1$. Since $x \in F^\times$. The map $F^\times \to F^\times$ defined by $y \mapsto xy$ is bijective. Hence, $\prod_{y \in F^\times} y = \prod_{y \in F^\times} xy = x^{p^n-1} \prod_{y \in F^\times} y$. But $\prod_{y \in F^\times} y \neq 0$, so $x^{p^n-1} = 1$ in F. Hence, $x^{p^n} = 0$ in F. ▫

Proposition 7.13. $\mathbb{F}_{p^m} \subset \mathbb{F}_{p^n}$ *if and only if m divides n.*

Proof Let us first prove that $\mathbb{F}_{p^m} \subset \mathbb{F}_{p^n}$ if m divides n. Let $x \in \mathbb{F}_{p^m}$, so that $x^{p^m} - x = 0$. Let us set $n = km, k \in \mathbb{N}$, so that $x^{p^n} = \left(x^{p^m}\right)^{p^{n-m}} = \left(x^{p^m}\right)^{p(k-1)m} = x^{p(k-1)m} = \left(x^{p^m}\right)^{p(k-2)m} = x^{p(k-2)m} = \cdots$. Through induction, we can finally get that $x^{p^n} = x$, which means that $x^{p^n} - x = 0$. Hence, $F_{p^n} \subset F_{p^m}$. For the other direction, assume $\mathbb{F}_{p^m} \in \mathbb{F}_{p^n}$. Then $\mathbb{F}_{p^m}$-vector spaces, $\mathbb{F}_{p^n} \cong \mathbb{F}_{p^m}^k$ for some $k \in \mathbb{N}$. Hence, $p^n = |\mathbb{F}_{p^n}| = |\mathbb{F}_{p^m}^k| = (p^m)^k = p^{mk}$, which proves that $m|n$, as desired.

7.4. Proof of theorem 5 (iii)

In this section, we prove Theorem 1.5 (iii). The main idea is to utilize Corollary 7.9, and to show that the field of rational numbers and finite fields admit field extensions of any degree. To prove this statement for the field of rational numbers, we will need the following well-known Eisenstein's criterion, whose proof can be found in [9].

Theorem 7.14 (Eisenstein's Criterion [9]). *Let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial with integral coefficients*

$$f(X) = X^n + a_{n-1}x^{n-1} + \cdots + a_1X + a_0, \text{where } a_i \in \mathbb{Z}.$$

Assume there exists a prime number p such that

(i) $p|a_i, i = 0,1, \ldots, n-1$,

(ii) $p^2 \nmid a_0$.

Then $f(X)$ is irreducible in $\mathbb{Q}[X]$.

Proposition 7.15. *The rational number field $\mathbb{Q}$ admits field extensions of any degree.*

Proof According to Theorem 7.14, $X^d - 2$ is an irreducible polynomial for any d. This single construction proves the existence of irreducible polynomials of any given degree in $\mathbb{Q}[X]$. In fact, let $\alpha \in \mathbb{C}$ be a root of $X^d - 2$ is the minimal polynomial of α. Hence, by Proposition 7.4 and Proposition 7.7 $\mathbb{Q}[\alpha]$ is a field whose extension degree over $\mathbb{Q}$ is d. Since d is arbitrary, we conclude that $\mathbb{Q}$ admits field extensions of any degree. ▫

Proposition 7.16. The *finite field $\mathbb{F}_q$ admits field extensions of any degree.*

By Proposition 7.13, since all natural numbers can be divided by 1, $\mathbb{F}_{q^d}$ is a field extension of $\mathbb{F}_q$ of degree d for any $d \in \mathbb{N}$. This directly proves that the finite field $\mathbb{F}_q$ admits field extensions of any degree. ▫

Proof of Theorem 1.5 (iii). According to Propositions 7.15 and 7.16, both the rational number field and finite fields admit field extensions of any degree. In this case, it can be easily concluded that $r\#_{\Bbbk}s = \max\{r, s\}$ for any $r, s \in \mathbb{N}$ by Corollary 7.9.

8. Conclusion

In this article, we have explored the existence of nonsingular bilinear maps between vector spaces over a field $\Bbbk$. The main focus of the study was the invariant $r\#_{\Bbbk}s$, which represents the minimal integer such that the condition $\mathcal{H}_{\Bbbk}(r, s, n)$ holds. We have provided some bounds for this invariant and discussed its dependence on the base field $\Bbbk$.

The main result of this research is presented in Theorem 1.5, which gives the value of $r\#_{\Bbbk}s$ for different types of fields $\Bbbk$. We proved that if $\Bbbk$ is an algebraically closed field, then $r\#_{\Bbbk}s = r + s - 1$. In the real number cases, we found that if the combinatorial number $\binom{r+s-2}{r-1}$ is odd, then $r\#_{\mathbb{R}}s = r + s - 1$. Finally, for the rational number field $\mathbb{Q}$ or a finite field $\mathbb{F}_q$, we established that $r\#_{\Bbbk}s = \max\{r, s\}$.

The proof of Theorem 1.5 (i) relied on algebraic geometry and used the fundamental theorem in algebraic geometry, Theorem 7, to derive the inequality $\dim W \geq \dim U + \dim V - 1$ for a nonsingular bilinear map $\phi: U \times V \to W$ over an algebraically closed field $\Bbbk$. This result has interesting applications in algebraic curves theory, as seen in the connection to Clifford's theorem.

The second proof of Theorem 1.5 (ii) made use of Hopf's theorem (Theorem 1.8) and some combinatorial observations. The theorem showed that whenever $n - s < k < r$, $\binom{n}{k}$ is even if there is a nonsingular bilinear map $\phi: \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$. This information was then used to derive $r\#_{\mathbb{R}}s = r + s - 1$ under certain conditions.

Furthermore, we provided another proof for Theorem 1.5 (ii) without relying on combinatorics, but rather through the construction of Grassmannian manifolds and determinantal varieties. This approach allowed us to show that under certain conditions, the degree of the determinantal variety in the projective space is odd, leading to the same result $r\#_{\mathbb{R}}s = r + s - 1$.

In the case of $\mathbb{Q}$ or a finite field $\mathbb{F}_q$, we showed that $r\#_{\mathbb{k}}s = \max\{r, s\}$, which was proven to be quite straightforward by considering the existence of field extensions of any degree.

In conclusion, this article successfully explores the existence of nonsingular bilinear maps over different fields, and provides insights into the invariant $r\#_{\mathbb{k}}s$. However, it is important to mention that the general calculation of $r\#_{\mathbb{k}}s$ remains challenging, as evidenced by the open problem of determining the precise value of $11\#_{\mathbb{R}}14$ in the domain. The research in this area may pave the way for further investigations and contribute to a deeper understanding of bilinear maps and their applications in various mathematical contexts.

8.1. Limitations of the study

While this article provides valuable insights into the existence of nonsingular bilinear maps and the behavior of the invariant $r\#_{\mathbb{k}}s$ for different fields, there are certain limitations to this study.

1. Complexity of Calculating $r\#_{\mathbb{k}}s$: The invariant $r\#_{\mathbb{k}}s$ is known to be quite challenging to calculate in general. Despite providing bounds and specific values for certain cases, a comprehensive formula or method for finding $r\#_{\mathbb{k}}s$ for all combinations of r and s remains an open problem.

2. Additional Conditions: In Theorem 1.5 (ii), we introduced an additional condition that relies on the combinatorial number $\binom{r+s-2}{r-1}$ being odd. While this condition was useful in obtaining the result for real number fields, it may not always be easy to determine for other fields, and it would be beneficial to explore alternative conditions for this case.

3. Generalization to Other Contexts: The study focused on nonsingular bilinear maps between vector spaces. However, bilinear maps have applications in diverse mathematical contexts, such as algebraic geometry, algebraic number theory, and representation theory. Generalizing the results of this study to other settings may open up new avenues for research.

8.2. Future trends

The research on nonsingular bilinear maps and the invariant $r\#_{\mathbb{k}}s$ holds great potential for future exploration in various directions. Here are some potential future trends that could be pursued:

1. Generalizing to Multilinear Maps: The study could be extended to investigate the existence and properties of nonsingular multilinear maps involving more than two vector spaces. Understanding the behavior of nonsingular multilinear maps and related invariants could lead to intriguing results and applications in higher-dimensional settings.

2. Connections with Representation Theory: Exploring the connections between nonsingular bilinear maps and representation theory could be a promising avenue. Investigating the role of nonsingular bilinear maps in the representation theory of Lie algebras and other algebraic structures may yield new insights.

3. Algebraic Topology and Bilinear Maps: The application of algebraic topology techniques to study nonsingular bilinear maps could provide new perspectives and proofs for the results presented in this article. Bridging the gap between algebraic topology and bilinear maps might lead to interesting connections and applications.

4. Computational Techniques: Developing computational methods to calculate $r\#_{\mathbb{k}}s$ for different fields and dimensions could be beneficial. Utilizing computer algebra systems and computational algebraic geometry tools might help to explore specific cases and patterns in the values of the invariant.

5. Applications in Physics and Engineering: Bilinear maps and related concepts have applications in physics and engineering. Exploring how the results of this study could be applied in these fields may lead to practical solutions and advancements.

In conclusion, the study of nonsingular bilinear maps and the invariant $r\#_{\mathbb{k}}s$ is an exciting and evolving area of research. By continuing to investigate these topics and exploring their connections to other mathematical and applied fields, researchers can contribute to a deeper understanding of bilinear maps and their significance in various contexts.

DOI: 10.54254/2753-8818/11/20230417

9. References

[1] Daniel B. Shapiro, *Compositions of Quadratic Forms*, 2000, de Gruyter Expositions in Mathematics 33.
[2] H. Hopf, *Ein topo1ogischer Beitrag zur reellen Algebra*, Comment. Math. Helv. 13 (1940/41), 219-239.
[3] L. Smith, *Nonsingular bilinear forms, generalized J homomorphisms, and the homotopy of spheres I*, 1978, Indiana Univ. Math. J. 27, 697–737.
[4] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of Algebraic Curves, Volume I*, 1985, Springer.
[5] D. Mumford, *The Red Book of Varieties and Schemes*, 1974, Lecture notes in mathematics 1358, Springer.
[6] J. Harris, L. W. Tu, *On symmetric and skew-symmetric determinantal varieties*, Topology, Volume 23, Issue 1, 1984, Pages 71-84.
[7] D. Huybrechts, *Complex Geometry, An Introduction*, Univertext, Springer.
[8] M. Reid, *Undergraduate Algebraic Geometry*, London Mathematical Society Student Texts 12, Cambridge University Press.
[9] M. Artin, *Algebra*, Pearson Prentice Hall, 2011.

Acknowledgment

Here I would like to express my sincere appreciation for my math teacher Weidong Hua, who introduced me to the realm of algebraic geometry, guided me through the learning of basic concepts, and offered me the inspiration for this article. I also appreciate my mother who provided infinite trust and support for me throughout the entire research process.